Research Article

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On tangent bundles of Walker four-manifolds

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Abstract: The aim of this study is to explore the complete lifts of almost Norden structures on tangent bundles of Walker four-manifolds. Furthermore, we examine the integrability conditions of the complete lifts J^C of the proper almost complex structure and identify the conditions for the isotropic Kähler-Norden property of the structure J^C on $(T(M_4), J^C, g^C)$. Finally, we investigate the holomorphic conditions of the Norden (Kähler-Norden) metrics g^C and discuss Quasi-Kähler-Norden metrics g^C on the tangent bundle $(T(M_4), \varphi^C, g^C)$ of Walker four-manifolds.

Keywords: Norden metrics, Walker four-manifolds, integrability conditions, proper almost complex structure, complete lifts, symbolic computation

MSC 2020: 53B30, 53C15, 53C50

1 Introduction

The aim of this study is to derive the most general outcomes within the tangent bundle by verifying the accuracy of computational results using computer programs. Thus, the appropriate almost complex structure and Norden-Walker metric are established on four-dimensional Norden-Walker manifolds through the assistance of complete lifts on the tangent bundle of this manifold.

$$J^C = \begin{pmatrix} J_i^h & 0 \\ \partial J_i^h & J_i^h \end{pmatrix}. \tag{1}$$

G^C of G belonging to $\Im_2^0(M)$ with local components G_{ji} can be expressed in matrix form as

$$G^C = \begin{pmatrix} \partial G_{ji} & G_{ji} \\ G_{ji} & 0 \end{pmatrix}. \tag{2}$$

Furthermore, it should be noted that complete lifts are defined by [1].

The notation $\Im_q^p(M_{2n})$ is used to denote the set of all tensor fields of type (p, q) on the manifold M_{2n}. It is specified that manifolds, tensor fields and connections associated with this notation are always assumed to be differentiable and of class C^∞. This means that these mathematical objects such as the manifold M_{2n}, the tensor fields and any connections are characterized by being smooth and possessing infinitely differentiable functions.

In summary, $\Im_q^p(M_{2n})$ represents the collection of tensor fields in the manifold M_{2n} of type (p, q) and it is assumed that these objects are differentiable and belong to the class C^∞.

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1.1 Norden (anti-Hermitian) and holomorphic Norden (Kähler-Norden) Metrics

A metric g is considered a Norden metric [2] if it satisfies the following condition:

$$g(JU, JV) = -g(U, V)$$

or equivalently

$$g(U, JV) = g(JU, V) \tag{3}$$

for any $U, V \in \Im_0^1(M_{2n})$, where $\Im_1^0(M_{2n})$ denotes the set of 1-form on the manifold (M_{2n}).

A Norden metric g is termed holomorphic if it satisfies the following condition:

$$(\varnothing_J g)(U, V, W) = 0 \tag{4}$$

for any $U, V, W \in \Im_0^1(M_{2n})$. Here $\varnothing_J g$ represents the Tachibana operator applied to g metric.

By putting $U = \partial_k$, $V = \partial_i$, $W = \partial_j$ in equation (4), the components $(\varnothing_J g)_{kij}$ of $\varnothing_J g$ according to the local coordinate system $x^1, \ldots, x^n$ can be expressed by

$$(\varnothing_J g)_{kij} = J_k^m \partial_m g_{ij} - J_i^m \partial_k g_{mj} + g_{mj}(\partial_i J_k^m - \partial_k J_i^m) + g_{im} \partial_j J_k^m.$$

If (M_{2n}, J, g) is a Norden manifold with a holomorphic Norden metric g, then it is referred to as a holomorphic Norden manifold.

1.2 Holomorphic tensor fields

We introduce an operator called the Tachibana operator:

$$\varnothing_J : \Im_q^0(M_{2n}) \longrightarrow \Im_{q+1}^0(M_{2n})$$

that acts on the pure tensor field ω defined in [2]

$$\begin{aligned}(\varnothing_J \omega)(U, V_1, V_2, \ldots, V_q) &= (JU)(\omega(V_1, V_2, \ldots, V_q)) - U(\omega(JV_1, V_2, \ldots, V_q)) \\ &\quad + \omega((L_{V_1}J)U, V_2, \ldots, V_q) + \ldots + \omega(V_1, V_2, \ldots, (L_{V_q}J)U),\end{aligned}$$

where L_V denotes Lie differentiation according to V.

