Research Article

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Regularity of weak solutions to the 3D stationary tropical climate model

https://doi.org/10.1515/math-2025-0137
received July 3, 2024; accepted February 5, 2025

Abstract: This article studies the regularity of weak solutions to the 3D stationary tropical climate model. We prove that when (U, V, θ) belongs to the homogeneous Morrey space $\dot{M}^{2,p}(\mathbb{R}^3)$ with $p > 3$, then $(U, V, \theta) \in C^\infty(\mathbb{R}^3)$.

Keywords: weak solutions, tropical climate model, regularity

MSC 2020: 35B65, 35Q35

1 Introduction

In this article, we consider the following 3D stationary tropical climate model:

$$\begin{cases} -\Delta U + \mathrm{div}(U \otimes U) + \mathrm{div}(V \otimes V) + \nabla P = 0, \\ -\Delta V + \mathrm{div}(U \otimes V) + \mathrm{div}(V \otimes U) + \nabla \theta = 0, \\ -\Delta \theta + U \cdot \nabla \theta + \mathrm{div} V = 0, \\ \mathrm{div} U = 0, \end{cases} \tag{1.1}$$

where $U(x) = (U_1, U_2, U_3)$, $V(x) = (V_1, V_2, V_3)$, $\theta = \theta(x)$, and $P = P(x)$ denote the barotropic mode of the velocity field, the first baroclinic mode of the velocity field, the scalar temperature, and the scalar pressure, respectively. The (i, j) component of matrix $U \otimes V$ is $U_i V_j$ with $i, j = 1, 2, 3$.

System (1.1) becomes the standard stationary tropical climate model when the term $\mathrm{div}(V \otimes U)$ in $(1.1)_2$ is replaced by $V \cdot \nabla U$. For the time-dependent version related to the standard stationary tropical climate model, i.e.,

$$\begin{cases} \partial_t u + \mu(-\Delta)^\alpha u + \mathrm{div}(u \otimes u) + \mathrm{div}(v \otimes v) + \nabla P = 0, \\ \partial_t v + \nu(-\Delta)^\beta v + \mathrm{div}(u \otimes v) + v \cdot \nabla u + \nabla \theta = 0, \\ \partial_t \theta + \eta(-\Delta)^\gamma \theta + u \cdot \nabla \theta + \mathrm{div} v = 0, \\ \mathrm{div} u = 0, \\ (u, v, \theta)|_{t=0} = (u_0, v_0, \theta_0) \end{cases} \tag{1.2}$$

with the parameters $\mu, \nu, \eta, \alpha, \beta, \gamma \geq 0$, there have been many studies on well-posedness and regularity. For the well-posedness results of system (1.2), when the viscosity depends on the temperature, Ye and Zhu [1]

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showed that there exists a global strong solution in $\mathbb{R}^2$ for the case $\mu = \nu = \eta = 0$. In this case, it is worth mentioning that system (1.2) reduces to the original tropical climate model derived by Frierson et al. [2]. For $\mu = \nu = 1$, $\eta = 0$, $0 \le \alpha \le 1$, $\beta \ge 1$ or $\alpha \ge 1$, $\beta \ge 0$, Ma et al. [3] obtained the local well-posedness of strong solutions in $\mathbb{R}^2$. Li and Titi [4] showed the existence and uniqueness of strong solutions in $\mathbb{R}^2$ with $\mu, \nu > 0$, $\eta = 0$, $\alpha = \beta = 1$. When $\mu, \nu, \eta > 0$, $\alpha = \beta = 5/4$, $\gamma = 0$, Li et al. [5] obtained the global well-posedness in $\mathbb{R}^3$. Zhang and Xu [6] established the global well-posedness of the classical solutions for system (1.2) with a full Laplacian term in $\mathbb{R}^3$. More 2D and 3D well-posedness results for system (1.2) were studied in [7–9].

For the regularity results of system (1.2), Ye [10] proved the global regularity in $\mathbb{R}^2$ with $\mu, \nu, \eta > 0, \alpha > 0$, $\beta = \gamma = 1$. When $\mu, \nu > 0, \eta = 0, \frac{1}{2} < \alpha < 1, 0 < \beta < 1, 2\alpha + \beta = 2$, a regularity criterion obtained by Bisconti [11] in $\mathbb{R}^2$. Wang et al. [12] and Wu [13] proved the global regularity for system (1.2) with a full Laplacian term in $\mathbb{R}^3$, respectively. For more 2D and 3D regularity results related to system (1.2), one could see [14–16].

It is worth mentioning that when $\partial_t u = \partial_t v = 0$, $\theta = 0$, and $\mu = \nu = \alpha = \beta = 1$, system (1.2) reduces to a system, which is similar to the standard stationary MHD system. Let us mention that Jarrín [17] obtained a criterion to improve the regularity of weak solutions to the standard stationary MHD system, provided that $(u, v) \in \dot{M}^{2,p}(\mathbb{R}^3)$.

To the best of our knowledge, with respect to the regularity of solutions to system (1.1), there are no corresponding results, which is our motivation in this article. Inspired by [17], we study the regularity of weak solutions to system (1.1) in the homogeneous Morrey space.

In this article, we write $\|(f_1, f_2, f_3)\|_X \coloneqq \|f_1\|_X + \|f_2\|_X + \|f_3\|_X$ and use $\|\cdot\|_X$ to denote $\|\cdot\|_{X(\mathbb{R}^3)}$. The notation $\mathcal{D}'(\mathbb{R}^3)$ is the dual space of $\mathcal{D}(\mathbb{R}^3) \coloneqq C_0^\infty(\mathbb{R}^3)$, where $C_0^\infty(\mathbb{R}^3)$ represents the set of infinitely differentiable functions with compact support in $\mathbb{R}^3$. The derivative of the multi-index α of the function f is $\partial^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$ with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. The positive constant c may change from line to line.

