Research Article

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Approximate multi-Cauchy mappings on certain groupoids

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Abstract: In this article, we give a representation of multi-Cauchy mappings on groupoids as an equation and then establish the (Hyers and Găvruţa) stability of such mappings on groupoids. In the case that the range is a subset of Banach space, the stability result will be different.

Keywords: Găvruţa stability, groupoid, Hyers-Ulam stability, multi-Cauchy mapping

MSC 2020: 39B52, 39B72, 39B82

1 Introduction

The first stability problem regarding the Cauchy functional equation from a group G into a metric group H has been proposed by Ulam [1] in 1940. One year later, Hyers [2] was the first author who answered Ulam's problem when G and H are Banach spaces as follows:

Let G and H be Banach spaces. Suppose that $\varphi : G \longrightarrow H$ fulfills

$$\|\varphi(a+b) - \varphi(a) - \varphi(b)\| \leq \delta,$$

for all $a, b \in G$ and for some $\delta \geq 0$. Then, there exists an additive mapping $\mathcal{A} : G \longrightarrow H$ such that $\|\varphi(x) - \mathcal{A}(x)\| \leq \delta$ for all $a \in G$. Next, some generalizations of Hyers' result for additive and linear mappings have been studied by Aoki [3], Rassias [4], and Găvruţa [5]. More information about the stability of miscellaneous functional equations on various spaces is available, for instance, in books [6–8].

Throughout this article, $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ are the sets of all positive integers, rationals, and real numbers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty)$. Moreover, for the set E, we denote $\overbrace{E \times E \times \ldots \times E}^{n\text{-times}}$ by E^n.

Over the last two decades, the stability problem for functional equations has been studied by the authors for multiple variables mappings like multi-additive mappings. Here, we indicate their definitions:

Let V be a commutative group, W be a linear space over $\mathbb{Q}$, and $n \in \mathbb{N}$ with $n \geq 2$. A mapping $f : V^n \longrightarrow W$ is called *multi-additive* if it satisfies

$$\mathcal{A}(x+y) = \mathcal{A}(x) + \mathcal{A}(y), \tag{1.1}$$

in each variable. It is shown in [9, Theorem 2] that a mapping f is multi-additive if and only if it satisfies

$$f(x_1 + x_2) = \sum_{i_1,\ldots,i_n \in \{1,2\}} f(x_{i_1 1}, \ldots, x_{i_n n}),$$

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where $x_i = (x_{i1}, \ldots, x_{in}) \in V^n$ with $i \in \{1, 2\}$. For more information about the structure of multi-additive mappings and their Ulam's stability, we refer to [9–11] and [12, Sections 13.4 and 17.2].

Let $(G, \bullet)$ and $(H, *)$ be groupoids equipped with binary operations. A mapping $\phi : G \longrightarrow H$ is a homomorphism between groupoids if

$$\phi(a \bullet b) = \phi(a) * \phi(b), \quad (a, b \in G). \tag{1.2}$$

A particular case of (1.2) is the famous Cauchy (additive) functional equation (1.1), where G and H are two semigroups with the operation +.

The stability of the Cauchy functional equation on square-symmetric groupoids was investigated by Páles et al. [13] who generalized the classical theorem of Hyers for the first time. Moreover, Kim [14] established some stability results of the Cauchy functional equation from square-symmetric groupoids into metric square-symmetric groupoids based on the control function proposed by Găvruţa. Some results on the stability of Cauchy and Jensen equations through a fixed point method were investigated in [15]; for more results on the stability of Cauchy equation, we refer to [16–18].

In this article, we introduce the multi-Cauchy mappings as a system of the Cauchy functional equations. Then, we reduce such system to obtain a single equation. Finally, we prove the stability of the multi-Cauchy functional equations on the powers of a square-symmetric groupoid.

2 Representation of multi-Cauchy mappings

Let $(G, \bullet)$ and $(H, *)$ be groupoids equipped with given binary operations. For each $a \in G$, set $2a \coloneqq a \bullet a$. Similarly, for each $h \in H$, put $2h \coloneqq h * h$. The binary operation $\bullet$ is said to be *square-symmetric* if $2(a \bullet b) \coloneqq 2a \bullet 2b$. It is obvious that a commutative semigroup is a square-symmetric groupoid. The converse, however, is not true in general. For example, put $G \coloneqq \mathbb{N}$, $r > 1$ a fixed element in G, the binary operation $a \bullet b \coloneqq a + rb \ \ (a, b \in G)$. Then, $(G, \bullet)$ is a square-symmetric groupoid and $\bullet$ is not associative.

Recall that a groupoid $(G, \bullet)$ is divisible ($\bullet$ is divisible) if for each $a \in G$, there exists a unique element $a' \in G$ such that $2a' = a$. For convenience, we will write $\frac{a}{2} \coloneqq a'$ or $\frac{1}{2}a \coloneqq a'$. To simplify the notation, for each a in a groupoid $G \coloneqq (G, \bullet)$ and each $n \in \mathbb{N}$, we write $2^0 a \coloneqq a$ and $2^{n+1}a \coloneqq 2(2^n a)$. If, in addition, G is divisible, then we also write $\frac{a}{2^0} \coloneqq a$ and $\frac{a}{2^{n+1}} \coloneqq \frac{1}{2}\left(\frac{a}{2^n}\right)$ for all $n \in \mathbb{N}$. To reach our aims in this article, we consider the following mapping:

$$\sigma_\bullet : G \longrightarrow G; \quad a \mapsto 2a \quad \text{and} \quad \lambda_* : H \longrightarrow H; \quad h \mapsto 2h.$$

Note that the square-symmetric of $\bullet$ implies that the mapping $\sigma_\bullet$ is an endomorphism. According to the aforementioned fact, the binary operation $\bullet$ such that $\sigma_\bullet$ is an automorphism (bijective) is divisible.