A holomorphic tensor field $\overset{*}{\omega}$ on $X_n(\mathbf{C})$ can be expressed on M_{2n} in the form of a pure tensor field ω, meeting the condition

$$(\varnothing_J \omega)(U, V_1, V_2, \ldots, V_q) = 0$$

for any $U, V_1, \ldots, V_q \in \Im_0^1(M_{2n})$.

2 Main results

2.1 Complete lifts of the Norden-Walker metrics

A Walker metric defined by [3]

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \alpha & \theta \\ 0 & 1 & \theta & \beta \end{pmatrix}, \tag{5}$$

where α, β, θ are smooth functions of the coordinates (x, y, z, t). The parallel null 2-plane D is spanned locally by $\{\partial_x, \partial_y\}$, where $\partial_x = \frac{\partial}{\partial_x}$, $\partial_y = \frac{\partial}{\partial_y}$.

The complete lift g^C of a neutral metric g on a four-manifolds M_4 with local components g_{ij} is defined by

$$g^C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \partial\alpha & \partial\theta & 1 & 0 & \alpha & \theta \\ 0 & 0 & \partial\theta & \partial\beta & 0 & 1 & \theta & \beta \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha & \theta & 0 & 0 & 0 & 0 \\ 0 & 1 & \theta & \beta & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

In addition, we obtain the inverse of the metric tensor g^C defined by (6) as follows:

$$(g^C)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\alpha & -\theta & 1 & 0 \\ 0 & 0 & 0 & 0 & -\theta & -\beta & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\alpha & -\theta & 1 & 0 & -\partial\alpha & -\partial\theta & 0 & 0 \\ -\theta & -\beta & 0 & 1 & -\partial\theta & -\partial\beta & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

where α, β, θ are smooth functions and the complete lifts $\alpha^C, \beta^C, \theta^C$ of the functions α, β, θ have the local expressions $\theta^C = y^i\partial_i\theta = \partial\theta$, $\beta^C = y^i\partial_i\beta = \partial\beta$, $\alpha^C = y^i\partial_i\alpha = \partial\alpha$ of the coordinates (x, y, z, t, k, l, m, n).

2.2 Complete lifts of the proper almost complex structure J

Consider Φ represents an almost complex structure on a Walker manifold M_4, meeting the condition

(i) $\Phi^2 = -I$,

(ii) $g(\Phi U, V) = g(U, \Phi V)$ (Nordenian property),

(iii) $\Phi\partial_x = \partial_y$, $\Phi\partial_y = -\partial_x$ (Φ induces a positive $\frac{\pi}{2}$-rotation on D).

It is evident that these three properties do not uniquely define Φ, i.e.,

$$\begin{cases} \Phi\partial_x = \partial_y, \Phi\partial_y = -\partial_x, \\ \Phi\partial_z = \omega\partial_x + \frac{1}{2}(\alpha+\beta)\partial_y - \partial_t, \Phi\partial_t = -\frac{1}{2}(\alpha+\beta)\partial_x + \omega\partial_y + \partial_z \end{cases}$$

and Φ has the local components

$$\Phi = (\Phi^i_j) = \begin{pmatrix} 0 & -1 & \omega & -\frac{1}{2}(\alpha+\beta) \\ 1 & 0 & \frac{1}{2}(\alpha+\beta) & \omega \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

in the context of the natural frame $\{\partial_x, \partial_y, \partial_z, \partial_t\}$, where $\omega = \omega(x, y, z, t)$ is an arbitrary function, we proceed to set $\omega = \theta$. Subsequently, the metric g defines a unique almost complex structure

$$J = (J^i_j) = \begin{pmatrix} 0 & -1 & \theta & -\frac{1}{2}(\alpha+\beta) \\ 1 & 0 & \frac{1}{2}(\alpha+\beta) & \theta \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{8}$$

(M_4, J, g) is denoted as an almost Norden-Walker manifold. In accordance with the terminology found in [4–8], we refer to J as the proper almost complex structure.

From equation (1), the complete lifts J^C of J on four-manifolds M_4 with local components J^i_j are determined by:

$$J^C = \begin{pmatrix} 0 & -1 & \theta & -\frac{1}{2}(\alpha+\beta) & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2}(\alpha+\beta) & \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial\theta & -\frac{1}{2}(\partial\alpha+\partial\beta) & 0 & -1 & \theta & -\frac{1}{2}(\alpha+\beta) \\ 0 & 0 & \frac{1}{2}(\partial\alpha+\partial\beta) & \partial\theta & 1 & 0 & \frac{1}{2}(\alpha+\beta) & \theta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{9}$$

where α, β, θ are smooth functions, the complete lifts $\alpha^C, \beta^C, \theta^C$ of these functions exhibit local expressions as $\theta^C = y^i\partial_i\theta = \partial\theta$, $\beta^C = y^i\partial_i\beta = \partial\beta$, $\alpha^C = y^i\partial_i\alpha = \partial\alpha$ in terms of the coordinates (x, y, z, t, k, l, m, n).