Our main result can be stated as follows. Here, the definitions of weak solutions, $\dot{M}^{2,p}(\mathbb{R}^3)$ and $\mathcal{W}^{k+2,p}(\mathbb{R}^3)$ will be given in the next section.

Theorem 1.1. *Let (U, V, θ) be a weak solution to system* (1.1). *Assume that $(U, V, \theta) \in \dot{M}^{2,p}(\mathbb{R}^3)$ with $p > 3$ and $P \in \mathcal{D}'(\mathbb{R}^3)$, then, for all $k \ge 0$ we have*

$$(U, V, \theta, P) \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$$

and

$$(\partial^\alpha U, \partial^\alpha V, \partial^\alpha \theta, \partial^\alpha P) \in C^{1-3/p}(\mathbb{R}^3), \quad |\alpha| \le k + 1.$$

Remark 1.1. In fact, the function (U, V, θ, P) belongs to $C^\infty(\mathbb{R}^3)$ since the exponent k could be large enough.

In order to prove Theorem 1.1, we need to overcome a difficulty, which is the estimation associated with $\nabla\theta$ and $\operatorname{div} V$ in system (1.1). In fact, Jarrín gave a way to deal with $(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G}))$ (see [17], pp. 6–7), i.e.,

$$\begin{aligned}
\sup_{0\le t\le T} \left\| \int_0^t e^{(t-s)\Delta}(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G}))\mathrm{d}s \right\|_{\dot{M}^{2,p}}
&\le \sup_{0\le t\le T} \int_0^t \|e^{(t-s)\Delta}(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G}))\|_{\dot{M}^{2,p}} \mathrm{d}s \\
&\le c \sup_{0\le t\le T} \|(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G}))\|_{\dot{M}^{2,p}} \left(\int_0^t \mathrm{d}s \right) \\
&\le cT \|(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G}))\|_{\dot{M}^{2,p}}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{0\le t\le T} t^{\frac{3}{2p}}\left\|\int_0^t e^{(t-s)\Delta}(\operatorname{div}(\mathbb{F}),\operatorname{div}(\mathbb{G}))\mathrm{d}s\right\|_{L^\infty}\\
&\le \sup_{0\le t\le T} t^{\frac{3}{2p}}\int_0^t \|e^{(t-s)\Delta}(\operatorname{div}(\mathbb{F}),\operatorname{div}(\mathbb{G}))\|_{L^\infty}\mathrm{d}s\\
&\le c\sup_{0\le t\le T} t^{\frac{3}{2p}}\int_0^t (t-s)^{-\frac{3}{2p}}\|(\operatorname{div}(\mathbb{F}),\operatorname{div}(\mathbb{G}))\|_{\dot M^{2,p}}\mathrm{d}s\\
&\le c\sup_{0\le t\le T} t^{\frac{3}{2p}}\|(\operatorname{div}(\mathbb{F}),\operatorname{div}(\mathbb{G}))\|_{\dot M^{2,p}}\left(\int_0^t (t-s)^{-\frac{3}{2p}}\mathrm{d}s\right)\\
&\le cT\|(\operatorname{div}(\mathbb{F}),\operatorname{div}(\mathbb{G}))\|_{\dot M^{2,p}},
\end{aligned}$$

where $(\mathbb{F},\mathbb{G})$ denote the external force and $p>3$. However, if we follow the above estimates, namely,

$$\sup_{0\le t\le T}\left\|\int_0^t e^{(t-s)\Delta}(\nabla\theta,\operatorname{div}V)\mathrm{d}s\right\|_{\dot M^{2,p}}\le cT\|(\nabla\theta,\operatorname{div}V)\|_{\dot M^{2,p}}$$

and

$$\sup_{0\le t\le T} t^{\frac{3}{2p}}\left\|\int_0^t e^{(t-s)\Delta}(\nabla\theta,\operatorname{div}V)\mathrm{d}s\right\|_{L^\infty}\le T\|(\nabla\theta,\operatorname{div}V)\|_{\dot M^{2,p}},$$

then the condition $(\nabla\theta,\operatorname{div}V)\in\dot M^{2,p}(\mathbb{R}^3)$ is required. To avoid this additional condition, in this article, we will choose another way to deal with these two terms, see the estimates of J_4 and K_3 below.

The rest of this article is divided into two sections. In Section 2, we will introduce some lemmas, definitions of weak solutions and function spaces. In Section 3, we will give the proof of Theorem 1.1 by three steps.

2 Preliminaries

In this section, we first recall the definitions of homogeneous Morrey space $\dot M^{r,p}(\mathbb{R}^3)$ and weak solutions to system (1.1). Then, we introduce the Sobolev-Morrey space $\mathcal{W}^{k,p}(\mathbb{R}^3)$ and some lemmas.

Definition 2.1. The function $(U,V,\theta)\in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ is called a weak solution to system (1.1) if it satisfies

$$\begin{cases}
\int_{\mathbb{R}^3}(-U\cdot\Delta\phi+(U\otimes U):\nabla\phi+(V\otimes V):\nabla\phi+P(\nabla\cdot\phi))\mathrm{d}x=0,\\
\int_{\mathbb{R}^3}(-V\cdot\Delta\phi+(U\otimes V):\nabla\phi+(V\otimes U):\nabla\phi+\theta(\nabla\cdot\phi))\mathrm{d}x=0,\\
\int_{\mathbb{R}^3}(-\theta\cdot\Delta\phi+(U\cdot\nabla\theta):\nabla\phi+V\cdot(\nabla\cdot\phi))\mathrm{d}x=0,\\
\operatorname{div}U=0,
\end{cases}$$

for any test function $\phi\in C_0^\infty(\mathbb{R}^3)$.

