Research Article

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Finite group with some $c^{\#}$-normal and S-quasinormally embedded subgroups

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Abstract: Let p be a prime that divides the order of a finite group G, and let P be a Sylow p-subgroup of G. Assume that d is the smallest number of generators of P and define $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$ as a collection of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$, the Frattini subgroup of P. This article focuses on the exploration of the structure of a finite group G for which every element of $\mathcal{M}_d(P)$ is either $c^{\#}$-normal or S-quasinormally embedded in G. Our results improve and generalize many known results.

Keywords: finite groups, $c^{\#}$-normal subgroups, S-quasinormally embedded subgroups, supersolvable groups

MSC 2020: 20D10, 20D20

1 Introduction

All groups are assumed to be finite. Our notation and terminology are standard (see, e.g., Robinson [1]).

Following Gaschütz [2], a CAP-subgroup of a group G is termed as a subgroup K of G, provided that K either covers or avoids every chief factor A/B within G, namely, either $KA = KB$ or $K \cap A = K \cap B$. As a generalization of CAP-subgroup, Wang and Wei [3] introduced the concept of $c^{\#}$-normal subgroup. A subgroup H of a group G is said to be a $c^{\#}$-normal subgroup of G, provided that there exists a normal subgroup M of G satisfying the conditions $G = HM$ and $H \cap M$ being a CAP-subgroup of G [3, Definition 2.3]. They investigated the impact of $c^{\#}$-normality of certain subgroups of prime power order on the p-supersolvability and p-nilpotency of a group. They proved the following: let N be normal subgroup of a p-solvable group G with G/N being p-supersolvable. If every maximal subgroup of a Sylow p-subgroup P of G is $c^{\#}$-normal in G, then G itself is p-supersolvable [4]. Furthermore, they also considered the special case, and they obtained the following result: consider a normal subgroup N of a group G with G/N being p-nilpotent, and denote by M a Sylow p-subgroup of N, where p is the smallest prime factor of dividing the order of G. Suppose that every maximal subgroup of M is a $c^{\#}$-normal subgroup of G, then G is p-nilpotent. It has been proved that the $c^{\#}$-normal subgroups provided better tools for us to study the structure of finite groups (e.g., [3–5]).

A subgroup L of a group G is termed S-quasinormal in G whenever it permutes with all Sylow subgroups of G. Since Kegel [6] introduced the concept of S-quasinormal subgroup, there has been much interest in investigating the topic on the S-quasinormality. In 1998, Ballester-Bolinches and Pedraza-Aguilera [7] introduced the concept of S-quasinormally embedded subgroups as a generalization of S-quasinormal subgroups. A subgroup L of a group G is termed S-quasinormally embedded in G if each Sylow subgroup of L is also a Sylow subgroup of an S-quasinormal subgroup of G [7, Definition]. It is proved in [7] that a group G is supersolvable if

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all of its Sylow subgroups' maximal subgroups are S-quasinormally embedded in G. Motivated by the result, Asaad and Heliel [8] showed that a group G is a p-nilpotent group if and only if each maximal subgroup of its Sylow p-subgroup is S-quasinormally embedded in G, where p is the smallest prime dividing $|G|$. These results have been generalized in several studies such as [7–13]. We note that S-quasinormally embedded subgroups and $c^{\#}$-normal subgroups are two different concepts.

Example 1.1. Let $G = A_5$ be the alternating group of degree 5, and let X be a Sylow subgroup of G. It is evident that X is an S-quasinormally embedded in G. We can see that $G = XG$ and $X \cap G = X$. But X neither covers $G/1$ nor avoids $G/1$. Thus, X is not a $c^{\#}$-normal subgroup of G.

Let $G = S_4$ be the symmetric group of degree 4 and $X = \langle(12)\rangle$. Then, $G = XA_4$ and so $X \cap A_4 = 1$, which implies that X is $c^{\#}$-normal in G. But X is not S-quasinormally embedded in G.