Remark 1. From (9), it can be deduced that when $\alpha = -\beta$ and $\theta = 0, J$ becomes integrable.

2.3 Integrability conditions of J^C

Theorem 1. *Let J be the proper almost complex structure on (M_4, J, g). Then, J^C is integrable if and only if the following PDEs hold*:

$$\begin{aligned} 0 &= -\theta_t + \theta\theta_x + (\partial\theta)\theta_k + (\partial\theta)_n - \theta(\partial\theta)_k + \frac{1}{2}[(\alpha+\beta)\theta_y + ((\partial\alpha) \\ &\quad + (\partial\beta))\theta_l - (\alpha+\beta)(\partial\theta)_l + (\alpha_z+\beta_z) - ((\partial\alpha)_m + (\partial\beta)_m)] \\ 0 &= \theta_z + \theta\theta_y + (\partial\theta)\theta_l - (\partial\theta)_m - \theta(\partial\theta)_l - \frac{1}{2}[(\alpha+\beta)\theta_x \\ &\quad - ((\partial\alpha) + (\partial\beta))\theta_k + (\alpha+\beta)(\partial\theta)_k + (\alpha_t+\beta_t) - ((\partial\alpha)_n + (\partial\beta)_n)] \\ 0 &= -\theta_n + \theta(\theta_k) + \frac{1}{2}(\alpha_m + \beta_m + (\alpha+\beta)\theta_l) \\ 0 &= -\theta_m - \theta(\theta_l) - \frac{1}{2}(\alpha_n + \beta_n - (\alpha+\beta)\theta_k) \end{aligned}$$

$$\begin{aligned} &(\partial\alpha)_k + (\partial\beta)_k + 2(\partial\theta)_l = 0, && (\partial\alpha)_l + (\partial\beta)_l - 2(\partial\theta)_k = 0 \\ &(\partial\alpha)_y + (\partial\beta)_y - 2(\partial\theta)_x = 0, && (\partial\alpha)_x + (\partial\beta)_x + 2(\partial\theta)_y = 0 \\ &\alpha_k + \beta_k + 2\theta_l = 0, && \alpha_l + \beta_l - 2\theta_k = 0 \\ &\alpha_x + \beta_x + 2\theta_y = 0, && \alpha_y + \beta_y - 2\theta_x = 0. \end{aligned} \tag{10}$$

Thus, we see that if $\alpha = -\beta$ and $\theta = 0$, then J^C is integrable.

Let $(T(M_4), J^C, g^C)$ be a Norden-Walker manifolds ($N_{J^C} = 0$) and $\alpha = \beta$. Then, equation (10) reduces to

$$\begin{matrix} \alpha_x = -\theta_y \\ \alpha_y = \theta_x \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_k = -\theta_l \\ \alpha_l = \theta_k, \end{matrix} \tag{11}$$

from which follows

$$\begin{matrix} \alpha_{xx} + \alpha_{yy} = 0 \\ \theta_{xx} + \theta_{yy} = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_{kk} + \alpha_{ll} = 0 \\ \theta_{ll} + \theta_{kk} = 0, \end{matrix}$$

e.g., the functions α and θ are harmonic according to the arguments x, y and k, l.

Theorem 2. *If $(T(M_4), J^c, g^c)$ is Norden manifold and $\alpha = \beta$, then α and θ are harmonic according to the arguments x, y, and k, l.*

2.4 Example of Norden-Walker metric

Suppose α equals β and let $h(x, y, k, l)$ represent a function of variables x, y, k and l that is harmonic with respect to α, for instance.

$$h(x, y, k, l) = e^x \cos y + e^k \cos l.$$

We define

$$\begin{aligned} \alpha = \partial(x, y, z, t, k, l, m, n) &= h(x, y, k, l) + \omega(z, t, m, n) \\ &= e^x \cos y + e^k \cos l + \omega(z, t, m, n), \end{aligned}$$

where ω represents any smooth functions z, t, m and n.

