Research Article

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Periodic or homoclinic orbit bifurcated from a heteroclinic loop for high-dimensional systems

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Abstract: Consider an autonomous ordinary differential equation in $\mathbb{R}^n$, which has a heteroclinic loop. Assume that the heteroclinic loop consists of two degenerate heteroclinic orbits γ_1, γ_2 and two saddle points with different Morse indices. The degenerate heteroclinic orbit in the sense that variational equation along the heteroclinic orbit γ_i has d_i ($d_i > 1, i = 1, 2$) linearly independent bounded solutions. By the different Morse indices and d_i, the heteroclinic loop is a heterodimensional loop, at the same time, it has high codimension in this situation. Applying Lin's method to the heteroclinic loop, we derived the bifurcation function. The zeros of this function correspond to the conditions under which periodic or homoclinic orbits can bifurcate from the high-codimension heteroclinic loop in the perturbed system.

Keywords: heteroclinic connections, Lin's method, exponential dichotomy, bifurcation

MSC 2020: 34C23, 34C25, 34C37

1 Introduction

Homoclinic and heteroclinic bifurcations play an important role in dynamical systems. More and more mathematicians have devoted themselves to study the bifurcation problems of homoclinic or heteroclinic orbits. An overview of homoclinic and heteroclinic bifurcation is given in [1]. A heteroclinic loop consists of two distinct hyperbolic saddle points and two heteroclinic orbits connecting them. When there are two saddle points with different Morse indices, this heteroclinic loop is called a heterodimensional loop. Otherwise, the heteroclinic loop is called a equidimensional loop [2]. Numerical and explicit examples of a heteroclinic loop are given in [3,4].

There is rich and complex recurrent dynamics near homoclinic or heteroclinic orbits. Hence, a central task is to find all orbits that stay near the homoclinic orbits or heteroclinic loop for all times. There are two different approaches to treat those problems. The first approach is to use Poincare or first-return maps. The existence of these special orbits is equivalent to the existence of the fixed points of Poincare or first-return maps. These methods are called geometric approaches. The second is the analytical approach. The core is using Lyapunov-Schmidt reduction. The heart of Lyapunov-Schmidt method is the Fredholm property. Chow et al. studied the persistence of the homoclinic orbit of the Duffing equation by the Fredholm property [5]. Following this work, many people have helped to develop the analytical approach to homoclinic or heteroclinic bifurcation problems. In 1990, Lin investigated the existence of periodic or aperiodic solutions near the heteroclinic chains for systems of ordinary differential equations and delay equations by analytical approach. This method was generalized by Fiedler, Vanderbauwhede, Sandstede, and many others as Lin's method [6]. The idea of Lin's method is to construct a sequence of piecewise continuous solutions near the original heteroclinic chain,

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and the bifurcation function can be obtained from these solutions. If the bifurcation function has zeros, then there exist periodic or aperiodic solutions near the heteroclinic chain. Lin's method can also be used in discrete dynamical systems, singularly perturbed systems, and numerical computation, cf. [7–9].

For the periodic or aperiodic solutions bifurcated from homoclinic orbit by analytical approach, refer [10–12]. Chow et al. [13] considered the equidimensional heteroclinic loop, which is the equilibria that form the heteroclinic loop that has the same dimension of the unstable manifold. Meanwhile, the authors considered non-degenerate heteroclinic orbit. They used geometric approach to seek homoclinic or periodic orbit bifurcated from a heteroclinic loop. Rademacher [14], studied the homoclinic orbit bifurcated from a heteroclinic loop with one equilibrium and one periodic orbit. They assumed that the unperturbed heteroclinic orbits are one- or two-dimensional. By exponential trichotomy and Lin' method, they found 1-homoclinic orbits near the heteroclinic loop. Jin et al. [15] considered an equidimensional loop for high-dimensional systems. They used local coordinate systems in a neighborhood of a heteroclinic loop to construct the Poincare maps and the bifurcation equations and then obtained the coexistence and coexistence regions of the 1-homoclinic loop, 1-periodic orbit, 2-homoclinic loop, and 2-periodic orbit near the heteroclinic loop. Zhu and Sun [16] considered the same subject that is homoclinic and periodic orbits bifurcated from the heteroclinic cycle connecting saddle-foci and saddle. Bykov cycle is a special heteroclinic cycle between two hyperbolic equilibria of saddle types p1 and p2, where one of the connections is transverse and isolated. Labouriau and Rodrigues [17] considered a differential equation in a three-dimensional manifold having a heteroclinic cycle that consists of two saddle-foci of different Morse indices whose one-dimensional invariant manifolds coincide and whose two-dimensional invariant manifolds intersect transversely. So, the heteroclinic cycle is defined by the presence of the Bykov cycle. They showed the existence of mixed dynamics in the neighborhood of the Bykov cycle. In the recent work by Knlbloch [18], this subject was extended to higher dimensions using Lin's method. Long and Xu [19] investigated the persistence of a heterodimensional loop under periodic perturbation. Under some conditions, the perturbed system can have a heteroclinic loop near the unperturbed heterodimensional loop. For more research results regarding the recurrent dynamic near heteroclinic loop, refer [20–24].

Based on the above background, we apply Lin's method to investigate periodic or homoclinic orbits near the heterodimensional loop under periodic perturbation for a high-dimensional system. We consider the following autonomous differential equation:

$$\dot{x}(t) = f(x(t)) \tag{1.1}$$

and its periodic perturbed equation is as follows:

$$\dot{x}(t) = f(x(t)) + \sum_{j=1}^{2} \mu_j g_j(x(t), \mu, t), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ and we give the following assumptions:

(H_1) $f \in C^3$.

(H_2) The unperturbed equation (1.1) has two distinct hyperbolic equilibria P_+ and P_-. Namely, $f(P_\pm) = 0$ and the eigenvalues of $Df(P_\pm)$ lie off the imaginary axis, where D denotes the derivative operator.