It is a natural question to ask how much information about the structure of a finite group we can obtain when a small quantity of $c^{\#}$-normal or S-quasinormally embedded maximal subgroups of Sylow subgroups. In order to use fewer $c^{\#}$-normal or S-quasinormally embedded subgroups to characterize the structure of a finite group G, we employ the following definition (refer to [14]).

Definition 1.2. Let P be a p-group with the smallest generator number d, and denote by $\mathcal{M}_d(P)$ a collection of maximal subgroups $\{P_1, P_2, \ldots, P_d\}$ of P, which satisfies $\bigcap_{i=1}^{d} P_i = \Phi(P)$, the Frattini subgroup of P.

For a given P, it is evident that $\mathcal{M}_d(P)$ is not uniquely determined. We know that P contains $\frac{p^d-1}{p-1}$ maximal subgroups and

$$\lim_{d\to\infty} \frac{(p^d-1)/(p-1)}{d} = \infty,$$

so

$$\frac{p^d-1}{p-1} \gg |\mathcal{M}_d(P)| = d.$$

In this article, we use the $c^{\#}$-normality or S-quasinormally embedding of maximal subgroups of Sylow subgroup P in $\mathcal{M}_d(P)$ to characterize the structure of a group G. We obtain results concerning the p-supersolvability, p-nilpotency, and supersolvability of G and generalize many known results.

2 Preliminaries

In this section, we show some lemmas, which are required in the proofs of our main results.

Lemma 2.1. [3, Lemma 2.5] *Let N be a normal subgroup of a group G, and let H be a $c^{\#}$-normal subgroup of G. Then, HN/N is a $c^{\#}$-normal subgroup of G/N if one of the following holds*:
(*a*) $N \leq H$.
(*b*) $(|H|, |N|) = 1$, *where* $(-, -)$ *denotes the greatest common divisor.*

Lemma 2.2. [7, Lemma 1] *Suppose that U is an S-quasinormally embedded subgroup of a group G, and K is a normal subgroup of G. Then,*
(*a*) *For any subgroup H of G such that $U \leq H \leq G$, U is S-quasinormally embedded in H.*
(*b*) *The subgroup UK is S-quasinormally embedded in G and the quotient group UK/K is S-quasinormally embedded in G/K.*

The following lemmas are related to S-quasinormal subgroups.

Lemma 2.3. [15, Theorem 1] *Given that H is an S-quasinormal subgroup of a group G, the quotient group H/H_G is nilpotent.*

Lemma 2.4. [16, Proposition B] *For a nilpotent subgroup H a group G, the equivalence of the following two statements holds*:
(*a*) *H is S-quasinormal within G.*
(*b*) *The Sylow subgroups of H are S-quasinormal within G.*

Lemma 2.5. [8, Lemma 2.6] *Let G_p be a Sylow p-subgroup of a group G, and let P be a maximal subgroup of G_p. The equivalence of the following two statements holds*:
(*a*) *P is normal within G.*
(*b*) *P is S-quasinormal within G.*

Lemma 2.6. [6] and [16] *Let H be an S-quasinormal subgroup of a group G. Then,*
(*a*) *If K is an S-quasinormal subgroup of G, then $H \cap K$ is also an S-quasinormal subgroup of G.*
(*b*) *If H is a p-subgroup, then $O^p(G) \le N_G(H)$.*

Lemma 2.7. [17] *Given that P is a Sylow p-subgroup of a group G and N is a normal subgroup of G with $P \cap N \le \Phi(P)$, it follows that N is p-nilpotent.*

Lemma 2.8. [18, I, Satz 17.4] *Let N be a normal abelian subgroup of a group G. Assume that $N \le M \le G$, where $|N|$ and the index of M in G are coprime. If N is complemented in M, then N is complemented in G.*

Lemma 2.9. [19, Lemma 2.6] *Assume that N is a non-trivial solvable normal subgroup of a group G. If every minimal normal subgroup of G that is contained in N is not contained in $\Phi(G)$, then the Fitting subgroup $F(N)$ of N is given by the direct product of those minimal normal subgroups of G contained in N.*