In that case, α is also harmonic according to x, y, k and l. We obtain

$$\begin{matrix} \alpha_x = e^x \cos y \\ \alpha_y = -e^x \sin y \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_k = e^k \cos l \\ \alpha_l = -e^k \sin l. \end{matrix}$$

From equations (11), we derive the partial differential equation (PDE) for θ to fulfill as

$$\begin{matrix} \theta_x = \alpha_y = -e^x \sin y \\ \theta_y = -\alpha_x = -e^x \cos y \end{matrix} \quad \text{and} \quad \begin{matrix} \theta_k = \alpha_l = -e^k \sin l \\ \theta_l = -\alpha_k = -e^k \cos l. \end{matrix}$$

For these PDE$_S$, we obtain a value of θ_1 by solutions of the equation $\theta_y = -\alpha_x = -e^x \cos y$ and a value of θ_2 by solutions of the equation $\theta_l = -\alpha_k = -e^k \cos l$. Thus, we obtain the results

$$\begin{aligned} \theta_1 &= -e^x \sin y + \beta_1(z, t, k, l, m, n) \\ \theta_2 &= -e^k \sin l + \beta_2(x, y, z, t, m, n), \end{aligned}$$

where β_1 and β_2 represent arbitrary smooth functions of z, t, k, l, m, n as well as x, y, z, t, m, n, respectively. Consequently, the complete lift of the Norden-Walker metric exhibits components in relation to θ_1 and θ_2, respectively.

$$g_1^C = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \partial(e^x \cos y + e^k \cos l + \omega(z,t,m,n)) & \partial(-e^x \sin y + \beta_1(z,t,k,l,m,n)) \\
0 & 0 & \partial(-e^x \sin y + \beta_1(z,t,k,l,m,n)) & \partial(e^x \cos y + e^k \cos l + \omega(z,t,m,n)) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & e^x \cos y + e^k \cos l + \omega(z,t,m,n) & -e^x \sin y + \beta_1(z,t,k,l,m,n) \\
0 & 1 & -e^x \sin y + \beta_1(z,t,k,l,m,n) & e^x \cos y + e^k \cos l + \omega(z,t,m,n)
\end{array}\right.$$

$$\left.\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & e^x \cos y + e^k \cos l + \omega(z,t,m,n) & -e^x \sin y + \beta_1(z,t,k,l,m,n) \\
0 & 1 & -e^x \sin y + \beta_1(z,t,k,l,m,n) & e^x \cos y + e^k \cos l + \omega(z,t,m,n) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right),$$

$$g_2^C = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \partial(e^x \cos y + e^k \cos l + \omega(z,t,m,n)) & \partial(-e^k \sin l + \beta_2(x,y,z,t,m,n)) \\
0 & 0 & \partial(-e^k \sin l + \beta_2(x,y,z,t,m,n)) & \partial(e^x \cos y + e^k \cos l + \omega(z,t,m,n)) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & e^x \cos y + e^k \cos l + \omega(z,t,m,n) & -e^k \sin l + \beta_2(x,y,z,t,m,n) \\
0 & 1 & -e^k \sin l + \beta_2(x,y,z,t,m,n) & e^x \cos y + e^k \cos l + \omega(z,t,m,n)
\end{array}\right.$$

$$\left.\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & e^x \cos y + e^k \cos l + \omega(z,t,m,n) & -e^k \sin l + \beta_2(x,y,z,t,m,n) \\
0 & 1 & -e^k \sin l + \beta_2(x,y,z,t,m,n) & e^x \cos y + e^k \cos l + \omega(z,t,m,n) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right).$$

2.5 Holomorphic conditions of g^C

If (M_4, J, g) represents an almost Norden-Walker manifold and

$$(\varnothing_J g)_{kij} = J_k^m \partial_m g_{ij} - J_i^m \partial_k g_{mj} + g_{mj}(\partial_i J_k^m - \partial_k J_i^m) + g_{im} \partial_j J_k^m = 0, \tag{12}$$

then, as demonstrated in Section 1, J is integrable and (M_4, J, g) is denoted as a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold.

By substituting equations (6) and (9) in (12), we observe the non-vanishing components of $(\varnothing_{J^C} g^C)_{kij}$.