(H_3) The unperturbed equation (1.1) has two heteroclinic solutions $\gamma_1(t)$ and $\gamma_2(t)$, which are asymptotic to equilibria P_+ and P_-, respectively. That is, $\dot{\gamma}_i(t) = f(\gamma_i(t))$, $i = 1, 2$ and

$$\lim_{t\to+\infty} \gamma_1(t) = P_+, \quad \lim_{t\to-\infty} \gamma_1(t) = P_-,$$
$$\lim_{t\to+\infty} \gamma_2(t) = P_-, \quad \lim_{t\to-\infty} \gamma_2(t) = P_+.$$

(H_4) $g_j \in C^3$, $g_j(P_\pm, \mu, t) = 0$, $g_j(x, 0, t) = 0$, and $g_j(x, \mu, t + 2) = g_j(x, \mu, t)$.

(H_5) $\dim(W^s(P_+)) = d_+$, $\dim(W^s(P_-)) = d_-$, where $W^s(P_+)$ and $W^s(P_-)$ are the stable manifolds of the equilibria P_+ and P_-, respectively.

(H_6)

$$\dim(T_{\gamma_1(0)} W^s(P_+) \cap T_{\gamma_1(0)} W^u(P_-)) = d_1$$

and

$$\dim(T_{\gamma_2(0)}W^s(P_-) \cap T_{\gamma_2(0)}W^u(P_+)) = d_2,$$

where $T_{\gamma_i(0)}W^{s/u}(P_\pm)$ is the tangent spaces of the corresponding invariant manifolds at $\gamma_i(0)$ and $d_i > 1$, $i = 1, 2$.

From $(H3)$, we know that the unperturbed equation (1.1) has a heteroclinic loop Γ. Refer Figure 1, where

$$\Gamma = \{P_-\} \cup \{\gamma_1(t) : t \in \mathbb{R}\} \cup \{P_+\} \cup \{\gamma_2(t) : t \in \mathbb{R}\}.$$

From $(H5)$, the hyperbolic equilibria P_+ and P_- can have different saddle point indices or Morse indices when $d_+ \neq d_-$ [14,18]. Long and Yang [25] defined the splitting index of heteroclinic orbit in the unperturbed heteroclinic chain. Analogously, we define splitting index of heteroclinic orbit γ_i as $S(\gamma_i)$, which are expressed as follows:

$$S(\gamma_1) = d_+ - d_- = s, \quad S(\gamma_2) = d_- - d_+ = -s.$$

The variational equation of (1.1) along the heteroclinic orbit γ_i is:

$$\dot{u}(t) = Df(\gamma_i(t))u(t). \tag{1.3}$$

Since $\gamma_i(t)$ is the heteroclinic solution of equation (1.1), $\gamma_i(t)$ is situated at the intersection of the stable manifold and the unstable manifold. From $(H6)$, we know equation (1.3) has d_i ($d_i > 1$) linearly independent bounded solutions, $i = 1, 2$. And the dimension of the intersection of the correspondence stable manifold and unstable manifold is d_i. If the intersection is non-transversal, the bifurcation phenomenon will occur in the original heteroclinic loop under periodic perturbation. By the definition of the splitting index of heteroclinic orbit γ_i and a simple calculation, we have the codimension of γ_i as

$$\begin{aligned}\operatorname{codim}(\gamma_i) &= n - \dim(T_{\gamma_i(0)}W^{s/u}(P_+) + T_{\gamma_i(0)}W^{u/s}(P_-)) \\ &= n - \dim(T_{\gamma_i(0)}W^{s/u}(P_+)) - \dim(T_{\gamma_i(0)}W^{u/s}(P_-)) + \dim(T_{\gamma_i(0)}W^{s/u}(P_+)) \cap (T_{\gamma_i(0)}W^{u/s}(P_-)) \\ &= d_i + (-1)^i s,\end{aligned}$$

for $i = 1, 2$. When $d_i + (-1)^i s > 0$, the intersection of the stable and unstable manifolds is non-transversal, $i = 1, 2$. The heteroclinic loop has high codimension in this situation. Therefore, under small perturbation, the heteroclinic loop is broken. So, in this study, we are mainly concerned with the heterodimensional loop bifurcation of high codimension under periodic perturbation. We will apply Lin's method to construct the periodic solution and homoclinic solution near the unperturbed heteroclinic loop.

The structure of this study is organized as follows. In Section 2, we first study the variational equation of (1.1) along the degenerate heteroclinic orbit γ_i and establish two-side exponential dichotomies. We introduce some notations and present the main result. In Section 3, we provide the proof of the main result. First, we study the existence of periodic solution near the heteroclinic loop Γ in Section 3.1. In Section 3.2, we construct the homoclinic solution near the heteroclinic loop Γ. Hence, under some conditions, periodic or homoclinic solution can bifurcated from heteroclinic loop Γ under periodic perturbation.

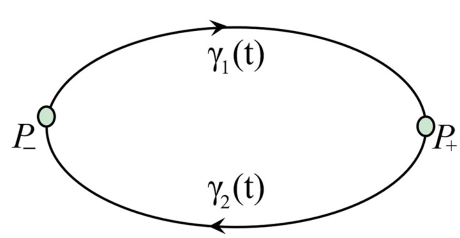

Figure 1: Heteroclinic loop Γ.

2 Preliminaries and main result

2.1 Lin's method

We give a brief description of the idea of Lin's method in this section, refer [6] for details. Lin's method is an analytical tool to deal with heteroclinic loop bifurcation.

We assume that for $\mu = 0$, a system in $\mathbb{R}^n$ has saddle points P_-, P_+ and the heteroclinic orbits are asymptotic to P_+ and P_-, respectively. The system is as follows:

$$x'(t) = f(x(t), \mu). \tag{2.1}$$

By the assumptions, we know that the system has a heteroclinic loop consisting of saddles P_-, P_+ and heteroclinic solutions $\gamma_1(t), \gamma_2(t)$.