For each $(t_1, \ldots, t_n), (s_1, \ldots, s_n) \in G^n$, we define the pointwise binary operation on G^n as follows:

$$(t_1, \ldots, t_n) \bullet (s_1, \ldots, s_n) = (t_1 \bullet s_1, \ldots, t_n \bullet s_n).$$

Consider the mapping $\Gamma_\bullet : G^n \longrightarrow G^n$ defined by $\Gamma_\bullet(s_1, \ldots, s_n) \coloneqq (\sigma_\bullet(s_1), \ldots, \sigma_\bullet(s_n))$. Moreover, for each $m \in \mathbb{N}$, we have

$$\Gamma_\bullet^m(s_1, \ldots, s_n) \coloneqq (\sigma_\bullet^m(s_1), \ldots, \sigma_\bullet^m(s_n)) = (2^m s_1, \ldots, 2^m s_n).$$

From now on, we denote $(2^m s_1, \ldots, 2^m s_n)$ by $2^m(s_1, \ldots, s_n)$ when no confusion can arise. It is easily seen that if $\sigma_\bullet$ is an automorphism, then so is $\Gamma_\bullet$.

A mapping $f : G^n \longrightarrow H$ is called *multi-Cauchy* (multi-additive) if, for each $j \in \{1, \ldots, n\}$ and all $a_i \in G$, the mapping $a \mapsto f(a_1, \ldots, a_{j-1}, a, a_{j+1}, \ldots, a_n)$ satisfies equation (1.2), i.e., for each $1 \le j \le n$,

$$f(a_1, \ldots, a_{j-1}, a_j \bullet a'_j, a_{j+1}, \ldots, a_n) = f(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n) * f(a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n).$$

Let $n \in \mathbb{N}$ with $n \ge 2$ and $a_i^n = (a_{i1}, a_{i2}, \ldots, a_{in}) \in G^n$, where $i \in \{1, 2\}$. We shall denote a_i^n by a_i when there is no risk of ambiguity. For $a_1, a_2 \in G^n$, we write $f(a_1) * f(a_2) \coloneqq *_{k \in \{1,2\}} f(a_k)$.

Assume that a square-symmetric groupoids G and H have the identities e_G and e_H, respectively. We consider the following condition on H:
(K) For $m \in \mathbb{N}$ and $h \in H$, if $mh = e_H$, then $h = e_H$.

There are plenty of known examples of groups with the condition (K), e.g., the group $(\mathbb{Z}, +)$. A mapping $f\colon G^n \longrightarrow H$ has the identity condition, if $f(a) = e_H$ for any $a \in G^n$ with at least one variable that is equal to e_G. It is easily checked that every multi-Cauchy mapping $f\colon G^n \longrightarrow H$ has identity condition provided that H has condition (K).

Set $\mathbf{n} := \{1, \dots, n\}$. For a subset $m = \{j_1, \dots, j_i\}$ of $\mathbf{n}$ with $1 \le j_1 < \dots < j_i \le n$ and $g = (g_1, \dots, g_n) \in V^n$,

$$_{m}g := (e_G, \dots, e_G, g_{j_1}, e_G, \dots, e_G, g_{j_i}, e_G, \dots, e_G) \in G^n$$

denotes the vector that coincides with g in exactly those components, which are indexed by the elements of m and whose other components are set equal e_G.

Lemma 2.1. *Let $(G, \bullet)$ be a groupoid and and $(H, *)$ be a commutative group. Suppose that a mapping $f\colon G^n \longrightarrow H$ satisfies*

$$f(a_1 \bullet a_2) = *_{j_1, \dots, j_n \in \{1,2\}}\, f(a_{j_1 1}, \dots, a_{j_n n}). \tag{2.1}$$

If H has the condition (K), then f has the identity condition.

Proof. We argue by induction on m that $f(_{m}g) = e_H$ for $0 \le m \le n-1$. Let $m = 0$. Substituting $a_1 = a_2 = (e_G, \dots, e_G)$ into (2.1), we have

$$f(e_G, \dots, e_G) = 2^n f(e_G, \dots, e_G).$$

Our assumption implies that $f(e_G, \dots, e_G) = e_H$. Assume that $f(_{m-1}a) = e_H$. We will prove that $f(_{m}a) = e_H$. Without loss of generality, we assume that the first m variables are not equal to e_G. Replacing (a_1, a_2) by $(_{m}a, e_G)$ in equation (2.1), we obtain $f(_{m}a) = 2^{n-m} f(_{m}a)$ and so $f(_{m}a) = e_H$. This completes the proof. □

Proposition 2.2. *Let $(G, \bullet)$ be a groupoid and and $(H, *)$ be a commutative group. If a mapping $f\colon G^n \longrightarrow H$ is multi-Cauchy, then it fulfills* (2.1). *The converse is true if H has the condition* (*K*).