Lemma 2.10. [1, Theorem 9.3.1] *Assume G is a π-separable group and $O_{\pi'}(G) = 1$, and it follows that $C_G(O_\pi(G)) \le O_\pi(G)$.*

Lemma 2.11. [20, Lemma 2.8] *Suppose G is a group and p is a prime number that divides the order of $|G|$ such that* $(|G|, p - 1) = 1$:
(1) *If N is normal in G of order p, then $N \le Z(G)$.*
(2) *If G has cyclic Sylow p-subgroup, then G is p-nilpotent.*
(3) *If $M \le G$ and $|G : M| = p$, then $M \trianglelefteq G$.*

3 Main results

In this section, we first study the p-supersolvability of a group G when all members of $\mathcal{M}_d(P)$ assumed to be $c^{\#}$-normal or S-quasinormally embedded in G.

Theorem 3.1. *Assume G is a group that is p-solvable, and denote by P a Sylow p-subgroup of G for a prime p dividing the order of G. Then, G is p-supersolvable if and only if all members in some fixed $\mathcal{M}_d(P)$ are either $c^{\#}$-normal or S-quasinormally embedded in G.*

Proof. We only need to prove the sufficiency by [4, Theorem 3.1]. Assume the theorem to be false, and consider G as a counterexample of minimal order. Let

$$\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}.$$

By our hypothesis, each P_i of $\mathcal{M}_d(P)$ is either $c^{\#}$-normal or S-quasinormally embedded in G, where $i = 1, \dots, d$. Without loss of generality, we assume that there exists an integer k with $1 \le k \le d$ such that for every m with $1 \le m \le k$, P_m is $c^{\#}$-normal in G, and for every n with $k+1 \le n \le d$, P_n is S-quasinormally embedded in G. We split the proof into the following steps:

(1) $O_{p'}(G) = 1$.

Otherwise, $O_{p'}(G) \ne 1$. Since $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$ and $PO_{p'}(G)/O_{p'}(G) \cong P$, we can see that $PO_{p'}(G)/O_{p'}(G)$ has the same smallest generator number d as P. Set

$$\mathcal{M}_d(PO_{p'}(G)/O_{p'}(G)) = \{P_1O_{p'}(G)/O_{p'}(G), \dots, P_dO_{p'}(G)/O_{p'}(G)\}.$$

Then, each $P_iO_{p'}(G)/O_{p'}(G)$ for $i \in \{1, \dots, d\}$ is either $c^{\#}$-normal or S-quasinormally embedded in $G/O_{p'}(G)$ by Lemmas 2.1 and 2.2. Consequently, $G/O_{p'}(G)$ fulfills the requirements stipulated in our theorem. Given the minimality of G, it is compelled that $G/O_{p'}(G)$ be p-supersolvable, which in turn implies that G must be p-supersolvable, leading to a contradiction.

(2) $\Phi(P)_G = 1$, in particular, $\Phi(O_p(G)) = 1$.

Suppose that $\Phi(P)_G \ne 1$. Then, it is clear that

$$\{P_1/\Phi(P)_G, \dots, P_d/\Phi(P)_G\} = \mathcal{M}_d(P/\Phi(P)_G).$$

Also, each $P_i/\Phi(P)_G$ for $i \in \{1, \dots, d\}$ is either $c^{\#}$-normal or S-quasinormally embedded in $G/\Phi(P)_G$ by Lemmas 2.1 and 2.2. Thus, the hypothesis of our theorem is automatically satisfied for $G/\Phi(P)_G$. So $G/\Phi(P)_G$ is p-supersolvable by the minimality of G. By [18, III, Satz 3.3], we have that $\Phi(P)_G \le \Phi(G)$ and so G is p-supersolvable, which contradicts the minimality of G.

(3) Any minimal normal subgroup of G that is contained within $O_p(G)$ possesses an order of p.