Definition 2.2. Let $1<r<p$ and $1<p<+\infty$. The homogeneous Morrey space $\dot M^{r,p}(\mathbb{R}^3)$ is the set of all functions $f\in L^r_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot M^{r,p}(\mathbb{R}^3)}:=\sup_{x_0\in\mathbb{R}^3,R>0}R^{\frac{3}{p}}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)}|f(x)|^r\mathrm{d}x\right)^{\frac{1}{r}}<+\infty,$$

where $B(x_0,R)$ is a ball of radius R centered at x_0.

The space $\dot{M}^{r,p}(\mathbb{R}^3)$ is a homogeneous space of degree $-\frac{3}{p}$ and satisfies the embedding relation $L^p(\mathbb{R}^3) \subset L^{p,q}(\mathbb{R}^3) \subset \dot{M}^{r,p}(\mathbb{R}^3)$. Here, $L^{p,q}(\mathbb{R}^3)$ is the Lorentz space and for more details of this space, one could refer to [18].

Definition 2.3. Let $p > 2$, $k \geq 0$. The Sobolev-Morrey space $\mathcal{W}^{k,p}(\mathbb{R}^3)$ is defined by

$$\mathcal{W}^{k,p}(\mathbb{R}^3) := \{\partial^\alpha f \in \dot{M}^{2,p}(\mathbb{R}^3) \text{ for all } |\alpha| \leq k\}.$$

Lemma 2.1. [17] *Let $f \in \dot{M}^{r,p}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. For all $1 \leq \sigma < +\infty$, there holds*

$$\|f\|_{\dot{M}^{r\sigma,p\sigma}} \leq c\|f\|_{\dot{M}^{r,p}}^{\frac{1}{\sigma}} \|f\|_{L^\infty}^{1-\frac{1}{\sigma}}.$$

Lemma 2.2. [19] *The space $\dot{M}^{r,p}(\mathbb{R}^3)$ is stable under convolution with functions in the space $L^1(\mathbb{R}^3)$, and we have*

$$\|f * g\|_{\dot{M}^{r,p}} \leq c\|f\|_{\dot{M}^{r,p}} \|g\|_{L_1}.$$

Lemma 2.3. [19] *Let $t > 0$ and h_t be the heat kernel: $h_t(x, y) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}}$, where $(x, y) \in \mathbb{R}^d$ with $d = 1, 2, \ldots, n$. Then, there holds*

$$t^{\frac{3}{2p}} \|h_t * f\|_{L^\infty} \leq c\|f\|_{\dot{M}^{r,p}}.$$

Lemma 2.4. [20] *For $i = 1, 2, 3$, $R_i := \frac{\partial_i}{\sqrt{-\Delta}}$ denotes the ith Riesz transform. Then,*

$$\|R_i R_j(f)\|_{\dot{M}^{r,p}} \leq c\|f\|_{\dot{M}^{r,p}}.$$

Lemma 2.5. [21] *Let $f \in \mathcal{S}'$ such that $\nabla f \in \dot{M}^{1,p}(\mathbb{R}^3)$ with $p > 3$, where $\mathcal{S}'$ is the collection of all continuous linear functional defined on $\mathcal{S}$, i.e., the tempered distributions space. There exists a constant $c > 0$ such that for all $x, y \in \mathbb{R}^3$, we have $|f(x) - f(y)| \leq c\|\nabla f\|_{\dot{M}^{1,p}} |x - y|^{1-3/p}$. Here, $\dot{M}^{1,p}(\mathbb{R}^3)$ is defined as the space of locally finite Borel measures $\mathrm{d}\mu$ such that*

$$\sup_{x_0 \in \mathbb{R}^3, R > 0} R^{\frac{3}{p}} \left(\frac{1}{|B(x_0, R)|} \int_{B(x_0,R)} d|\mu|(x) \right) < +\infty.$$

Now, we introduce the following lemma, which is a simple result of the Banach fixed point theorem.

Lemma 2.6. [22,23] *Let Y be a Banach space. Given a bilinear from $H : Y \times Y \to Y$ such that $\|H(x_1, x_2)\|_Y \leq c_0\|x_1\|_Y \|x_2\|_Y$ for some constant $c_0 > 0$ and for all $x_1, x_2 \in Y$, we have the following assertions for the equation:*

$$u = z + H(u, u). \tag{2.1}$$

Suppose that $z \in B_\varepsilon(0) := \{f \in Y : \|f\|_Y < \varepsilon\}$ for some $\varepsilon \in (0, \frac{1}{4c_0})$, then (2.1) *has a solution $u \in B_{2\varepsilon}(0) := \{f \in Y : \|f\|_Y < 2\varepsilon\}$. It means that the unique solution lies in the ball $\overline{B_{2\varepsilon}(0)}$. Moreover, the solution u depends continuously on z, that is, when $\bar{z} \in B_\varepsilon(0)$, $\bar{u} \in B_{2\varepsilon}(0)$, and $\bar{u} = \bar{z} + H(\bar{u}, \bar{u})$, it follows that*

$$\|u - \bar{u}\|_Y \leq \frac{1}{1 - 4\varepsilon c_0} \|z - \bar{z}\|_Y. \tag{2.2}$$

We can conclude the local well-posedness of the solution from (2.2) *when $c_0 = cT^a$ for some $a > 0$.*

3 The proof of main result

Proof of Theorem 1.1. For the sake of clarity, the proof is divided into three steps. In step 1, we prove the local existence and uniqueness of solution (u, v, θ) to system (3.1), i.e., Proposition 3.1. In step 2, we focus on system (1.1) and obtain the boundedness of (U, V, θ) in $\mathbb{R}^3$, i.e., Proposition 3.2. In step 3, we first study the regularity of the solution (U, V, θ) to system (1.1), then from the relation between (U, V) and pressure P (see (3.17)), we are able to obtain the regularity of P.