We apply Lin's method to construct Lin orbits x_1, x_2 that are composed of piecewise continuous orbits $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}$ near the heteroclinic loop, which is characterized by a gap on $t = 0$ in a distinguished direction. When gaps ζ_i disappear, Lin orbits can glue together. So, we can obtain the existence of homoclinic or periodic orbits equivalent to the existence of the zeros of the corresponding bifurcation function. That is,

$$\begin{aligned}
&x_{i,1}' = f(x_{i,1}, \mu), \quad t \in [-T, 0],\\
&x_{i,2}' = f(x_{i,2}, \mu), \quad t \in [0, T],\\
&x_{i,1}(0), x_{i,2}(0) \in \Sigma_{P_\pm \to P_\pm},\\
&x_{1,2}(T) = x_{2,1}(-T), \quad x_{1,1}(-T) = x_{2,2}(-T),\\
&\|x_{i1}(t) - \gamma_i(t)\| < \varepsilon, \quad t \in [-T, 0],\\
&\|x_{i2}(t) - \gamma_i(t)\| < \varepsilon, \quad t \in [0, T],\\
&x_{i1}(0) - x_{i2}(0) \in R\psi_i(0).
\end{aligned}$$

There are constants $0 < \varepsilon \ll 1$, $T_0 \gg 1$, such that the boundary-value problem has a unique solution $\{x_i\}_{i=1,2}$ with $T > T_0$ and $|x_i| < \varepsilon$, and the solution is smooth. we can obtain the bifurcation function as follows:

$$\zeta_i \coloneqq \langle \psi_i(0), x_{i1}(0) - x_{i2}(0) \rangle.$$

That is, the existence of perodic orbits is equivalent to the existence of $\zeta_i \coloneqq 0$.

2.2 Exponential dichotomy

Since P_+ and P_- are hyperbolic equilibria of equation (1.1), at the same time

$$\begin{aligned}
&\lim_{t\to+\infty} Df(\gamma_1(t)) = Df(P_+), \quad \lim_{t\to-\infty} Df(\gamma_1(t)) = Df(P_-),\\
&\lim_{t\to+\infty} Df(\gamma_2(t)) = Df(P_-), \quad \lim_{t\to-\infty} Df(\gamma_2(t)) = Df(P_+).
\end{aligned}$$

By the exponential dichotomy roughness theorem, we know (1.3) has two-side exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. Most of our analysis depends on the exponential dichotomy of the linear variational equation around the heteroclinic orbit γ_i, so these basic properties are given in detail, respectively. For a detailed analysis, we refer Coppel [26] and Palmer [27]. We start with the following lemma (refer [28] for similar results in a different setting).

Lemma 2.1. *Assume that* ($H5$) *and* ($H6$) *hold. There exists a fundamental matrix solution* U_1 *for the variational equation*

$$\dot{u}(t) = Df(\gamma_1(t))u(t), \tag{2.2}$$

where constants $M > 0$, $K_0 > 0$, $n \times n$ *nonsingular matrix* C, *projections* P^1_{ss}, P^+_{us}, P^-_{su}, P^1_{uu} *with* $P^1_{ss} + P^+_{us} + P^-_{su} + P^1_{uu} = I$, I *be the* $n \times n$ *unit matrix, and the following hold:*

$$\begin{aligned}
&(i)\ |U_1(t)(P^1_{ss} + P^+_{us})U_1^{-1}(s)| \leqslant K_0 e^{2M(s-t)}, \quad 0 < s \leqslant t,\\
&(ii)\ |U_1(t)(P^-_{su} + P^1_{uu})U_1^{-1}(s)| \leqslant K_0 e^{2M(t-s)}, \quad 0 < t \leqslant s,\\
&(iii)\ |U_1(t)(P^1_{ss} + P^-_{su})U_1^{-1}(s)| \leqslant K_0 e^{2M(t-s)}, \quad t \leqslant s < 0,\\
&(iv)\ |U_1(t)(P^+_{us} + P^1_{uu})U_1^{-1}(s)| \leqslant K_0 e^{2M(s-t)}, \quad s \leqslant t < 0,\\
&(v)\ \lim_{t\to+\infty} U_1(t)(P^1_{ss} + P^+_{us})U_1^{-1}(t) = C\begin{pmatrix} I^+_s & 0\\ 0 & 0\end{pmatrix}C^{-1},\\
&(vi)\ \lim_{t\to+\infty} U_1(t)(P^-_{su} + P^1_{uu})U_1^{-1}(t) = C\begin{pmatrix} 0 & 0\\ 0 & I^1_u\end{pmatrix}C^{-1},\\
&(vii)\ \lim_{t\to-\infty} U_1(t)(P^1_{ss} + P^-_{su})U_1^{-1}(t) = C\begin{pmatrix} 0 & 0\\ 0 & I^-_u\end{pmatrix}C^{-1},\\
&(viii)\ \lim_{t\to-\infty} U_1(t)(P^+_{us} + P^1_{uu})U_1^{-1}(t) = C\begin{pmatrix} I^1_s & 0\\ 0 & 0\end{pmatrix}C^{-1},
\end{aligned} \tag{2.3}$$

where I^+_s, I^1_u, I^-_u, I^1_s *are the* $d_+ \times d_+$, $(n - d_- - s) \times (n - d_- - s)$, $(n - d_-) \times (n - d_-)$, $(d_+ - s) \times (d_+ - s)$ *unit matrixes, respectively.*

Moreover, $\operatorname{rank}(P^1_{ss}) = d_1$, $\operatorname{rank}(P^1_{uu}) = d_1 - s$.

For the variational equation

$$\dot{u}(t) = Df(\gamma_2(t))u(t), \tag{2.4}$$

we have analogous two-side exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. Without loss of generality, we take these constants to be the same as above. Then, we have the following Lemma.