By statement (1) and the p-solvability of G, it follows that $O_p(G) > 1$. Consider N to be a minimal normal subgroup of G that is contained within $O_p(G)$. By hypotheses, every P_m is $c^{\#}$-normal in G, and hence, there exists a normal subgroup K_m of G such that $G = P_mK_m$ and $P_m \cap K_m$ is a CAP-subgroup of G for all $m \in \{1, \dots, k\}$, i.e., $P_m \cap K_m$ covers or avoids $N/1$. Suppose that there exists some P_m such that $P_m \cap K_m$ avoids $N/1$. Then, $P_m \cap K_m \cap N = 1$. By the minimal normality of N, we can see that either $N \cap K_m = 1$ or $N \cap K_m = N$. If $N \cap K_m = 1$, then NK_m/K_m is a minimal normal subgroup of G/K_m. But G/K_m is a p-group as $G = P_mK_m$, which means that $N \cong NK_m/K_m$ is of order p. If $N \cap K_m = N$, then $N \cap P_m = 1$. This derives that $P = P_m \times N$, and thus, $|N| = p$. Now we may assume that every $P_m \cap K_m$ covers $N/1$ for all $m \in \{1, \dots, k\}$. Then, $N \le P_m \cap K_m$ and so

$$N \le \bigcap_{m=1}^{k} (P_m)_G,$$

where $(P_m)_G$ is the core of P_m in G.

By our assumption, for each $n \in \{k+1, \dots, d\}$, P_n is S-quasinormally embedded in G, and consequently, there exists an S-quasinormal subgroup M_n of G for which P_n serves as a Sylow p-subgroup. Therefore, we may apply Lemmas 2.3 and 2.4 to see that $M_n/(M_n)_G$ is nilpotent and all Sylow subgroups of $M_n/(M_n)_G$ are S-quasinormal in $G/(M_n)_G$. In particular, $P_n(M_n)_G/(M_n)_G$ is S-quasinormal in $G/(M_n)_G$. It follows from Lemma 2.5 that $P_n(M_n)_G/(M_n)_G$ is normal in $G/(M_n)_G$. We obtain that $P_n \le (M_n)_G$ and so $P \cap (M_n)_G = P_n$.

Let

$$T = \left(\bigcap_{m=1}^{k} (P_m)_G\right) \cap \left(\bigcap_{n=k+1}^{d} (M_n)_G\right),$$

then $T \trianglelefteq G$. By the minimal normality of N, we have that $N \cap (M_n)_G = N$ or 1. If $N \cap (M_n)_G = N$ for all $n \in \{k+1, \dots, d\}$, then $N \le (M_n)_G$, and consequently, $N \le T$. We obtain that

$$\begin{aligned} T \cap P &= \left(\bigcap_{m=1}^{k} (P_m)_G\right) \cap \left(\bigcap_{n=k+1}^{d} (M_n)_G\right) \cap P = \left(\bigcap_{m=1}^{k} (P_m)_G\right) \cap \left(\bigcap_{n=k+1}^{d} (M_n)_G \cap P\right) \\ &= \left(\bigcap_{m=1}^{k} (P_m)_G\right) \cap \left(\bigcap_{n=k+1}^{d} P_n\right) \le \bigcap_{i=1}^{d} P_i = \Phi(P). \end{aligned}$$

From Lemma 2.7, we deduce that T is p-nilpotent, and hence, by statement (1), T is a p-group. Statement (2) implies that $T = N = 1$, which is a contradiction. Therefore, there exists some M_i such that $N \cap (M_i)_G = 1$, where $i \in \{k+1, \ldots, d\}$. Therefore, $|N| = p$.

(4) No counterexample exists.