Step 1. The local existence and uniqueness of solution (u, v, θ) **to system** (3.1).

We consider the following time-dependent version of system (1.1):

$$\begin{cases} \partial_t u - \Delta u + \operatorname{div}(u \otimes u) + \operatorname{div}(v \otimes v) + \nabla P = 0, \\ \partial_t v - \Delta v + \operatorname{div}(u \otimes v) + \operatorname{div}(v \otimes u) + \nabla \theta = 0, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta + \operatorname{div} v = 0, \\ \operatorname{div} u = 0, \\ (u, v, \theta)|_{t=0} = (u_0, v_0, \theta_0). \end{cases} \tag{3.1}$$

Now, we give the following proposition:

Proposition 3.1. *Let* $(u_0, v_0, \theta_0) \in \dot{M}^{2,p}(\mathbb{R}^3)$ *with* $p > 3$. *There exists a unique solution* $(u, v, \theta) \in C_*([0, T_0], \dot{M}^{2,p}(\mathbb{R}^3))$ *to system* (3.1), *where* T_0 *is a positive time, depending on* $\|(u_0, v_0, \theta_0)\|_{\dot{M}^{2,p}}$ *such that*

$$\sup_{0 \le t \le T_0} t^{\frac{3}{2p}} \|(u, v, \theta)(t, \cdot)\|_{L^\infty} < +\infty.$$

Here, $C_*([0, T], \dot{M}^{2,p}(\mathbb{R}^3))$ *denotes the functional space of bounded and weak-*continuous functions on* $[0, T]$ *with values in the homogeneous Morrey space* $\dot{M}^{2,p}(\mathbb{R}^3)$.

Proof. Applying Leray's projector $\mathbb{P} = I + \nabla(-\Delta)^{-1}\operatorname{div}$ to $(3.1)_1$, then by Duhamel's formulae, we obtain

$$\begin{aligned} u(t, \cdot) &= e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P}(\operatorname{div}(u \otimes u))(s, \cdot) \mathrm{d}s + \int_0^t e^{(t-s)\Delta} \mathbb{P}(\operatorname{div}(v \otimes v))(s, \cdot) \mathrm{d}s \\ &\coloneqq I_1 + I_2 + I_3, \end{aligned} \tag{3.2}$$

$$\begin{aligned} v(t, \cdot) &= e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} (\operatorname{div}(u \otimes v))(s, \cdot) \mathrm{d}s + \int_0^t e^{(t-s)\Delta} (\operatorname{div}(v \otimes u))(s, \cdot) \mathrm{d}s + \int_0^t e^{(t-s)\Delta} \nabla \theta(s, \cdot) \mathrm{d}s \\ &\coloneqq J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \theta(t, \cdot) &= e^{t\Delta} \theta_0 + \int_0^t e^{(t-s)\Delta} (u \cdot \nabla \theta)(s, \cdot) \mathrm{d}s + \int_0^t e^{(t-s)\Delta} \operatorname{div} v(s, \cdot) \mathrm{d}s \\ &\coloneqq K_1 + K_2 + K_3, \end{aligned} \tag{3.4}$$

where (I_1, J_1, K_1) is the linear term and $(I_2, I_3, J_2, J_3, J_4, K_2, K_3)$ is the bilinear term.

Exploiting Picard's fixed point argument, we introduce the following Banach space:

$$E_T \coloneqq \left\{ f \in C_*([0, T], \dot{M}^{2,p}(\mathbb{R}^3)) : \sup_{0 \le t \le T} t^{\frac{3}{2p}} \|f(t, \cdot)\|_{L^\infty} < +\infty \right\},$$

with the norm

$$\|f\|_{E_T} \coloneqq \sup_{0 \le t \le T} \|f(t, \cdot)\|_{\dot{M}^{2,p}} + \sup_{0 \le t \le T} t^{\frac{3}{2p}} \|f(t, \cdot)\|_{L^\infty}.$$

We will estimate $(I_1, I_2, I_3, J_1, J_2, J_3, J_4, K_1, K_2, K_3)$ in the space E_T. As $(u_0, v_0, \theta_0) \in \dot{M}^{2,p}(\mathbb{R}^3)$, by Lemma 2.2 and Lemma 2.3, it is easy to obtain

$$\|(I_1, J_1, K_1)\|_{E_T} \le c\|(u_0, v_0, \theta_0)\|_{\dot{M}^{2,p}}. \tag{3.5}$$

Next, we will show that

$$\|(I_2, I_3, J_2, J_3, J_4, K_2, K_3)\|_{E_T} \le cT^{\frac{1}{2}-\frac{3}{2p}}\|(u, v, \theta)\|_{E_T}^2 + cT^{\frac{1}{2}}\|(v, \theta)\|_{E_T}, \tag{3.6}$$

where $p > 3$.

We first give the following estimate, which will be used frequently in subsequent estimates:

$$\begin{aligned}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}s^q}\mathrm{d}s &= \int_0^{t/2} \frac{1}{(t-s)^{\frac{1}{2}}s^q}\mathrm{d}s + \int_{t/2}^t \frac{1}{(t-s)^{\frac{1}{2}}s^q}\mathrm{d}s\\ &\le c\left(t^{-\frac{1}{2}}\int_0^{t/2}\frac{1}{s^q}\mathrm{d}s + t^{-q}\int_{t/2}^t\frac{1}{(t-s)^{\frac{1}{2}}}\mathrm{d}s\right)\\ &\le ct^{\frac{1}{2}-q},\end{aligned}$$

where $0 \le q < 1$. For I_2, recalling that $\mathbb{P}$ is continuous in $\dot{M}^{2,p}(\mathbb{R}^3)$, then by Lemma 2.2 and the well-known estimate on the heat kernel $\|\nabla h_{(t-s)}(\cdot)\|_{L^1} \le \frac{c}{(t-s)^{\frac{1}{2}}}$, we obtain