Lemma 2.2. *Assume that* (*H*5) *and* (*H*6) *hold. There exists a fundamental matrix solution* U_2 *for* (2.4), *constants* $M > 0$, $K_0 > 0$, $n \times n$ *nonsingular matrix* C, *projections* P^2_{ss}, P^-_{us}, P^2_{uu}, P^-_{su} *such that* $P^2_{ss} + P^-_{us} + P^2_{uu} + P^+_{su} = I$, I *be the* $n \times n$ *unit matrix, and the following hold*:

$$\begin{aligned}
&(i)\ |U_2(t)(P^2_{ss} + P^-_{us})U_2^{-1}(s)| \leqslant K_0 e^{2M(s-t)}, \quad 0 < s \leqslant t,\\
&(ii)\ |U_2(t)(P^+_{su} + P^2_{uu})U_2^{-1}(s)| \leqslant K_0 e^{2M(t-s)}, \quad 0 < t \leqslant s,\\
&(iii)\ |U_2(t)(P^2_{ss} + P^+_{su})U_2^{-1}(s)| \leqslant K_0 e^{2M(t-s)}, \quad t \leqslant s < 0,\\
&(iv)\ |U_2(t)(P^-_{us} + P^2_{uu})U_2^{-1}(s)| \leqslant K_0 e^{2M(s-t)}, \quad s \leqslant t < 0,\\
&(v)\ \lim_{t\to\infty} U_2(t)(P^2_{ss} + P^-_{us})U_2^{-1}(t) = C\begin{pmatrix} I^-_s & 0\\ 0 & 0\end{pmatrix}C^{-1},\\
&(vi)\ \lim_{t\to\infty} U_2(t)(P^+_{su} + P^2_{uu})U_2^{-1}(t) = C\begin{pmatrix} 0 & 0\\ 0 & I^2_u\end{pmatrix}C^{-1},\\
&(vii)\ \lim_{t\to-\infty} U_2(t)(P^2_{ss} + P^+_{su})U_2^{-1}(t) = C\begin{pmatrix} 0 & 0\\ 0 & I^+_u\end{pmatrix}C^{-1},\\
&(viii)\ \lim_{t\to-\infty} U_2(t)(P^-_{us} + P^2_{uu})U_2^{-1}(t) = C\begin{pmatrix} I^2_s & 0\\ 0 & 0\end{pmatrix}C^{-1},
\end{aligned} \tag{2.5}$$

where I^-_s, I^2_u, I^+_u, I^2_s *are the* $d_- \times d_-$, $(n - d_+ + s) \times (n - d_+ + s)$, $(n - d_+) \times (n - d_+)$, $(d_- + s) \times (d_- + s)$ *unit matrixes, respectively.*

Moreover, $\operatorname{rank}(P^2_{ss}) = d_2$, $\operatorname{rank}(P^2_{uu}) = d_2 + s$.

2.3 Statement of the main result

Before presenting the main result, we introduce some notations. Let u_k^i denote the kth column of U_i. Renumbering if necessary, we can assume that

$$P_{uu}^i = \begin{pmatrix} I_{d_i+(-1)^i s} & 0 & 0 \\ 0 & 0_{d_i} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_{ss}^i = \begin{pmatrix} 0_{d_i+(-1)^i s} & 0 & 0 \\ 0 & I_{d_i} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $I_{d_i+(-1)^i s}$ and $0_{d_i+(-1)^i s}$ are $(d_i + (-1)^i s) \times (d_i + (-1)^i s)$ identity and zero matrixes, respectively, for $i = 1, 2$. Then, we have

$$P_{uu}^i U_i = (u_1^i, u_2^i, \ldots, u_{d_i+(-1)^i s}^i, 0, \ldots, 0),$$
$$P_{ss}^i U_i = (0, \ldots, 0, u_{d_i+(-1)^i s+1}, u_{d_i+(-1)^i s+2}, \ldots, u_{d_i+(-1)^i s+d_i}, 0, \ldots, 0),$$

for $i = 1, 2$.

Let U_i^{-1} denote the inverse of U_i. Then, we have $U_i^{-1}U_i = I$. Differentiating with respect to t, we obtain $\dot{U}_i^{-1}U_i + U_i^{-1}\dot{U}_i = 0$ and hence, $\dot{U}_i^{-1} = -U_i^{-1}\dot{U}_i U_i^{-1} = -U_i^{-1}Df(\gamma_i)$. Then, $(\dot{U}_i^{-1})^T = -Df(\gamma_i)^T (U_i^{-1})^T$, where T denotes the transpose of a matrix. Hence, $(U_i^{-1})^T$ is a fundamental matrix solution of the adjoint equation (2.4). Let $(u_j^i)^\perp$ denote the jth row of U_i^{-1}. $(u_j^i)^\perp$ can be obtained from the fundamental matrix solution of the adjoint equation. Clearly, $\langle (u_p^i)^\perp, u_q^i \rangle = \delta_{pq}$, the Kronecker delta. By the definition of P_{uu}^i, we have

$$P_{uu}^i U_i^{-1} = ((u_1^{i\perp})^T, (u_2^{i\perp})^T, \ldots, (u_{d_i+(-1)^i s}^{i\perp})^T, 0, \ldots, 0)^T,$$

for $i = 1, 2$.

Let

$$a_{j,k}^i(\alpha_i) = \int_{-\infty}^{+\infty} \langle (u_j^i)^\perp(s), g_k(\gamma_i(s), 0, s + \alpha_i) \rangle \mathrm{d}s$$
$$b_{j,pq}^i = \int_{-\infty}^{+\infty} \langle (u_j^i)^\perp(s), D_{11} f(\gamma_i(s)) u_p^i(s) u_q^i(s) \rangle \mathrm{d}s,$$

$i = 1, 2$, $j = 1, \ldots, d_i + (-1)^i s$, $p, q = d_i + (-1)^i s + 1, \ldots, d_i + (-1)^i s + d_i - 1$. Using those notations, we let

$$M_j^i(\beta, \mu, \alpha_i) = \sum_{k=1}^{2} a_{j,k}^i(\alpha_i)\mu_k + \frac{1}{2}\sum_{p=1}^{d_i-1}\sum_{q=1}^{d_i-1} b_{j,pq}^i \beta_p^i \beta_q^i,$$

for $i = 1, 2$, $j = 1, \ldots, d_i + (-1)^i s$, $p, q = d_i + (-1)^i s + 1, \ldots, d_i + (-1)^i s + d_i - 1$. And define $M^i(\beta, \mu, \alpha_i) : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^{d_i+(-1)^i s}$ by

$$M^i(\beta, \mu, \alpha_i) = (M_1^i(\beta, \mu, \alpha_i), \ldots, M_{d_i+(-1)^i s}^i(\beta, \mu, \alpha_i)).$$

Further, we let $M : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{d_1-s} \times \mathbb{R}^{d_2+s}$ be given by

$$M(\beta, \mu, \alpha) = (M^1(\beta, \mu, \alpha_1), M^2(\beta, \mu, \alpha_2)), \tag{2.6}$$

where $\alpha = (\alpha_1, \alpha_2)$.