Proof. Let f be multi-Cauchy mapping. We prove by induction on n that it satisfies equation (2.1). For $n = 1$, f fulfills (1.2). Assume that (2.1) is valid for some positive integer $n-1$. We will show that it is true for n. We have

$$\begin{aligned} f(a_1^n \bullet a_2^n) &= f(a_1^{n-1} \bullet a_2^{n-1}, a_{1n}) * f(a_1^{n-1} \bullet a_2^{n-1}, a_{2n}) \\ &= (*_{j_1, \dots, j_n \in \{1,2\}} f(a_{j_1 1}, \dots, a_{j_{n-1}, n-1}, a_{1n})) * (*_{j_1, \dots, j_n \in \{1,2\}} f(a_{j_1 1}, \dots, a_{j_{n-1}, n-1}, a_{2n})) \\ &= *_{j_1, \dots, j_n \in \{1,2\}} f(a_{j_1 1}, \dots, a_{j_n n}), \end{aligned}$$

which shows that (2.1) holds for n. For the converse, by substituting $a_2 = (e_H, \dots, e_H, a_{2j}, e_H, \dots, e_H)$ into (2.1) and using Lemma 2.1, we obtain

$$f(a_{11}, \dots, a_{1,j-1}, a_{1j} \bullet a_{2j}, \dots, a_{1n}) = f(a_1) * f(a_{11}, \dots, a_{1,j-1}, a_{2j}, \dots, a_{1n}),$$

and so the proof is now complete. □

3 Stability results for (2.1)

In this section, we prove some stabilities regarding equation (2.1).

Let (X, d) be a metric space. The Lipschitz modulus of a mapping $\phi : X \longrightarrow X$ is denoted by $\mathrm{Lip}(\phi)$ and defined by

$$\mathrm{Lip}(\phi) := \sup\left\{\frac{\mathrm{d}(\phi(x), \phi(y))}{\mathrm{d}(x, y)} \middle| x, y \in X, x \ne y\right\}.$$

It is simply to check that $\mathrm{Lip}(\phi \circ \psi) \le \mathrm{Lip}(\phi)\mathrm{Lip}(\psi)$.

One of the main goals of this article is the upcoming theorem, which is the Găvruţa stability of equation (2.1).

Theorem 3.1. *Let* $(G, \bullet)$ *be a square-symmetric groupoid and* $(H, *, d)$ *be a complete metric divisible semigroup. Suppose that a mapping* $f : G^n \longrightarrow H$ *fulfills*

$$d(f(a_1 \bullet a_2), *_{j_1,\ldots,j_n \in\{1,2\}} f(a_{j_1 1}, \ldots, a_{j_n n})) \le \varphi(a_1, a_2), \tag{3.1}$$

for all $a_1, a_2 \in G^n$ *in which* $\varphi : G^n \times G^n \longrightarrow \mathbb{R}_+$ *is a function that satisfies*

$$\sum_{m=0}^{\infty} \mathrm{Lip}(\lambda_*^{-m})\varphi(2^m a_1, 2^m a_2) < \infty, \tag{3.2}$$

for all $a_1, a_2 \in G^n$. *Then, for each* $a \in G^n$,

$$F(a) \coloneqq \lim_{m\to\infty} \lambda_*^{-m+1} \circ f(2^m a) \tag{3.3}$$

converges and the mapping $F : G^n \longrightarrow H$ *is a solution of* (2.1) *such that*

$$d(f(a), F(a)) \le \mathrm{Lip}(\lambda_*^{-1})\Phi(a), \tag{3.4}$$

for all $a \in G^n$, *where* $\Phi(a) \coloneqq \sum_{m=0}^{\infty}\mathrm{Lip}(\lambda_*^{-m})\varphi(2^m a, 2^m a)$. *In addition, if* $\{\mathrm{Lip}(\lambda_*^{-m})\mathrm{Lip}(\lambda_*^{m})\}$ *is bounded, then* F *is a unique solution of* (2.1) *for which the mapping* $a \mapsto d(f(a), F(a))$ *is bounded with bound* $\mathrm{Lip}(\lambda_*^{-1})\Phi(a)$.

Proof. Consider a family of mappings $F_m : G^n \longrightarrow H$ defined via $F_0 \coloneqq f$ and $F_m \coloneqq \lambda_*^{-m} \circ f \circ \Gamma_\bullet^m$, for all $m \in \mathbb{N}$. We shall show that for each fixed and arbitrary $a \in G^n$, the sequence $\{F_m\}$ is convergent. Replacing a_1 and a_2 by $2^m a_1$ in (3.1), we obtain

$$d(f(2^m a_1 \bullet 2^m a_1), f(2^m a_1) * f(2^m a_1)) \le \varphi(2^m a_1, 2^m a_1),$$

for all $a_1 \in G^n$. For the rest of proof, we set a_1 by a unless otherwise stated explicitly. The last inequality can be rewritten as form

$$d(f(2^{m+1}a), \lambda_* \circ f(2^m a)) \le \varphi(2^m a, 2^m a). \tag{3.5}$$