Theorem 3. *If (M_4, J, g) denotes an almost Norden-Walker manifold, then the triple $(T(M_4), J^C, g^C)$ constitutes a holomorphic Norden (Kähler-Norden) manifold if and only if the following PDEs are satisfied*:

$$\alpha_x = \alpha_y = \alpha_k = \alpha_l = \alpha_m = \alpha_n = 0$$

$$(\partial\alpha)_x = (\partial\alpha)_y = (\partial\alpha)_k = (\partial\alpha)_l = 0$$

$$\beta_x = \beta_y = \beta_k = \beta_l = \beta_m = \beta_n = 0$$

$$(\partial\beta)_x = (\partial\beta)_y = (\partial\beta)_z = (\partial\beta)_k = (\partial\beta)_l = 0$$
$$\theta_x = \theta_y = \theta_k = \theta_l = \theta_m = \theta_n = 0$$
$$(\partial\theta)_x = (\partial\theta)_y = (\partial\theta)_k = (\partial\theta)_l = 0$$

$$\begin{aligned}
&(\partial\alpha)_t - 2(\partial\theta)_z = 0, && 2\theta_t + (\partial\beta)_m - 2(\partial\theta)_n = 0\\
&\beta_z - \theta_t + (\partial\theta)_n = 0, && \beta_z - \alpha_z + (\partial\alpha)_m + (\partial\beta)_m = 0\\
&2\theta_z - (\partial\alpha)_n = 0, && \alpha_t - \beta_t - 4\theta_z + (\partial\alpha)_n + (\partial\beta)_n = 0\\
&\alpha_z + \beta_z - (\partial\alpha)_m = 0, && \alpha_t + \beta_t - 2(\partial\theta)_m - (\partial\beta)_n = 0\\
&\theta_z - \alpha_t + (\partial\theta)_m = 0.
\end{aligned}$$

2.6 Quasi-Kähler-Norden-Walker manifold

A Norden-Walker manifold (M, J, g) meeting the condition $\varnothing_k g_{ij} + 2\nabla_k G_{ij}$ being zero is termed a quasi-Kähler manifold, where G is defined by $G_{ij} = J_i^m g_{mj}$ [9]. Furthermore, for the covariant derivative ∇G of the associated metric G set $(\nabla G)_{ijk} = \nabla_i G_{jk}$ on the Norden-Walker four-manifolds.

Theorem 4. *A triplet $(T(M_4), J^C, g^C)$ forms a quasi-Kähler-Norden manifold if and only if the following PDEs are satisfied*:

$$\alpha_x = \alpha_y = \alpha_k = \alpha_l = \alpha_m = \alpha_n = 0$$
$$(\partial\alpha)_x = (\partial\alpha)_y = (\partial\alpha)_k = (\partial\alpha)_l = 0$$
$$\beta_x = \beta_y = \beta_k = \beta_l = \beta_m = \beta_n = 0$$
$$(\partial\beta)_x = (\partial\beta)_y = (\partial\beta)_z = (\partial\beta)_k = (\partial\beta)_l = 0$$
$$\theta_x = \theta_y = \theta_k = \theta_l = \theta_m = \theta_n = 0$$
$$(\partial\theta)_x = (\partial\theta)_y = (\partial\theta)_k = (\partial\theta)_l$$

$$\begin{aligned}
&3(\partial\theta)_z - 2(\partial\alpha)_t = 0, && 3\theta_z - \alpha_t - (\partial\alpha)_n + (\partial\theta)_m = 0\\
&2(\partial\theta)_z - (\partial\alpha)_t = 0, && (\partial\alpha)_m - \theta_t - \alpha_z + (\partial\theta)_n = 0\\
&\beta_z - \theta_t + (\partial\theta)_n = 0, && \beta_z + \theta_t - (\partial\theta)_n + (\partial\beta)_m = 0\\
&\theta_z - \alpha_t + (\partial\theta)_m = 0, && -\beta_t - \theta_z + (\partial\beta)_n + (\partial\theta)_m = 0.
\end{aligned}$$

2.7 Isotropic Kähler-Norden structures J^C on $T(M_4)$

A proper almost complex structure on a Norden-Walker manifold (M_4, J, g) is termed isotropic Kähler if $\|\nabla J^2\| = 0$ but $\nabla J \neq 0$. In addition, the complete lift of connection coefficient is written as

$${}^C\Gamma_{i,j}^k = \frac{1}{2}(g^C)^{ks}(\partial_i(g^C)_{sj} + \partial_j(g^C)_{is} - \partial_s(g^C)_{ij}).$$