Our main result can be stated as follows.

Theorem 1. *Assume that* (*H*1)–(*H*5) *hold. Let* $M(\beta, \mu, \alpha)$ *be as in* (2.6). *If there are some points* $(\beta_0, \mu_0, \alpha_0) \in \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2$, *such that*

$$M(\beta_0, \mu_0, \alpha_0) = 0$$

and

$$D_{(\beta,\mu)} M(\beta_0, \mu_0, \alpha_0)$$

is a nonsingular $(d_1 + d_2) \times (d_1 + d_2)$ *matrix, then there exists constants* $T_0 > 0$, $r_0 > 0$ *and a differentiable function* $\hat{\psi}^* : B_2(\alpha_0, r_0) \times (T_0, +\infty) \to \overline{B}_1(r_0)$, *where* $B_2(\alpha_0, r_0) \subset \mathbb{R}^2$, $\overline{B}_1(r_0) \subset \mathbb{R}^{d_1+d_2}$, *such that the perturbed equation* (1.2) *has periodic or homoclinic solution* $\gamma(t)$ *near the unperturbed heteroclinic loop* Γ *when* $\mu = \mu_0 + \hat{\psi}^*$.

The proof establishing the existence of a periodic solution bifurcating from the heteroclinic loop Γ is detailed in Section 3.1. Figure 2 illustrates the associated bifurcation phenomenon. The proof of the homoclinic solution bifurcated from heteroclinic loop Γ is given in Section 3.2. And, the Figure 3 illustrates the associated bifurcation phenomenon.

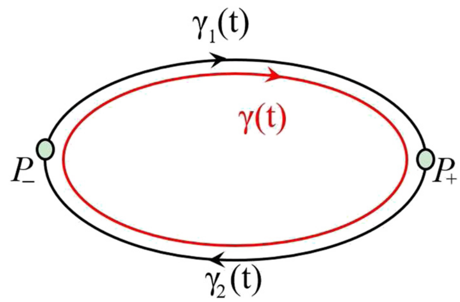

Figure 2: The periodic solution bifurcated from heteroclinic loop Γ.

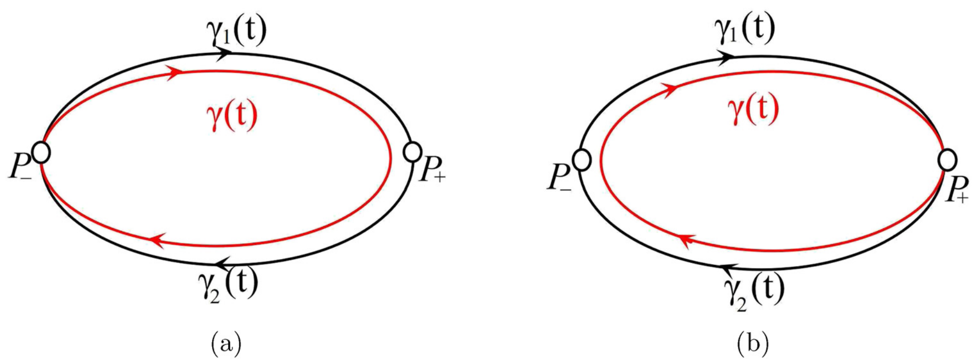

Figure 3: The homoclinic solution bifurcated from heteroclinic loop Γ. (a) Homoclinic solution $\gamma(t)$ asymptotic to hyperbolic equilibrium P_- near the unperturbed heteroclinic loop Γ and (b) homoclinic solution $\gamma(t)$ asymptotic to hyperbolic equilibrium P_+ near the unperturbed heteroclinic loop Γ.

3 Proof of Theorem 1

For the proof of the conclusion of Theorem 1, we apply Lin's method for constructing Lin orbits x_i, $i = 1, 2$ that are composed of piecewise continuous orbits $x_{i,1}$, $x_{i,2}$, $i = 1, 2$ near the heteroclinic loop, which is characterized by a gap ζ_i on $\Sigma_{P_\pm \to P_\pm}$ in a distinguished direction Z. The orbit $x_1 = \{x_{1,1}, x_{2,2}\}$ starts in $\Sigma_{P_- \to P_+}$, follows $\Sigma_{P_- \to P_+}$ until it reaches a neighborhood of P_+ follows then $\Sigma_{P_+ \to P_-}$, and $x_2 = \{x_{1,2}, x_{2,1}\}$ starts in $\Sigma_{P_+ \to P_-}$, stays further close to $\Sigma_{P_+ \to P_-}$ until it reaches a neighborhood of P_-, follows then $\Sigma_{P_- \to P_+}$ again, and terminates finally in $\Sigma_{P_- \to P_+}$. Because the periodic and homoclinic solutions are constructed in different ways, we divide the proof of Theorem 1 into two parts, which prove that the periodic solution can be bifurcated from heteroclinic loop Γ in Section 3.1 and the homoclinic solution bifurcated from heteroclinic loop Γ in Section 3.2, respectively. First, we seek the periodic solution near the unperturbed heteroclinic loop Γ.

3.1 Periodic solution bifurcated from heteroclinic loop Γ

In this section, our objective is to find periodic solution near the unperturbed heteroclinic loop Γ for equation (1.2). The Lin orbits have been glued together at $t = T$ and $t = -T$, with a gap at $t = 0$. When the gap disappears, they can be glued together to form a periodic solution near the heteroclinic loop Γ.

We define functions $b_1 : \mathbb{R}^{d_1+d_2-2} \times (0, +\infty) \to \mathbb{R}^n$ by

$$b_1(\beta, t) = -(\gamma_1(t) - \gamma_2(-t)) - (\gamma_1(-t) - \gamma_2(t)) - \left(\sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t) - \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(-t)\right) - \left(\sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(-t) - \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(t)\right),$$

and $b_2 : \mathbb{R}^{d_1+d_2-2} \times (0, +\infty) \to \mathbb{R}^n$ by

$$b_2(\beta, t) = -(\gamma_1(t) - \gamma_2(-t)) + (\gamma_1(-t) - \gamma_2(t)) - \left(\sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t) - \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(-t)\right) + \left(\sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(-t) - \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(t)\right),$$

where $\beta = (\beta_1^1, \dots, \beta_{d_1-1}^1, \beta_1^2, \dots, \beta_{d_2-1}^2)$. For $i = 1, 2$, note that

$$|b_i(\beta, t)| = O(e^{-Mt}),$$

uniformly with respect to β from any bounded subset of $\mathbb{R}^{d_1+d_2-2}$.