It follows from (3.5) that

$$\begin{aligned} d(F_{m+1}(a), F_m(a)) &= d(\lambda_*^{-(m+1)} \circ f(2^{m+1}a), \lambda_*^{-m+1} \circ \lambda_* \circ f(2^m a)) \\ &\le \mathrm{Lip}(\lambda_*^{-(m+1)})d(f(2^m a), \lambda_* \circ f(2^m a)) \\ &\le \mathrm{Lip}(\lambda_*^{-(m+1)})\varphi(2^m a, 2^m a) \\ &\le \mathrm{Lip}(\lambda_*^{-1})\mathrm{Lip}(\lambda_*^{-m})\varphi(2^m a, 2^m a), \end{aligned}$$

for all $a \in G^n$. It concludes from the aforementioned relation that for each $l, k \in \mathbb{N}$ with $k > l$,

$$d(F_k(a), F_l(a)) \le \sum_{j=l}^{k-1} d(F_{j+1}(a), F_j(a)) \le \sum_{j=l}^{k-1} \mathrm{Lip}(\lambda_*^{-(j+1)})\varphi(2^j a, 2^j a).$$

The convergence of series (3.2) implies that the last term in the aforementioned inequalities goes to zero as $l \to \infty$, which shows that $\{F_m\}$ is a Cauchy sequence, and hence, it converges to mapping $F : G^n \longrightarrow H$ as defined in (3.3), which is also well defined. Furthermore, for each $m \in \mathbb{N}$, we have

$$\begin{aligned} d(F_{m+1}(a), f(a)) &\le \sum_{j=0}^{m} d(F_{j+1}(a), F_j(a)) \\ &\le \sum_{j=0}^{m} \mathrm{Lip}(\lambda_*^{-(j+1)})\varphi(2^j a, 2^j a) \\ &\le \mathrm{Lip}\lambda_*^{-1} \sum_{j=0}^{m} \mathrm{Lip}(\lambda_*^{-j})\varphi(2^j a, 2^j a). \end{aligned}$$

Taking $m \to \infty$, we reach to (3.4). Replacing (a_1, a_2) by $(2^m a_1, 2^m a_2)$ in (3.1) and using the endomorphism of $\Gamma_\bullet$, we find

$$d(f \circ (\Gamma_\bullet^m(a_1 \bullet a_2)), *_{j_1,\dots,j_n \in \{1,2\}} f(2^m a_{j_1 1}, \dots, 2^m a_{j_n n})) \leq \varphi(2^m a_1, 2^m a_2),$$

for all $a_1, a_2 \in G^n$. On the other hand, λ_* is an automorphism of $(H, *)$ and so λ_*^{-m} is an automorphism as well, for all $m \in \mathbb{N}$. This discussion necessitates that

$$\begin{aligned} &d(F_m(a_1 \bullet a_2), *_{j_1,\dots,j_n \in \{1,2\}} F_m(a_{j_1 1}, \dots, a_{j_n n})) \\ &\quad = d(\lambda_*^{-m}(f \circ \Gamma_\bullet^m(a_1 \bullet a_2)), *_{j_1,\dots,j_n \in \{1,2\}} \lambda_*^{-m} \circ f \circ \Gamma_\bullet^m(a_{j_1 1}, \dots, a_{j_n n})) \\ &\quad = d(\lambda_*^{-m}(f \circ \Gamma_\bullet^m(a_1 \bullet a_2)), \lambda_*^{-m}(*_{j_1,\dots,j_n \in \{1,2\}} f \circ \Gamma_\bullet^m(a_{j_1 1}, \dots, a_{j_n n}))) \\ &\quad \leq \operatorname{Lip}(\lambda_*^{-m}) d(f \circ \Gamma_\bullet^m(a_1 \bullet a_2), *_{j_1,\dots,j_n \in \{1,2\}} f \circ \Gamma_\bullet^m(a_{j_1 1}, \dots, a_{j_n n})) \\ &\quad \leq \operatorname{Lip}(\lambda_*^{-m}) \varphi(2^m a_1, 2^m a_2). \end{aligned}$$

Letting the limit as $m \to \infty$ in the aforementioned relation and applying the definition of F_m, we see that F satisfies (2.1). For the uniqueness of F, assume that $\mathcal{F}$ is an arbitrary solution of (2.1) such that the mapping $a \mapsto d(f(a), \mathcal{F}(a))$, is bounded with bound $\operatorname{Lip}\lambda_*^{-1}\Phi(a)$. Substituting $a_1 = a_2 := a$ into (2.1), we obtain

$$F \circ \Gamma_\bullet = \lambda_* \circ F, \quad \mathcal{F} \circ \Gamma_\bullet = \lambda_* \circ \mathcal{F}.$$

One can prove by induction that

$$\lambda_*^{-m} \circ F \circ \Gamma_\bullet^m = F, \quad \lambda_*^{-m} \circ \mathcal{F} \circ \Gamma_\bullet^m = \mathcal{F}.$$