$$\begin{aligned}\sup_{0\le t\le T}\|I_2\|_{\dot{M}^{2,p}} &= \sup_{0\le t\le T}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}(\operatorname{div}(u\otimes u))(s,\cdot)\mathrm{d}s\right\|_{\dot{M}^{2,p}}\\ &\le c\sup_{0\le t\le T}\int_0^t\|e^{(t-s)\Delta}(\operatorname{div}(u\otimes u))(s,\cdot)\|_{\dot{M}^{2,p}}\mathrm{d}s\\ &\le c\sup_{0\le t\le T}\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}s^{\frac{3}{2p}}}\left(s^{\frac{3}{2p}}\|u(s,\cdot)\|_{L^\infty}\|u(s,\cdot)\|_{\dot{M}^{2,p}}\right)\mathrm{d}s\\ &\le cT^{\frac{1}{2}-\frac{3}{2p}}\|u\|_{E_T}^2.\end{aligned}$$

Using the Hölder inequality, the facts that $e^{(t-s)\Delta}\mathbb{P}(\operatorname{div}(\cdot))$ can be written as a matrix of convolution operators and $\|K_{i,j}(t-s,\cdot)\|_{L^1} \le \frac{c}{(t-s)^{\frac{1}{2}}}$, with $K_{i,j}$ being the kernel of $e^{(t-s)\Delta}\mathbb{P}(\operatorname{div}(\cdot))$ (see Proposition 11.1 of [23]), we obtain

$$\begin{aligned}\sup_{0\le t\le T} t^{\frac{3}{2p}}\|I_2\|_{L^\infty} &= \sup_{0\le t\le T} t^{\frac{3}{2p}}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}(\operatorname{div}(u\otimes u))(s,\cdot)\mathrm{d}s\right\|_{L^\infty}\\ &\le \sup_{0\le t\le T} t^{\frac{3}{2p}}\int_0^t\|e^{(t-s)\Delta}\mathbb{P}(\operatorname{div}(u\otimes u))(s,\cdot)\|_{L^\infty}\mathrm{d}s\\ &\le c\sup_{0\le t\le T} t^{\frac{3}{2p}}\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}s^{\frac{3}{p}}}\left(s^{\frac{3}{2p}}\|u(s,\cdot)\|_{L^\infty}\right)^2\mathrm{d}s\\ &\le cT^{\frac{1}{2}-\frac{3}{2p}}\|u\|_{E_T}^2,\end{aligned}$$

which together with the estimate of $\|I_2\|_{\dot{M}^{2,p}}$ yields $\|I_2\|_{E_T} \le cT^{\frac{1}{2}-\frac{3}{2p}}\|u\|_{E_T}^2$. Observing that $U\cdot\nabla\theta = \operatorname{div}(U\theta)$ in K_2, then similar to the estimate of I_2, we have

$$\|(I_3, J_2, J_3, K_2)\|_{E_T} \le cT^{\frac{1}{2}-\frac{3}{2p}}\|(u, v, \theta)\|_{E_T}^2.$$

As stated above, the estimation of J_4 cannot follow directly from the estimate given by Jarrín (see pp. 6–7 in [17]), and we deal with J_4 as

$$
\begin{aligned}
\sup_{0\le t\le T}\|J_4\|_{\dot{M}^{2,p}} &= \sup_{0\le t\le T}\left\|\int_0^t e^{(t-s)\Delta}\nabla\theta(s,\cdot)\mathrm{d}s\right\|_{\dot{M}^{2,p}}\\
&\le \sup_{0\le t\le T}\int_0^t\|e^{(t-s)\Delta}\nabla\theta(s,\cdot)\|_{\dot{M}^{2,p}}\mathrm{d}s\\
&\le c\left(\sup_{0\le t\le T}\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}}\mathrm{d}s\right)\|\theta\|_{E_T}\\
&\le cT^{\frac{1}{2}}\|\theta\|_{E_T}
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{0\le t\le T} t^{\frac{3}{2p}}\|J_4\|_{L^\infty} &= \sup_{0\le t\le T} t^{\frac{3}{2p}}\left\|\int_0^t e^{(t-s)\Delta}\nabla\theta(s,\cdot)\mathrm{d}s\right\|_{L^\infty}\\
&\le c\sup_{0\le t\le T} t^{\frac{3}{2p}}\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}s^{\frac{3}{2p}}}\left(s^{\frac{3}{2p}}\|\theta(s,\cdot)\|_{L^\infty}\right)\mathrm{d}s\\
&\le c\sup_{0\le t\le T} t^{\frac{3}{2p}}\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}s^{\frac{3}{2p}}}\mathrm{d}s\|\theta\|_{E_T}\\
&\le cT^{\frac{1}{2}}\|\theta\|_{E_T},
\end{aligned}
$$

which yields that $\|J_4\|_{E_T}\le cT^{\frac{1}{2}}\|\theta\|_{E_T}$. Similarly, we directly obtain $\|K_3\|_{E_T}\le cT^{\frac{1}{2}}\|v\|_{E_T}$.

The above estimates show that (3.6) holds. This together with (3.5) and Lemma 2.6 gives the local existence and uniqueness of solution (u, v, θ) to system (3.1) for a time $0 < T_0(u_0, v_0, \theta_0) < +\infty$ small enough, which completes the proof of Proposition 3.1. □

Step 2. The boundness of (U, V, θ).

Proposition 3.2. *Let* (U, V, θ) *be a weak solution to system* (1.1). *If* $(U, V, \theta)\in\dot{M}^{2,p}(\mathbb{R}^3)$ *with* $p > 3$, *then we have* $(U, V, \theta)\in L^\infty(\mathbb{R}^3)$.