Consider N as a minimal normal subgroup of G such that $N \leq O_p(G)$. Then, statement (3) and Lemma 2.8 are combined to give that N is complemented in G. We obtain that $N \cap \Phi(G) = 1$ and so $O_p(G) \cap \Phi(G) = 1$. We apply Lemma 2.9 to give that

$$O_p(G) = N_1 \times N_2 \times \ldots \times N_s,$$

where each N_i (for $i = 1, \ldots, s$) is a minimal normal subgroup of G of order p. Since

$$G/C_G(N_i) \cong \mathrm{Aut}(N_i)$$

and $\mathrm{Aut}(N_i)$ is a cyclic group of order $p-1$, we conclude that

$$G/C_G(O_p(G)) = G/\bigcap_{i=1}^{r} C_G(N_i)$$

is p-supersolvable. Given that G is p-solvable and $O_{p'}(G) = 1$, it follows that

$$C_G(O_p(G)) \leq O_p(G)$$

by Lemma 2.10. It follows that $G/O_p(G)$ is p-supersolvable. Now, every chief factor of G contained in $O_p(G)$ is of order p; and hence, every p-chief factor of G is cyclic. It follows that G is p-supersolvable, which is a final contradiction. □

As immediate consequences of Theorem 3.1, we have the following.

Corollary 3.2. [4, Theorem 3.1] *Consider G as a p-solvable group, and denote by P a Sylow p-subgroup of G, where p is a prime number that divides the order of G. Then, G is p-supersolvable if and only if all members in some fixed $\mathcal{M}_d(P)$ are $c^{\#}$-normal in G.*

Corollary 3.3. *Consider G as a p-solvable group, and denote by P a Sylow p-subgroup of G, where p is a prime number that divides the order of G. Then, G is p-supersolvable if and only if all members in some fixed $\mathcal{M}_d(P)$ are S-quasinormally embedded in G.*

Remark 3.4. The assumption of "G is p-solvable" in Theorem 3.1 is indispensable. Indeed, $G = A_5$ is a counterexample for $p = 5$.

Revising the method used in the proof of Theorem 3.1, we can obtain some results. In the following theorem, we replace the condition "G is p-solvable" in Theorem 3.1 by "$(|G|, p-1) = 1$."

Theorem 3.5. *Let G be a group, p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$, and P be a Sylow p-subgroup of G. Then, G is a p-nilpotent group if and only if all members in some fixed $\mathcal{M}_d(P)$ are either $c^{\#}$-normal or S-quasinormally embedded in G.*

Proof. Applying [4, Theorem 3.2], it is only the sufficiency of the condition that is in doubt. Assume the result is incorrect and consider G as a counterexample possessing the smallest possible order. Let

$$\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}.$$

With the same arguments as in the proof of Theorem 3.1, we can obtain the statements (1)–(4).

(1) $O_{p'}(G) = 1$.

(2) $\Phi(P)_G = 1$, in particular, $\Phi(O_p(G)) = 1$.

(3) Let N be a p-group. If N is a minimal normal subgroup of G, then $|N| = p$.
(4) Any minimal normal subgroup of G is a subgroup of $O_p(G)$.

Let N be a minimal normal subgroup of G. Then, statement (1) means that $p||N|$. Let $P_i \in \mathcal{M}_d(P)$. By hypotheses, P_i is either $c^\#$-normal or S-quasinormally embedded in G. Given that P_i is $c^\#$-normal in G, a normal subgroup K of G exists, satisfying $G = P_iK$ and $P_i \cap K$ being a CAP-subgroup of G, i.e., $P_i \cap K$ covers or avoids $N/1$. If $P_i \cap K$ covers $N/1$, then $N \leq P_i$ and so $N \leq O_p(G)$. If $P_i \cap K$ avoids $N/1$, then $P_i \cap K \cap N = 1$. By the minimal normality of N, we can see that either $N \cap K = 1$ or $N \cap K = N$. If $N \cap K = 1$, then NK/K is a minimal normal subgroup of G/K. It follows that N has order p and so $N \leq O_p(G)$. If $N \cap K = N$, then $N \cap P_i = 1$. Consequently, $|P_iN|_p = |P_i||N|_p$ and so $|N|_p = p$. Lemma 2.11 and $(|G|, p-1) = 1$ are combined to give that N is p-nilpotent. Thus, N is a p-group by statement (1). Hence, $N \leq O_p(G)$.