Now, for each $a \in G^n$ we have

$$\begin{aligned} d(F(a), \mathcal{F}(a)) &= d(\lambda_*^{-m} \circ F(2^m a), \lambda_*^{-m} \circ \mathcal{F}(2^m a)) \\ &\leq \operatorname{Lip}(\lambda_*^{-m}) d(F(2^m a), \mathcal{F}(2^m a)) \\ &\leq \operatorname{Lip}(\lambda_*^{-m})[d(F(2^m a), f(2^m a)) + d(f(2^m a), \mathcal{F}(2^m a))] \\ &\leq 2\operatorname{Lip}(\lambda_*^{-m}) \sum_{j=0}^{\infty} \operatorname{Lip}(\lambda_*^{-(j+1)}) \varphi(\Gamma_\bullet^j(\Gamma_\bullet^m)(a), \Gamma_\bullet^j(\Gamma_\bullet^m)(a)) \\ &\leq 2\operatorname{Lip}(\lambda_*^{-m}) \operatorname{Lip}(\lambda_*^{m}) \sum_{j=0}^{\infty} \operatorname{Lip}(\lambda_*^{-(m+j+1)}) \varphi(2^{m+j} a, 2^{m+j} a) \\ &= 2\operatorname{Lip}(\lambda_*^{-m}) \operatorname{Lip}(\lambda_*^{m}) \sum_{j=m}^{\infty} \operatorname{Lip}(\lambda_*^{-(j+1)}) \varphi(2^j a, 2^j a). \end{aligned}$$

Due to the boundedness of $\{\operatorname{Lip}(\lambda_*^{-m})\operatorname{Lip}(\lambda_*^{m})\}$, it follows that $F = \mathcal{F}$, when $m \to \infty$. This completes the proof. □

The next corollary is a direct consequence of Theorem 3.1 concerning the stability of equation (2.1).

Corollary 3.2. *Let $(G, \bullet)$ be a square-symmetric groupoid and $(H, *, d)$ be a complete metric divisible semigroup. Let $f : G^n \longrightarrow H$ satisfy* (3.1) *and $\varphi : G^n \times G^n \longrightarrow \mathbb{R}_+$ be a function such that for some real number L with* $0 \leq L < 1$, *the inequality*

$$d(\lambda_*^{-m}(h_1)), \lambda_*^{-m}(h_2)\varphi(2^m a_1, 2^m a_2) \leq Ld(\lambda_*^{-m+1}(h_1)), \lambda_*^{-m+1}(h_2)\varphi(2^{m-1} a_1, 2^{m-1} a_2), \tag{3.6}$$

holds for all $a_1, a_2 \in G^n$, $h_1, h_2 \in H$, and $n \in \mathbb{N}$. Then, for each $a \in G^n$, limit (3.3) *exists and the mapping $F : G^n \longrightarrow H$ is a unique solution of* (2.1) *such that*

$$d(f(a), F(a)) \leq \frac{1}{1-L}\varphi(a, a),$$

for all $a \in G^n$.

Proof. Substituting $a_1 = a_2 = a$ and $m = 1$ into (3.6), we have

$$d(\lambda_*^{-1}(h_1)), \lambda_*^{-1}(h_2)\varphi(2a, 2a) \le Ld(h_1, h_2)\varphi(a, a),$$

or equivalently,

$$\mathrm{Lip}\,(\lambda_*^{-1})\varphi(2a, 2a) \le L\varphi(a, a),$$

for all $a \in G^n$. By induction and using (3.6), one can obtain

$$\mathrm{Lip}\,(\lambda_*^{-m})\varphi(2^m a, 2^m a) \le L^m\varphi(a, a),$$

for all $a \in G^n$. Therefore, the bound of (3.4) defined in Theorem 3.1 does not exceed $\frac{1}{1-L}\mathrm{Lip}\,(\lambda_*^{-1})\varphi(a, a)$. □

In the following result, we show Hyers' stability of equation (2.1).

Corollary 3.3. *Let* $\delta > 0$. *Let* $(G, \bullet)$ *be a square-symmetric groupoid and* $(H, *, d)$ *be a complete metric divisible semigroup. Suppose that a mapping* $f : G^n \longrightarrow H$ *fulfills*

$$d(f(a_1 \bullet a_2), *_{j_1,\dots,j_n \in\{1,2\}} f(a_{j_1 1}, \dots, a_{j_n n})) \le \delta,$$

for all $a_1, a_2 \in G^n$. *Then, for each* $a \in G^n$,

$$F(a) \coloneqq \lim_{m\to\infty} \lambda_*^{-m+1} \circ f(2^m a)$$

converges, and the mapping $F : G^n \longrightarrow H$ *is a solution of* (2.1) *such that*

$$d(f(a), F(a)) \le \delta\mathrm{Lip}(\lambda_*^{-1}) \sum_{m=0}^{\infty} \mathrm{Lip}(\lambda_*^{-m}),$$

for all $a \in G^n$. *Moreover,* F *is a unique solution of* (2.1) *such that the mapping* $a \mapsto d(f(a), F(a))$, *is bounded.*

Proof. Let $\varphi(a_1, a_2) = \delta$. Then, the result will be found by Theorem 3.1. □

In analogy with Theorem 3.1, we have the next alternative result for the stability of (2.1). In this theorem, we add the condition divisibility for the square-symmetric groupoid $(G, \bullet)$.