For $i = 1$ or $i = 2$, we suppose $x_i(t)$ is a solution of equation (1.2). Fix $T \in \mathbb{N}$ and take the change in variable

$$x_1(t + \alpha_1) = \gamma_1(t) + y_1(t) + \sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t) + \frac{1}{2} b_1(\beta, T), \tag{3.1}$$

$$x_2(t + \alpha_2) = \gamma_2(t) + y_2(t) + \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(t) + \frac{1}{2T} b_2(\beta, T)t. \tag{3.2}$$

Then, $y_i(t)$ satisfies the following equation:

$$\dot{y}_i = Df(\gamma_i)y_i + \widetilde{g}_i(y_i, \beta, \mu, \alpha_i), \tag{3.3}$$

with

$$\begin{aligned} &\widetilde{g}_1(y_1, \beta, \mu, \alpha_1, T)(t) \\ &= f(\gamma_1(t) + y_1(t) + \sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t) + \frac{1}{2} b_1(\beta, T)) - f(\gamma_1(t)) \\ &\quad - Df(\gamma_1(t))y_1(t) - \sum_{i=1}^{d_1-1} \beta_i^1 Df(\gamma_1(t)) u_{d_1-s+i}^1(t) \\ &\quad + \sum_{j=1}^{2} \mu_j g_j\left(\gamma_1(t) + y_1(t) + \sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t) + \frac{1}{2} b_1(\beta, T), \mu, t + \alpha_1\right). \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\widetilde{g}_2(y_2, \beta, \mu, \alpha_2, T)(t) \\ &= f\left(\gamma_2(t) + y_2(t) + \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(t) + \frac{1}{2T} b_2(\beta, T)t\right) - f(\gamma_2(t)) \\ &\quad - Df(\gamma_2(t))y_2(t) - \sum_{i=1}^{d_2-1} \beta_i^2 Df(\gamma_2(t)) u_{d_2+s+i}^2(t) - \frac{1}{2T} b_2(\beta, T) \\ &\quad + \sum_{j=1}^{2} \mu_j g_j\left(\gamma_2(t) + y_2(t) + \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(t) + \frac{1}{2T} b_2(\beta, T)t, \mu, t + \alpha_2\right). \end{aligned} \tag{3.5}$$

By the definition of the function $b_i(\beta, t)$, if $y_1(T) = y_2(-T)$ and $y_1(-T) = y_2(T)$, then $x_1(T + \alpha_1) = x_2(-T + \alpha_2)$ and $x_1(T + \alpha_1) = x_2(-T + \alpha_2)$. Hence, under the above change in variable, we will seek solution $y_i(t)$ of equation (3.3) satisfying $y_1(T) = y_2(-T)$ and $y_1(-T) = y_2(T)$.

Let $D_i g$ or $D_{ij} h$ denote the derivatives of a multivariate function g with respect to its ith or the i, jth variables. Based on the formula of $\widetilde{g}_i(y_i, \beta, \mu, \alpha_i, T)$, the following lemma is derived through a straightforward computation.

Lemma 3.1. *For any $i = 1$ or $i = 2$, the function $\widetilde{g}_i(y_i, \beta, \mu, \alpha_i, T)$ satisfies the following properties:*

$$(i)\quad \lim_{T\to\infty} \widetilde{g}_i(0,0,0, \alpha_i, T) = 0;\ \lim_{T\to\infty} D_1\widetilde{g}_i(0,0,0, \alpha_i, T) = 0;$$

$$(ii)\quad \lim_{T\to\infty} D_{11}\widetilde{g}_i(0,0,0, \alpha_i, T) = D_{11} f(\gamma_i);$$

$$(iii)\quad \lim_{T\to\infty} \frac{\partial \widetilde{g}_i}{\partial \mu_j}(0,0,0, \alpha_i, T)(t) = g_j(\gamma_i, 0, t + \alpha_i);$$

$$(iv)\quad \lim_{T\to\infty} \frac{\partial \widetilde{g}_i}{\partial \beta_j^i}(0,0,0, \alpha_i, T) = 0;$$

$$\lim_{T\to\infty} \frac{\partial^2 \widetilde{g}_i}{\partial \beta_j^i \beta_k^i}(0,0,0, \alpha_i, T) = D_{11} f(\gamma_i) u_j^i u_k^i.$$

Let $C^1([a, b], \mathbb{R}^n)$ be the Banach space of C^1 functions, which take $[a, b]$ into $\mathbb{R}^n$, and we equipped norm $\|z\| = \max_{t\in[a,b]}|z(t)|$, for $z \in C^1([a, b], \mathbb{R}^n)$. Assume

$$\mathbb{X}_T = C^1([-T, 0], \mathbb{R}^n), \quad \mathbb{Y}_T = C^1([0, T], \mathbb{R}^n).$$

For any $\eta_i \in \mathbb{R}^n$ and $\xi_i \in \mathbb{R}^n$, $i = 1, 2, 3, 4$, consider the functions $z_1, z_3 \in \mathbb{X}_T$ and $z_2, z_4 \in \mathbb{Y}_T$ by

$$z_1(t) = U_1(t)P_{su}^- \xi_1 + U_1(t)\int_0^t (P_{ss}^1 + P_{su}^-)U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s + U_1(t)(P_{uu}^1 + P_{us}^+)U_1^{-1}(-T)\eta_1$$
$$+ U_1(t)\int_{-T}^t (P_{uu}^1 + P_{us}^+)U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s,$$

$$z_2(t) = U_1(t)P_{us}^+ \xi_2 + U_1(t)\int_0^t (P_{ss}^1 + P_{us}^+)U_1^{-1}(s)\widetilde{g}_1(z_2, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s + U_1(t)(P_{uu}^1 + P_{su}^-)U_1^{-1}(T)\eta_2$$
$$+ U_1(t)\int_T^t (P_{uu}^1 + P_{su}^-)U_1^{-1}(s)\widetilde{g}_1(z_2, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s,$$