Due to ${}^C\Gamma_{i,j}^k = {}^C\Gamma_{j,i}^k$, we only need to consider ${}^C\Gamma_{i,j}^k(i\langle j)$. Let $(g^C)^{-1}$ be the inverse of the metric tensor g^C defined by (6) and the covariant derivative ${}^C\nabla J^C$ of the almost complex structure is expressed as $({}^C\nabla J^C)_{ij}^k = {}^C\nabla_i (J^C)_j^k$. If the following conditions hold:

$$\alpha_x = \beta_x = \theta_x = (\partial\alpha)_x = (\partial\beta)_x = (\partial\theta)_x = 0,$$
$$\alpha_y = \beta_y = \theta_y = (\partial\alpha)_y = (\partial\beta)_y = (\partial\theta)_y = 0,$$

$$
\begin{aligned}
&(\partial\beta)_z = 0,\\
&\alpha_k = \beta_k = \theta_k = (\partial\alpha)_k = (\partial\beta)_k = (\partial\theta)_k = 0,\\
&\alpha_l = \beta_l = \theta_l = (\partial\alpha)_l = (\partial\beta)_l = (\partial\theta)_l = 0,
\end{aligned}
\tag{13}
$$

then, the non-vanishing components of $({}^C\nabla(J^C))^k_{ij} = {}^C\nabla_i(J^C)^k_j$ are as follows:

$$
\begin{aligned}
&{}^C\nabla_z(J^C)^x_z = \frac{1}{2}[3\theta_z - \alpha_t - (\partial\alpha)_n + (\partial\theta)_m]\\
&{}^C\nabla_z(J^C)^x_t = \frac{1}{2}[\alpha_z + \theta_t - (\partial\alpha)_m - (\partial\theta)_n]\\
&{}^C\nabla_z(J^C)^y_z = \frac{1}{2}[(\partial\alpha)_m - \theta_t - \alpha_z - (\partial\theta)_n]\\
&{}^C\nabla_z(J^C)^y_n = \theta_m - \alpha_n\\
&{}^C\nabla_z(J^C)^k_z = 2(\partial\theta)_z - (\partial\alpha)_t\\
&{}^C\nabla_t(J^C)^x_z = \frac{1}{2}[\beta_z + \theta_t + (\partial\beta)_m - (\partial\theta)_n]\\
&{}^C\nabla_t(J^C)^x_t = \frac{1}{2}[\beta_t + \theta_z - (\partial\beta)_n - (\partial\theta)_m]\\
&{}^C\nabla_t(J^C)^x_m = \beta_m - \theta_n\\
&{}^C\nabla_m(J^C)^x_z = \frac{1}{2}[3\theta_m - \alpha_n]\\
&{}^C\nabla_m(J^C)^x_t = \frac{1}{2}[-\alpha_m - \theta_n]\\
&{}^C\nabla_m(J^C)^k_z = \theta_z - \alpha_t + (\partial\theta)_m\\
&{}^C\nabla_n(J^C)^x_t = \frac{1}{2}[-\theta_m - \beta_n]\\
&{}^C\nabla_n(J^C)^y_t = \frac{1}{2}[\beta_m + \theta_n]\\
&{}^C\nabla_n(J^C)^k_z = \beta_z - \theta_t + (\partial\theta)_n
\end{aligned}
\tag{14}
$$

If we utilize (6), (7), (10), and (11), then we can deduce that

$$\|{}^C\nabla(J^C)^2\| = (g^C)^{ij}(g^C)^{kl}(g^C)_{ms}({}^C\nabla J^C)^m_{ik}({}^C\nabla J^C)^s_{jl} = 0.$$

Theorem 5. *The isotropic Kähler property is satisfied by the complete lifts J^C of the proper almost complex structure on the almost Norden manifold $(T(M_4), J^C, g^C)$.*

3 Conclusion

This study first examined the complete lifts of almost Norden structures on tangent bundles of Walker four-manifolds. Then, the integrability conditions of the complete lifts J^C of the proper almost complex structure are investigated and the conditions for the isotropic Kähler-Norden property of the structure J^C on $(T(M_4), J^C, g^C)$ are identified. Finally, the holomorphic conditions of the Norden (Kähler-Norden) metrics g^C are studied and Quasi-Kähler-Norden metrics g^C on the tangent bundle $(T(M_4), \varphi^C, g^C)$ of four-Walker-manifolds are discussed.

The obtained results on tangent bundle of four-Walker-manifolds are more generalized with respect to past studies and this generalization is achieved through symbolic computation. Future studies can also be carried out on the curvature tensor.

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