Theorem 3.4. *Let* $(G, \bullet)$ *be a square-symmetric divisible groupoid and* $(H, *, d)$ *be a complete metric divisible semigroup. Suppose that a mapping* $f : G^n \longrightarrow H$ *fulfills*

$$d(f(a_1 \bullet a_2), *_{j_1,\dots,j_n \in\{1,2\}} f(a_{j_1 1}, \dots, a_{j_n n})) \le \varphi(a_1, a_2), \tag{3.7}$$

for all $a_1, a_2 \in G^n$ *in which* $\varphi : G^n \times G^n \longrightarrow \mathbb{R}_+$ *is a function satisfying*

$$\sum_{m=0}^{\infty} \mathrm{Lip}(\lambda_*^{m})\varphi(2^{-m}a_1, 2^{-m}a_2) < \infty, \tag{3.8}$$

for all $a_1, a_2 \in G^n$. *Then, for each* $a \in G^n$,

$$F(a) \coloneqq \lim_{m\to\infty} \lambda_*^{m+1} \circ f(2^{-m}a), \tag{3.9}$$

converges and the mapping $F : G^n \longrightarrow H$ *is a solution of* (2.1) *such that*

$$d(f(a), F(a)) \le \mathrm{Lip}\lambda_*^{-1}\Psi(a), \tag{3.10}$$

for all $a \in G^n$, *where* $\Psi(a) \coloneqq \sum_{m=0}^{\infty}\mathrm{Lip}\,(\lambda_*^{m})\varphi(2^{-m}a, 2^{-m}a)$. *In addition, if* $\{\mathrm{Lip}\,(\lambda_*^{-m})\mathrm{Lip}\,(\lambda_*^{m})\}$ *is bounded, then* F *is a unique solution of* (2.1) *for which the mapping* $a \mapsto d(f(a), F(a))$, *is bounded with bound* $\mathrm{Lip}\lambda_*^{-1}\Psi(a)$.

Proof. Define the mappings $F_m : G^n \longrightarrow H$ defined via $F_0 \coloneqq f$ and $F_m \coloneqq \lambda_*^m \circ f \circ \Gamma_\bullet^{-m}$, for all $m \in \mathbb{N}$. Replacing a_1 and a_2 by $2^{-m}a_1$ in (3.7), we have

$$d(f(2^{-m}a_1 \bullet 2^{-m}a_1), f(2^{-m}a_1) * f(2^{-m}a_1)) \leq \varphi(2^{-m}a_1, 2^{-m}a_1),$$

for all $a_1 \in G^n$. For the rest of proof, we set a_1 by a unless otherwise stated explicitly. We reform the last relation as follows:

$$d(f(2^{-m+1}a), \lambda_* \circ f(2^{-m}a)) \leq \varphi(2^{-m}a, 2^{-m}a). \tag{3.11}$$

By (3.11), we have

$$\begin{aligned} d(F_{m-1}(a), F_m(a)) &= d(\lambda_*^{m-1} \circ f(2^{-m+1}a), \lambda_*^{m-1} \circ \lambda_* \circ f(2^{-m}a)) \\ &\leq \mathrm{Lip}(\lambda_*^{m-1}) d(f(2^{-m+1}a), \lambda_* \circ f(2^{-m}a)) \\ &\leq \mathrm{Lip}(\lambda_*^{m-1})\varphi(2^{-m}a, 2^{-m}a), \end{aligned}$$

for all $a \in G^n$. A direct result from the aforementioned relation shows that for each $l, k \in \mathbb{N}$,

$$\begin{aligned} d(F_k(a), F_{k+l}(a)) &\leq \sum_{j=l}^{l-1} d(F_{k+j}(a), F_{k+j+1}(a)) \\ &\leq \sum_{j=0}^{l-1} \mathrm{Lip}(\lambda_*^{k+j})\varphi(2^{-k-j-1}a, 2^{-k-j-1}a) \\ &\leq \sum_{j=k}^{k+l-1} \mathrm{Lip}(\lambda_*^{-1})\mathrm{Lip}(\lambda_*^{j+1})\varphi(2^{-j-1}a, 2^{-j-1}a). \end{aligned}$$

It follows from the convergence of series (3.8) that the last term in the aforementioned inequalities intends to zero as $k \to \infty$, which shows that $\{F_m\}$ is a Cauchy sequence in H. Therefore, a mapping $F : G^n \longrightarrow H$ as defined in (3.9) is well defined. Moreover, for each $m \in \mathbb{N}$, we have

$$\begin{aligned} d(F_{m+1}(a), f(a)) &\leq \sum_{j=0}^{m} d(F_{j+1}(a), F_j(a)) \\ &\leq \sum_{j=0}^{m} \mathrm{Lip}(\lambda_*^{j})\varphi(2^{-j-1}a, 2^{-j-1}a) \\ &\leq \sum_{j=1}^{\infty} \mathrm{Lip}(\lambda_*^{j-1})\varphi(2^{-j}a, 2^{-j}a) \\ &\leq \mathrm{Lip}\lambda_*^{-1} \sum_{j=1}^{\infty} \mathrm{Lip}(\lambda_*^{j})\varphi(2^{-j}a, 2^{-j}a). \end{aligned}$$