Now, we may assume that all members of $\mathcal{M}_d(P)$ are S-quasinormally embedded in G and $N_p = N \cap P \nleq \Phi(P)$ by Lemma 2.7. Without loss of generality, we assume that $N_p \nleq P_1 \in \mathcal{M}_d(P)$. Hence, there exists an S-quasinormal subgroup H of G, for which P_1 is a Sylow p-subgroup. Therefore, P_1H_G/H_G is a Sylow p-subgroup of H/H_G. By Lemmas 2.3 and 2.4, H/H_G is nilpotent and P_1H_G/H_G is an S-quasinormal subgroup of H/H_G. Since P_1H_G/H_G is a maximal subgroup of a Sylow p-subgroup of G/H_G, we see that P_1H_G/H_G is normal in G/H_G by Lemma 2.5. Hence, P_1H_G is normal in G, which implies that $P_1 \leq H_G$. By the minimal normality of N, we obtain that $N \cap H_G = N$ or 1. If $N \cap H_G = N$, then $N_p \leq P \cap H_G = P_1$, which is a contradiction. Therefore, $N \cap H_G = 1$ and so $N \cap P_1 = 1$. Consequently, $|N|_p = p$. It means that N is p-nilpotent, and we apply statement (1) to give that $N \leq O_p(G)$.

(5) The final contradiction.

By statements (2) and (4), we obtain that $O_p(G) \neq 1$ and $O_p(G)$ is elementary abelian. Applying Lemmas 2.8 and 2.9, we can see that

$$O_p(G) = N_1 \times N_2 \times \ldots \times N_r,$$

where N_i is a minimal normal subgroup of G with $|N_i| = p$. Furthermore, $O_p(G)$ has a complement K in G, i.e.,

$$G = O_p(G) \rtimes K.$$

Let

$$T = \bigcap_{i=1}^{r} C_G(N_i).$$

Since $N_i \leq Z(P)$, we have $P \leq T$. If $T \cap K \neq 1$, then there exists a minimal normal subgroup L of G such that $L \leq T \cap K$ and $L \nleq O_p(G)$, which contradicts with statement (4). Hence, $T \cap K = 1$. Moreover, we obtain that $P = O_p(G)$. Set

$$P_i = N_1 \times N_2 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_r.$$

Then, P_i is normal in G and $|G/P_i|_p = p$. By Lemma 2.11, G/P_i is p-nilpotent. Furthermore,

$$G/\bigcap_{i=1}^{r} P_i$$

is p-nilpotent, but $\bigcap_{i=1}^{r} P_i = 1$. It follows that G is p-nilpotent, which a final contradiction. □

Therefore, from Theorem 3.5, we obtain

Corollary 3.6. [4, Theorem 3.2] and [5, Theorem 3.1] *Let p be a prime divisor of the order of a group G with* $(|G|, p-1) = 1$, *and let P be a Sylow p-subgroup of G. Then, G is a p-nilpotent group if and only if all members in some fixed $\mathcal{M}_d(P)$ are $c^\#$-normal in G.*

Corollary 3.7. [10, Theorem 3.1] *Let p be a prime divisor of the order of a group G with* $(|G|, p-1) = 1$, *and let P be a Sylow p-subgroup of G. Then, G is a p-nilpotent group if and only if all members in some fixed* $\mathcal{M}_d(P)$ *are S-quasinormally embedded in G.*

Combining Theorems 3.1 and 3.5, we obtain the following result.

Theorem 3.8. *Consider an arbitrary prime divisor p of the order of a group, and denote by P a Sylow p-subgroup of G. A group G is supersolvable if and only if all members in some fixed $\mathcal{M}_d(P)$ are either $c^{\#}$-normal or S-quasinormally embedded in G.*

Proof. Only the sufficiency needs to be shown. Let p be the smallest prime of the order of G. Then, from Theorem 3.5 G is p-nilpotent. It follows that G is solvable. We apply Theorem 3.1 to see that for arbitrary prime divisor of $|G|$, G is p-supersolvable, and thus, G is supersolvable. □

In view of Theorem 3.8, we have the following.