$$z_3(t) = U_2(t)P_{su}^+ \xi_3 + U_2(t)\int_0^t (P_{ss}^2 + P_{su}^+)U_2^{-1}(s)\widetilde{g}_2(z_3, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s + U_2(t)(P_{uu}^2 + P_{us}^-)U_2^{-1}(-T)\eta_3$$
$$+ U_2(t)\int_{-T}^t (P_{uu}^2 + P_{us}^-)U_2^{-1}(s)\widetilde{g}_2(z_3, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s,$$

$$z_4(t) = U_2(t)P_{us}^- \xi_4 + U_2(t)\int_0^t (P_{ss}^2 + P_{us}^-)U_2^{-1}(s)\widetilde{g}_2(z_4, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s + U_2(t)(P_{uu}^2 + P_{su}^+)U_2^{-1}(T)\eta_4$$
$$+ U_2(t)\int_T^t (P_{uu}^2 + P_{su}^+)U_2^{-1}(s)\widetilde{g}_2(z_4, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s.$$

Lemma 3.2. *Given η_i and ξ_i, the functions z_1, z_2, z_3, z_4 are solutions of equation* (3.3) *for* $i = 1, 2, 3, 4$.

Proof. For given η_i and ξ_i, by the definition of $z_i(t)$, if $z_i(t)$ is a continuous solution of the above integral equation, then taking derivatives with respect to t, a simple computation shows that $z_i(t)$ is a solution of equation (3.3) near the heteroclinic loop Γ, $i = 1, 2, 3, 4$.

Next we will prove that the above integral equation has a continuous solution. Using the equation z_i, we define the operator F_i by the right hand side of equation z_i. Next we will show that the operator F_i has a fixed point in the space $\mathbb{X}_T$ or $\mathbb{Y}_T$ for $i = 1, 2, 3, 4$. We only prove the situation of the operator F_1 in the space $\mathbb{X}_T$. Other proofs are similar.

Let the constants K_0 and M be given in (2.3). By Lemma 3.1 and smoothness of f and g_j, there are constants δ, δ_1, δ_2, and T_0 such that

$$\|\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)\| \leq \frac{\delta M}{2K_0}$$
$$\|\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T) - \widetilde{g}_1(\widetilde{z}_1, \beta, \mu, \alpha_1, T)\| \leq \frac{M}{2K_0}\|z_1 - \widetilde{z}_1\|$$

for $t \in [-T, 0]$, $(z_1, \widetilde{z}_1) \in \overline{B}(0, \delta)$, and $(\beta, \mu, \alpha_1, T) \in \overline{B}_1(0, \delta_1) \times \overline{B}_2(0, \delta_2) \times \mathbb{R} \times (T_0, +\infty)$, where $\overline{B}(0, \delta)$, $\overline{B}_1(0, \delta_1)$, and $\overline{B}_2(0, \delta_2)$ are closed subsets with radius $\delta > 0$, $\delta_1 > 0$, and $\delta_2 > 0$ centered at the origin of $\mathbb{X}_T$, $\mathbb{R}^{d_1+d_2-2}$ and $\mathbb{R}^2$, repetitively.

With this choice of δ and for any $\eta_1 \in \mathbb{R}^n$ and $\xi_1 \in \mathbb{R}^n$ with $\|\eta_1\| \leq \frac{\delta}{4K_0}$ and $\|\xi_1\| \leq \frac{\delta}{4K_0}$, define a Banach space

$$\mathfrak{X}_T(\xi_1, \eta_1) = \{z_1(t) | z_1(t) \in \overline{B}(0, \delta), P_{su}^{-} z_1(0) = P_{su}^{-}\xi_1, \quad \text{and} \quad (P_{uu}^1 + P_{us}^{+}) z_1(-T) = U_1(-T)(P_{uu}^1 + P_{us}^{+}) U_1^{-1}(-T)\eta_1\}$$

For any $z_1(t) \in \mathfrak{X}_T(\xi_1, \eta_1)$, define

$$F_1(z_1)(t) = U_1(t)P_{su}^{-}\xi_1 + U_1(t)\int_0^t (P_{ss}^1 + P_{su}^{-})U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s + U_1(t)(P_{uu}^1 + P_{us}^{+})U_1^{-1}(-T)\eta_1$$
$$+ U_1(t)\int_{-T}^t (P_{uu}^1 + P_{us}^{+})U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s,$$

for $t \in [-T, 0]$. It is easy to see that $P_{su}^{-}F_1(z_1)(0) = P_{su}^{-}\xi_1$ and $(P_{uu}^1 + P_{us}^{+})F_1(z_1)(-T) = U_1(-T)(P_{uu}^1 + P_{us}^{+})U_1^{-1}(-T)\eta_1$. By the exponential dichotomy of the variational equation, we obtain

$$\begin{aligned}
\|F_1(z_1)(t)\| &\leq \|U_1(t)P_{su}^{-}\xi_1\| + \|U_1(t)(P_{uu}^1 + P_{us}^{+})U_1^{-1}(-T)\eta_1\| \\
&\quad + \int_0^t \|U_1(t)(P_{ss}^1 + P_{su}^{-})U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\|\mathrm{d}s \\
&\quad + \int_{-T}^t \|U_1(t)(P_{uu}^1 + P_{us}^{+})U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\|\mathrm{d}s, \\
&\leq K_0 e^{2Mt}\|\xi_1\| + \int_0^t K_0 e^{2M(t-s)}\|\widetilde{g}_1\|\mathrm{d}s + K_0 e^{-2M(T+t)}\|\eta_1\| + \int_{-T}^t K_0 e^{2M(s-t)}\|\widetilde{g}_1\|\mathrm{d}s \\
&\leq K_0\|\xi_1\| + \frac{K_0}{2M}\|\widetilde{g}_1\| + K_0\|\eta_1\| + \frac{K_0}{2M}\|\widetilde{g}_1\| \\
&\leq \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4} = \delta,
\end{aligned} \tag{3.6}$$

for $t \in [-T, 0]$. Thus, $\|F_1(z_1)(t)\| \leq \delta$, so $F_1 : \mathfrak{X}_T(\xi_1, \eta_1) \to \mathfrak{X}_T(\xi_1, \eta_1)$.