Letting $m \to \infty$, we obtain (3.10). Replacing (a_1, a_2) by $(2^{-m}a_1, 2^{-m}a_2)$ in (3.7) and using the endomorphism of $\Gamma_\bullet^m$, we obtain

$$\begin{aligned} & d(F_m(a_1 \bullet a_2), *_{j_1,\dots,j_n \in \{1,2\}} F_m(a_{j_1 1}, \dots, a_{j_n n})) \\ &= d(\lambda_*^m(f \circ \Gamma_\bullet^{-m}(a_1 \bullet a_2)), *_{j_1,\dots,j_n \in \{1,2\}} \lambda_*^m \circ f \circ \Gamma_\bullet^{-m}(a_{j_1 1}, \dots, a_{j_n n})) \\ &= d(\lambda_*^m(f \circ \Gamma_\bullet^{-m}(a_1 \bullet a_2)), \lambda_*^m(*_{j_1,\dots,j_n \in \{1,2\}} f \circ \Gamma_\bullet^{-m}(a_{j_1 1}, \dots, a_{j_n n}))) \\ &\leq \mathrm{Lip}(\lambda_*^m) d(f \circ \Gamma_\bullet^{-m}(a_1 \bullet a_2), *_{j_1,\dots,j_n \in \{1,2\}} f \circ \Gamma_\bullet^{-m}(a_{j_1 1}, \dots, a_{j_n n})) \\ &\leq \mathrm{Lip}(\lambda_*^m)\varphi(2^{-m}a_1, 2^{-m}a_2). \end{aligned}$$

Taking $m \to \infty$ in the aforementioned relation and using the definition of F_m, we find that F fulfills (2.1). Here, we show the uniqueness of F. Assume that $\mathcal{F}$ is an arbitrary solution of (2.1) such that the mapping $a \mapsto d(f(a), \mathcal{F}(a))$, is bounded with bound $\mathrm{Lip}\lambda_*^{-1}\Psi(a)$. Similar to the proof of Theorem 3.1, one can prove by induction that

$$\lambda_*^m \circ F \circ \Gamma_\bullet^{-m} = F, \quad \lambda_*^m \circ \mathcal{F} \circ \Gamma_\bullet^{-m} = \mathcal{F}.$$

Note that $\{\mathrm{Lip}(\lambda_*^{-m})\mathrm{Lip}(\lambda_*^m)\}$ is bounded and it is assumed that $\mathcal{F}$ is another solution of (2.1) such that $a \mapsto d(f(a)), d(\mathcal{F}(a))$ is bounded with bound $\mathrm{Lip}\lambda_*^{-1}\Psi(a)$. Thus, for each $a \in G^n$, we have

$$\begin{aligned} d(F(a), \mathcal{F}(a)) &= d(\lambda_*^m \circ F(2^{-m}a), \lambda_*^m \circ \mathcal{F}(2^{-m}a)) \\ &\le \mathrm{Lip}(\lambda_*^m) d(F(2^{-m}a), \mathcal{F}(2^{-m}a)) \\ &\le \mathrm{Lip}(\lambda_*^m)[d(F(2^{-m}a), f(2^{-m}a)) + d(f(2^{-m}a), \mathcal{F}(2^{-m}a))] \\ &\le 2\mathrm{Lip}(\lambda_*^{-1})\mathrm{Lip}(\lambda_*^m)\sum_{j=0}^{\infty} \mathrm{Lip}(\lambda_*^j)\varphi(\Gamma_\bullet^{-j}(\Gamma_\bullet^{-m})(a), \Gamma_\bullet^{-j}(\Gamma_\bullet^{-m})(a)) \\ &\le 2\mathrm{Lip}(\lambda_*^{-1})\mathrm{Lip}(\lambda_*^m)\mathrm{Lip}(\lambda_*^{-m})\sum_{j=0}^{\infty} \mathrm{Lip}(\lambda_*^{m+j})\varphi(2^{-m-j}a, 2^{-m-j}a) \\ &= 2\mathrm{Lip}(\lambda_*^{-1})\mathrm{Lip}(\lambda_*^m)\mathrm{Lip}(\lambda_*^{-m})\sum_{j=m+1}^{\infty} \mathrm{Lip}(\lambda_*^j)\varphi(2^{-j}a, 2^{-j}a). \end{aligned}$$

Since the sequence $\{\mathrm{Lip}\lambda_*^{-m}\mathrm{Lip}(\lambda_*^m)\}$ is bounded, this proves that $F = \mathcal{F}$, and therefore, the proof is finished. □

Corollary 3.5. *Let* $(G, \bullet)$ *be a square-symmetric groupoid and* $(H, *, d)$ *be a complete metric divisible semigroup. Let* $f: G^n \longrightarrow H$ *satisfy* (3.7) *and* $\varphi: G^n \times G^n \longrightarrow \mathbb{R}_+$ *be a function such that for some real number* L *with* $0 \le L < 1$, *the inequality*

$$d(\lambda_*^m(h_1)), \lambda_*^m(h_2)\varphi(2^{-m}a_1, 2^{-m}a_2) \le Ld(\lambda_*^{m-1}(h_1)), \lambda_*^{m-1}(h_2)\varphi(2^{-m+1}a_1, 2^{-m+1}a_2) \tag{3.12}$$

holds for all $a_1, a_2 \in G^n$, $h_1, h_2 \in H$, *and* $n \in \mathbb{N}$ *Then, for each* $a \in G^n$, *limit* (3.9) *exists and the mapping* $F: G^n \longrightarrow H$ *is a unique solution of* (2.1) *such that*

$$d(f(a), F(a)) \le \frac{L}{1-L}\varphi(a, a),$$

for all $a \in G^n$.