Proof. First, applying Leray's projector $\mathbb{P}$ to system $(1.1)_1$, we obtain

$$
\begin{cases}
\partial_t U - \Delta U + \mathbb{P}(\mathrm{div}(U\otimes U)) + \mathbb{P}(\mathrm{div}(V\otimes V)) = 0,\\
\partial_t V - \Delta V + \mathrm{div}(U\otimes V) + \mathrm{div}(V\otimes U) + \nabla\theta = 0,\\
\partial_t\theta - \Delta\theta + U\cdot\nabla\theta + \mathrm{div}V = 0,\\
\mathrm{div}U = 0,
\end{cases}
\tag{3.7}
$$

where $\partial_t U = \partial_t V = 0$ and $\partial_t\theta = 0$ due to (U, V, θ) being the time-independent function.

Next, operating $(3.7)_1$, $(3.7)_2$, and $(3.7)_3$ by $e^{(t-s)\Delta}$ and integrating over $[0, t]$ with respect to s, we obtain

$$
U = e^{t\Delta}U + \int_0^t e^{(t-s)\Delta}\mathbb{P}(\mathrm{div}(U\otimes U))\mathrm{d}s + \int_0^t e^{(t-s)\Delta}\mathbb{P}(\mathrm{div}(V\otimes V))\mathrm{d}s,
\tag{3.8}
$$

$$
V = e^{t\Delta}V + \int_0^t e^{(t-s)\Delta}\mathrm{div}(U\otimes V)\mathrm{d}s + \int_0^t e^{(t-s)\Delta}\mathrm{div}(V\otimes U)\mathrm{d}s + \int_0^t e^{(t-s)\Delta}\nabla\theta\mathrm{d}s,
\tag{3.9}
$$

and

$$\theta = e^{t\Delta}\theta + \int_0^t e^{(t-s)\Delta}(U\cdot\nabla\theta)\mathrm{d}s + \int_0^t e^{(t-s)\Delta}\mathrm{div}\, V\mathrm{d}s. \tag{3.10}$$

Since (3.8), (3.9), and (3.10) are structurally similar to (3.2), (3.3), and (3.4), respectively, we could obtain the existence and uniqueness of solution (U, V, θ) to system (3.7) by the same method as in Step 1. Then, by Proposition 3.1, we have

$$\sup_{0\le t\le T_0} t^{\frac{3}{2p}}\|(U, V, \theta)\|_{L^\infty} < +\infty.$$

As (U, V, θ) does not depend on the time variable, hence $(U, V, \theta) \in L^\infty(\mathbb{R}^3)$. □

Step 3. The regularity of weak solution (U, V, θ) **and pressure P**.

Applying Leray's projector $\mathbb{P}$ to system $(1.1)_1$, then we can write $(1.1)_1$, $(1.1)_2$, and $(1.1)_3$ as, respectively,

$$\partial^\alpha U = -\frac{1}{-\Delta}\mathbb{P}(\partial^\alpha \mathrm{div}(U\otimes U)) - \frac{1}{-\Delta}\mathbb{P}(\partial^\alpha \mathrm{div}(V\otimes V)) \coloneqq A_1 + A_2,$$
$$\partial^\alpha V = -\frac{1}{-\Delta}(\partial^\alpha \mathrm{div}(U\otimes V)) - \frac{1}{-\Delta}(\partial^\alpha \mathrm{div}(V\otimes U)) - \frac{1}{-\Delta}(\partial^\alpha\nabla\theta) \coloneqq B_1 + B_2 + B_3,$$

and

$$\partial^\alpha\theta = -\frac{1}{-\Delta}(\partial^\alpha(U\cdot\nabla\theta)) - \frac{1}{-\Delta}(\partial^\alpha\mathrm{div}V) \coloneqq D_1 + D_2.$$

Claim: The following estimates hold:

$$\|\partial^\alpha U\|_{\dot M^{2\sigma,p\sigma}} < +\infty, \tag{3.11}$$

$$\|\partial^\alpha V\|_{\dot M^{2\sigma,p\sigma}} < +\infty, \tag{3.12}$$

and

$$\|\partial^\alpha \theta\|_{\dot M^{2\sigma,p\sigma}} < +\infty, \tag{3.13}$$

where $1 \le \sigma < +\infty$ and $1 \le |\alpha| \le k + 2$ with $k \ge 0$.

Proof of Claim. The proof is divided into four cases: $|\alpha| = 1, 2, k + 1$ and $k + 2$ with $k \ge 0$. We first consider the cases $|\alpha| = 1$ and 2, and then we complete the proofs of the cases $|\alpha| = k + 1$ and $k + 2$ with the aid of a recurrence hypothesis that $(\partial^\alpha U, \partial^\alpha V, \partial^\alpha\theta) \in \dot M^{2\sigma,p\sigma}(\mathbb{R}^3)$ for all $1 \le \sigma < +\infty$ and $1 \le |\alpha| \le k$.

In the case of $|\alpha| = 1$. For A_1, by Lemmas 2.1, 2.4, Hölder inequalities, Proposition 3.2, and the fact that $\mathbb{P}$ is continuous in $\dot M^{r,p}(\mathbb{R}^3)$, we obtain

$$\|A_1\|_{\dot M^{2\sigma,p\sigma}} \le c\|U\otimes U\|_{\dot M^{2\sigma,p\sigma}} \le c\|U\|^{\frac{1}{\sigma}}_{\dot M^{2,p}}\|U\|^{2-\frac{1}{\sigma}}_{L^\infty} < +\infty. \tag{3.14}$$

Similar to (3.14), we infer

$$\|A_2\|_{\dot M^{2\sigma,p\sigma}} \le c\|V\otimes V\|_{\dot M^{2\sigma,p\sigma}} \le c\|V\|^{\frac{1}{\sigma}}_{\dot M^{2,p}}\|V\|^{2-\frac{1}{\sigma}}_{L^\infty} < +\infty.$$