Corollary 3.9. [4, Theorem 3.5] *A group G is supersolvable if and only if all members in some fixed $\mathcal{M}_d(P)$ are $c^{\#}$-normal in G, for every Sylow subgroup P of G.*

Corollary 3.10. [10, Theorem 3.2] *A group G is supersolvable if and only if for each Sylow subgroup P of G, all members in some fixed $\mathcal{M}_d(P)$ are S-quasinormally embedded in G.*

It is recalled that a formation $\mathfrak{F}$ is defined as a class of groups that is closed under the operations of taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is termed saturated if $G \in \mathfrak{F}$ holds whenever $G/\Phi(G) \in \mathfrak{F}$ is satisfied. Let $\mathcal{U}$ be the class of all groups that are supersolvable. It is evident that $\mathcal{U}$ constitutes a saturated formation. We also discover that aforementioned results cannot be extended to saturated formation, specifically:

Define $\mathfrak{F}$ as a saturated formation encompassing $\mathcal{U}$. Consider K as a normal subgroup of a group G with the property that $G/K \in \mathfrak{F}$. Assume that for every prime p dividing the order of K, $P \in \mathrm{Syl}_p(K)$, and all members in some fixed $\mathcal{M}_d(P)$ are either $c^{\#}$-normal or S-quasinormally embedded in G. But $G \notin \mathfrak{F}$.

Example 3.11. The formation function, denoted by $f(p)$, is defined as follows:

$$f(p) = \{\text{the class of } p'\text{-groups for any prime } p\}.$$

Furthermore, let $\mathfrak{F}$ be the locally defined formation based on the set $\{f(p)\}$. Suppose that K/L is a p-chief factor of a supersolvable group X, then $X/C_X(K/L)$ is a cyclic group of order dividing $p-1$ and so $X/C_X(K/L) \in f(p)$. It follows that $X \in \mathfrak{F}$, and hence, $\mathcal{U} \subseteq \mathfrak{F}$. The inclusion of $A_4 \in \mathfrak{F}$ is evident.

Let $P = \langle a, b, c\rangle$ be an elementary abelian group of order 3^3, and let $u, v \in \mathrm{Aut}(P)$ such that

$$u = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} a & b & c \\ b & c^{-1} & a^{-1} \end{pmatrix}.$$

Then, $u^3 = v^3 = (uv)^2 = 1$, which means that $A = \langle u, v\rangle \cong A_4$. Set $G = P \rtimes A$, the semidirect product of P and A. We can see that P is an irreducible and faithful A_4-module over $GF(p)$, implying that P is a minimal normal subgroup of G with $C_A(P) = 1$. Given that $A_4 \in \mathfrak{F}$ and $G/P \cong A \cong A_4$, we conclude that $G/P \in \mathfrak{F}$. Define $M = PS$, with S is a Sylow 2-subgroup of G. Note that $O^3(G) \leq M \trianglelefteq G$. Since S is an elementary abelian group of order 4, we derive that any minimal normal subgroup of M contained in P has order 3. Invoking Maschke's theorem [1, Theorem 8.1.2], we deduce that P is a completely reducible S-module. Consequently, P admits a decomposition as $P = \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle$, with $\langle a_i\rangle$ $(i = 1, 2, 3)$ being S-invariant. Let $P_i = \langle a_j | j \neq i\rangle$. Then,

$$\mathcal{M}_d(P) = \{P_1, P_2, P_3\},$$

and each P_i in $\mathcal{M}_d(P)$ is S-quasinormally embedded within G. Furthermore, P is identified as a 3-chief factor of G and $G/C_G(P) \cong A_4$, which is not a $3'$-group. Thus, $G \notin \mathfrak{F}$.

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