Proof. Substituting $a_1 = a_2 = a$ and $m = 1$ into (3.12), we obtain

$$d(\lambda_*(h_1)), \lambda_*(h_2)\varphi(2^{-1}a, 2^{-1}a) \le Ld(h_1, h_2)\varphi(a, a),$$

for all $a \in G^n$. It follows from the aforementioned equality that

$$\mathrm{Lip}(\lambda_*)\varphi(2^{-1}a, 2^{-1}a) \le L\varphi(a, a),$$

for all $a \in G^n$. By induction, we obtain, due to (3.12)

$$\mathrm{Lip}(\lambda_*^m)\varphi(2^{-m}a, 2^{-m}a) \le L^m\varphi(a, a),$$

for all $a \in G^n$. Hence, the bound of (3.4) defined in Theorem 3.4 does not exceed $\frac{L}{1-L}\mathrm{Lip}(\lambda_*^{-1})\varphi(a, a)$. □

In analogue to Corollary 3.3, equation (2.1) has Hyers' stability as follows.

Corollary 3.6. *Let* $\delta > 0$. *Let* $(G, \bullet)$ *be a square-symmetric groupoid and* $(H, *, d)$ *be a complete metric divisible semigroup. Suppose that a mapping* $f: G^n \longrightarrow H$ *fulfills*

$$d(f(a_1 \bullet a_2), *_{j_1, \ldots, j_n \in \{1,2\}} f(a_{j_1 1}, \ldots, a_{j_n n})) \le \delta,$$

for all $a_1, a_2 \in G^n$. *Then, for each* $a \in G^n$,

$$F(a) := \lim_{m\to\infty} \lambda_*^{m+1} \circ f(2^{-m}a),$$

converges, and the mapping $F : G^n \longrightarrow H$ *is a solution of* (2.1) *such that*

$$d(f(a), F(a)) \leq \delta \mathrm{Lip}(\lambda_*^{-1}) \sum_{m=0}^{\infty} \mathrm{Lip}(\lambda_*^{m}),$$

for all $a \in G^n$. *Moreover,* F *is a unique solution of* (2.1) *for which the mapping* $a \mapsto d(f(a), F(a))$, *is bounded.*

Proof. Letting $\varphi(a_1, a_2) = \delta$, we obtain the desired result from Theorem 3.4. □

Let X be a Banach space over $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$. We define the binary operation $*$ on X by $x * y = rx + sy$, where $r, s \in \mathbb{K}$, which are fixed. For the element $(a_{l_1 1}, \dots, a_{l_n n}) \in G^n$, we put $t_i = \mathrm{Card}\{l_j : \ l_j = 1\}$. Clearly, $0 \leq s_i \leq n$. With these explanations, for a multi-Cauchy mapping $f : G^n \longrightarrow X$, equation (2.1) converts to

$$f(a_1 \bullet a_2) = \sum_{\substack{0 \leq t_i \leq n \\ l_1, \dots, l_n \in \{1,2\}}} r^{t_i} s^{n-t_i} f(a_{l_1 1}, \dots, a_{l_n n}), \tag{3.13}$$

for all $a_1, a_2 \in G^n$.

Corollary 3.7. *Let* $(G, \bullet)$ *be a square-symmetric groupoid and* X *be a Banach space over* $\mathbb{K}$. *Suppose that a mapping* $f : G^n \longrightarrow X$ *fulfills*

$$\left\| f(a_1 \bullet a_2) - \sum_{\substack{0 \leq t_i \leq n \\ l_1, \dots, l_n \in \{1,2\}}} r^{t_i} s^{n-t_i} f(a_{l_1 1}, \dots, a_{l_n n}) \right\| \leq \varphi(a_1, a_2),$$

for all $a_1, a_2 \in G^n$ *in which* $\varphi : G^n \times G^n \longrightarrow \mathbb{R}_+$ *is a function satisfying one of the following conditions*:
(*i*) $|r + s| \neq 0$ *and* $\sum_{m=0}^{\infty} |r + s|^{-m} \varphi(2^m a_1, 2^m a_2) < \infty$;
(*ii*) $(G, \bullet)$ *is divisible and* $\sum_{m=1}^{\infty} |r + s|^{m} \varphi(2^{-m} a_1, 2^{-m} a_2) < \infty$,

for all $a_1, a_2 \in G^n$. *Then, there exists a uniquely determined solution of* (3.13) *such that*

$$\|f(a) - F(a)\| \leq T,$$

for all $a \in G^n$, *where*

$$T := \begin{cases} \displaystyle\sum_{m=0}^{\infty} |r + s|^{-m} \varphi(2^m a_1, 2^m a_2), & \text{if } (i) \text{ holds}, \\ \displaystyle\sum_{m=1}^{\infty} |r + s|^{m} \varphi(2^m a_1, 2^{-m} a_2), & \text{if } (ii) \text{ holds}. \end{cases}$$

Proof. It is easily checked that $\mathrm{Lip}(\lambda_*^{m}) = |r + s|^m$ and $\mathrm{Lip}(\lambda_*^{-m}) = |r + s|^{-m}$. Therefore, the desired results can be obtained by Theorems 3.1 and 3.4. □

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