Regarding B_1, we conclude from Lemmas 2.1, 2.4, Hölder inequalities, and Proposition 3.2 that

$$\begin{aligned}\|B_1\|_{\dot M^{2\sigma,p\sigma}} &= \left\|-\frac{1}{-\Delta}(\partial^\alpha\mathrm{div}(U\otimes V))\right\|_{\dot M^{2\sigma,p\sigma}}\\ &\le c\|U\otimes V\|_{\dot M^{2\sigma,p\sigma}}\\ &\le c\|U\|_{\dot M^{2\sigma,p\sigma}}\|V\|_{L^\infty}\\ &\le c\|U\|^{\frac{1}{\sigma}}_{\dot M^{2,p}}\|U\|^{1-\frac{1}{\sigma}}_{L^\infty}\|V\|_{L^\infty}\\ &< +\infty.\end{aligned} \tag{3.15}$$

By a similar argument to (3.15), B_2 and B_3 are handled as

$$\|(B_2, B_3)\|_{\dot{M}^{2\sigma,p\sigma}} \le c\left(\|U\|_{\dot{M}^{2,p}}^{\frac{1}{\sigma}}\|U\|_{L^\infty}^{1-\frac{1}{\sigma}}\|V\|_{L^\infty} + \|\theta\|_{\dot{M}^{2,p}}^{\frac{1}{\sigma}}\|\theta\|_{L^\infty}^{1-\frac{1}{\sigma}}\right) < +\infty.$$

With respect to D_1, applying Lemmas 2.1, 2.4, Hölder inequalities, and Proposition 3.2, one deduces

$$\|D_1\|_{\dot{M}^{2\sigma,p\sigma}} = \left\|-\frac{1}{-\Delta}\partial^\alpha \operatorname{div}(U\theta)\right\|_{\dot{M}^{2\sigma,p\sigma}} \le c\|U\|_{\dot{M}^{2,p}}^{\frac{1}{\sigma}}\|U\|_{L^\infty}^{1-\frac{1}{\sigma}}\|\theta\|_{L^\infty} < +\infty.$$

Similarly, it holds that

$$\|D_2\|_{\dot{M}^{2\sigma,p\sigma}} \le c\|V\|_{\dot{M}^{2\sigma,p\sigma}} \le c\|V\|_{\dot{M}^{2,p}}^{\frac{1}{\sigma}}\|V\|_{L^\infty}^{1-\frac{1}{\sigma}} < +\infty.$$

In the case of $|\alpha| = 2$. We split $\alpha = \alpha_1 + \alpha_2$ with $|\alpha_1| = 1$, and $|\alpha_2| = 1$. For A_1, B_1, and D_1, we write

$$\begin{cases} A_1 = -\frac{1}{-\Delta}\mathbb{P}(\partial^{\alpha_1}\operatorname{div}\partial^{\alpha_2}(U \otimes U)), \\ B_1 = -\frac{1}{-\Delta}(\partial^{\alpha_1}\operatorname{div}\partial^{\alpha_2}(U \otimes V)), \\ D_1 = -\frac{1}{-\Delta}(\partial^{\alpha_1}\operatorname{div}\partial^{\alpha_2}(U\theta)). \end{cases} \tag{3.16}$$

Recalling the Leibniz rule, for $i, j = 1, 2, 3$, we obtain $\partial^{\alpha_2}(U_iV_j) = V_j\partial^{\alpha_2}U_i + U_i\partial^{\alpha_2}V_j$. Using Lemmas 2.1, 2.4, Hölder inequalities, and Proposition 3.2, it follows that

$$\begin{aligned}\|(A_1, B_1, D_1)\|_{\dot{M}^{2\sigma,p\sigma}} &\le c\|(U\partial^{\alpha_2}U, U\partial^{\alpha_2}V, V\partial^{\alpha_2}U, U\partial^{\alpha_2}\theta, \theta\partial^{\alpha_2}U)\|_{\dot{M}^{2\sigma,p\sigma}} \\ &\le c\|\partial^{\alpha_2}U\|_{\dot{M}^{2\sigma,p\sigma}}\|\partial^{\alpha_2}V\|_{\dot{M}^{2\sigma,p\sigma}}\|\partial^{\alpha_2}\theta\|_{\dot{M}^{2\sigma,p\sigma}}\|U\|_{L^\infty}\|V\|_{L^\infty}\|\theta\|_{L^\infty} \\ &< +\infty.\end{aligned}$$

Similarly, the following estimate holds:

$$\begin{aligned}\|(A_2, B_2, B_3, D_2)\|_{\dot{M}^{2\sigma,p\sigma}} &\le c\|(V\partial^{\alpha_2}V, U\partial^{\alpha_2}V, V\partial^{\alpha_2}U, \partial^{\alpha_2}\theta, \partial^{\alpha_2}V)\|_{\dot{M}^{2\sigma,p\sigma}} \\ &\le c\|\partial^{\alpha_2}U\|_{\dot{M}^{2\sigma,p\sigma}}\|\partial^{\alpha_2}V\|_{\dot{M}^{2\sigma,p\sigma}}\|\partial^{\alpha_2}\theta\|_{\dot{M}^{2\sigma,p\sigma}}\|U\|_{L^\infty}\|V\|_{L^\infty} \\ &< +\infty.\end{aligned}$$

In the case of $|\alpha| = k + 1$. We split $\alpha = \alpha_1 + \alpha_2$ with $|\alpha_1| = 1$ and $|\alpha_2| = k$. For A_1, B_1, and D_1, we deduce

$$\begin{cases} A_1 = -\frac{1}{-\Delta}\mathbb{P}(\partial^{\alpha_1}\operatorname{div}\partial^{\alpha_2}(U \otimes U)), \\ B_1 = -\frac{1}{-\Delta}(\partial^{\alpha_1}\operatorname{div}\partial^{\alpha_2}(U \otimes V)), \\ D_1 = -\frac{1}{-\Delta}(\partial^{\alpha_1}\operatorname{div}\partial^{\alpha_2}(U\theta)). \end{cases}$$

Applying the Leibniz rule, one has

$$\begin{cases} \partial^{\alpha_2}(U_iU_j) = \sum_{|\beta|\le k} c_{\alpha_2,\beta}\partial^\beta U_i\partial^{\alpha_2-\beta}U_j, \\ \partial^{\alpha_2}(U_iV_j) = \sum_{|\beta|\le k} c_{\alpha_2,\beta}\partial^\beta U_i\partial^{\alpha_2-\beta}V_j, \\ \partial^{\alpha_2}(U_i\theta) = \sum_{|\beta|\le k} c_{\alpha_2,\beta}\partial^\beta U_i\partial^{\alpha_2-\beta}\theta, \end{cases}$$

where $i, j = 1, 2, 3$ and $c_{\alpha_2,\beta} > 0$. Making use of the recurrence hypothesis, Lemmas 2.1, 2.4, Hölder inequalities, and Proposition 3.2, we obtain

$$\|(A_1, B_1, D_1)\|_{\dot{M}^{2\sigma,p\sigma}} \le c\left\|\sum_{|\beta|\le k}(\partial^\beta U_i\partial^{\alpha_2-\beta}U_j, \partial^\beta U_i\partial^{\alpha_2-\beta}V_j, \partial^\beta U_i\partial^{\alpha_2-\beta}\theta)\right\|_{\dot{M}^{2\sigma,p\sigma}}$$

$$\begin{aligned}
&\le c \sum_{1\le|\beta|\le k-1} (\|\partial^\beta U\|_{\dot M^{2\sigma,p\sigma}} \|\partial^{\alpha_2-\beta} U\|_{\dot M^{2\sigma,p\sigma}} \|\partial^{\alpha_2-\beta} V\|_{\dot M^{2\sigma,p\sigma}} \\
&\quad \times \|\partial^{\alpha_2-\beta}\theta\|_{\dot M^{2\sigma,p\sigma}} + \|(\partial^k U, \partial^k V, \partial^k\theta)\|_{\dot M^{2\sigma,p\sigma}} \|U\|_{L^\infty}\|V\|_{L^\infty}\|\theta\|_{L^\infty}) \\
&< +\infty.
\end{aligned}$$

Similarly, it holds that

$$\begin{aligned}
\|(A_2, B_2, B_3, D_2)\|_{\dot M^{2\sigma,p\sigma}} &\le c \left\| \sum_{|\beta|\le k} (\partial^\beta V_i \partial^{\alpha_2-\beta} V_j, \partial^\beta V_i \partial^{\alpha_2-\beta} U_j, \partial^{\alpha_2}\theta, \partial^{\alpha_2} V) \right\|_{\dot M^{2\sigma,p\sigma}} \\
&\le c \sum_{1\le|\beta|\le k-1} (\|\partial^\beta V\|_{\dot M^{2\sigma,p\sigma}} \|\partial^{\alpha_2-\beta} U\|_{\dot M^{2\sigma,p\sigma}} \|\partial^{\alpha_2-\beta} V\|_{\dot M^{2\sigma,p\sigma}} \\
&\quad + \|(\partial^k U, \partial^k V)\|_{\dot M^{2\sigma,p\sigma}} \|U\|_{L^\infty}\|V\|_{L^\infty} + \|(\partial^{\alpha_2} V, \partial^{\alpha_2}\theta)\|_{\dot M^{2\sigma,p\sigma}}) \\
&< +\infty.
\end{aligned}$$

In the case of $|\alpha| = k + 2$. We repeat again the above estimates to obtain $(\partial^\alpha U, \partial^\alpha V, \partial^\alpha\theta) \in \dot M^{2\sigma,p\sigma}(\mathbb{R}^3)$ for $|\alpha| = k + 2$. Therefore, the claim holds. □

Next, we study the pressure term in system (1.1). Applying the operator $\partial^\alpha \mathrm{div}$ to system $(1.1)_1$, it holds that

$$\partial^\alpha P = \frac{1}{-\Delta}(\mathrm{div}\partial^\alpha \mathrm{div}(U \otimes U)) + \frac{1}{-\Delta}(\mathrm{div}\partial^\alpha \mathrm{div}(V \otimes V)), \tag{3.17}$$

where $|\alpha| \le k + 2$ with $k \ge 0$. By (3.11), (3.12), the Leibniz rule, Lemmas 2.1 and 2.4, Hölder inequalities, and Proposition 3.2, we obtain $\partial^\alpha P \in \dot M^{2\sigma,p\sigma}(\mathbb{R}^3)$ for all $1 \le \sigma < +\infty$.

The above estimates show that $(U, V, \theta, P) \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$ with $p > 3$, $k \ge 0$. Moreover, we can conclude from the embedding $\dot M^{2,p}(\mathbb{R}^3) \subset \dot M^{1,p}(\mathbb{R}^3)$ and Lemma 2.5 that $(\partial^\alpha U, \partial^\alpha V, \partial^\alpha\theta, \partial^\alpha P) \in C^{1-3/p}(\mathbb{R}^3)$ for all $p > 3$ and $|\alpha| \le k + 1$ with $k \ge 0$. The proof of Theorem 1.1 is completed. □

Acknowledgements: The authors are grateful to the anonymous referees for their very helpful comments and suggestions that greatly improve the quality and presentation of the original manuscript.

Funding information: This work was supported by research supported by the Natural Science Foundation of China (Grant No. 11901346).

Author contributions: All authors contributed equally to this work. All authors read and approved the final manuscript.

Conflict of interest: The authors state no conflicts of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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