For any $(z_1, \widetilde{z}_1) \in \mathfrak{X}_T(\xi_1, \eta_1)$, we have

$$\|F_1(z_1)(t) - F_1(\widetilde{z}_1)(t)\|$$
$$= \left\| \int_0^t U_1(t)(P_{ss}^1 + P_{su}^{-})U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s \right.$$

$$
\begin{aligned}
&- \int_0^t U_1(t)(P_{ss}^1 + P_{su}^-)U_1^{-1}(s)\widetilde{g}_1(\tilde{z}_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s \\
&+ \int_{-T}^t U_1(t)(P_{uu}^1 + P_{us}^+)U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s \\
&- \int_{-T}^t U_1(t)(P_{uu}^1 + P_{us}^+)U_1^{-1}(s)\widetilde{g}_1(\tilde{z}_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s \Bigg\| \\
\le& \int_0^t \|U_1(t)(P_{ss}^1 + P_{su}^-)U_1^{-1}(s)\|\|\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T) - \widetilde{g}_1(\tilde{z}_1, \beta, \mu, \alpha_1, T)\|\mathrm{d}s \\
&+ \int_{-T}^t \|U_1(t)(P_{uu}^1 + P_{us}^+)U_1^{-1}(s)\|\|\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T) - \widetilde{g}_1(\tilde{z}_1, \beta, \mu, \alpha_1, T)\|\mathrm{d}s \\
\le& \frac{k_0}{M}\|\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T) - \widetilde{g}_1(\tilde{z}_1, \beta, \mu, \alpha_1, T)\| \\
\le& \frac{1}{2}\|z_1 - \tilde{z}_1\|,
\end{aligned}
$$

for $t \in [-T, 0]$. Thus, F_1 is a contraction on $\mathfrak{X}_T(\xi_1, \eta_1)$ and there is a unique fixed point $z_1^*(\beta, \mu, \alpha_1, T) \in \mathfrak{X}_T(\xi_1, \eta_1)$, for $(\beta, \mu, \alpha_1, T) \in \overline{B}_1(0, \delta_1) \times \overline{B}_2(0, \delta_2) \times \mathbb{R} \times (T_0, +\infty)$. Therefore, we can prove that there exists a solution z_1^* of equation (3.3) on $[-T, 0]$. Similarly, we can prove that there exists a solution z_2^* of equation (3.3) on $[0, T]$, and a solution z_3^* of equation (3.3) on $[-T, 0]$, and a solution z_4^* of equation (3.3) on $[0, T]$.

Thus, we conclude that z_1, z_2, z_3, z_4 are said to be four piecewise continuous solutions of equation (3.3) near the heteroclinic Γ (Figure 4).

The proof is complete. □

If we can seek some η_i and ξ_i such that $z_1(0) = z_2(0)$, $z_3(0) = z_4(0)$, then $z_1(t)$ and $z_2(t)$ stick together at $t = 0$ and $z_3(t)$ and $z_4(t)$ stick together at $t = 0$. So, $z_1(t)$ and $z_2(t)$ can form a solution $y_1(t)$ of equation (3.3) in $[-T, T]$ near the heteroclinic orbit γ_1, and $z_3(t)$ and $z_4(t)$ form a solution $y_2(t)$ of equation (3.3) in $[-T, T]$ near the heteroclinic orbit γ_2. Moreover, if $z_4(T) = z_1(-T)$ and $z_2(T) = z_3(-T)$, then $y_1(t)$ and $y_2(t)$ satisfy $y_1(-T) = y_2(T)$ and $y_1(T) = y_2(-T)$. Hence, equation (3.3) has a periodic solution $y(t)$ with $4T$ period consisting of $z_1(t)$, $z_2(t)$, $z_3(t)$, and $z_4(t)$.

$$
y(\beta, \mu, \alpha, T)(t) = \begin{cases} y_1(\beta, \mu, \alpha_1, T)(t) = \begin{cases} z_1(\beta, \mu, \alpha_1, T)(t), & t \in [-T, 0], \\ z_2(\beta, \mu, \alpha_1, T)(t), & t \in [0, T], \end{cases} \\ y_2(\beta, \mu, \alpha_2, T)(t) = \begin{cases} z_3(\beta, \mu, \alpha_2, T)(t), & t \in [-T, 0], \\ z_4(\beta, \mu, \alpha_2, T)(t), & t \in [0, T]. \end{cases} \end{cases}
$$

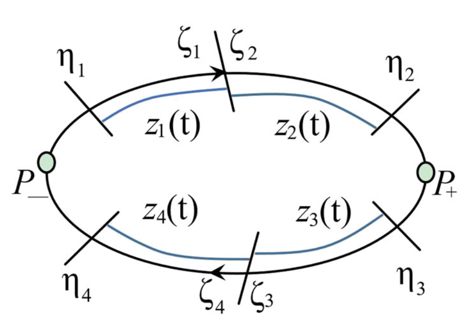

Figure 4: $z_1(t),\ldots,z_4(t)$ near the heteroclinic loop Γ.

By the definition of $b_i(\beta, t)$ and the change in variables (3.1) and (3.2), we have $x_1(T + \alpha_1) = x_2(-T + \alpha_2)$, $x_1(-T + \alpha_1) = x_2(T + \alpha_2)$. Hence, equation (1.2) has a periodic solution with $4T$ period consisting of $x_1(t)$ and $x_2(t)$ near the heteroclinic loop.

Next we will seek some η_i and ξ_i such that $z_1(0) = z_2(0)$, $z_3(0) = z_4(0)$, $z_2(T) = z_3(-T)$, and $z_4(T) = z_1(-T)$ hold. We can decompose $z_1(0) = z_2(0)$ into the following three equations:

$$P_{su}^{-}\xi_1 = P_{su}^{-}U_1^{-1}(T)\eta_2 + \int_T^0 P_{su}^{-}U_1^{-1}(s)\tilde{g}_1 \mathrm{d}s, \tag{3.7}$$

$$P_{us}^{+}\xi_2 = P_{us}^{+}U_1^{-1}(-T)\eta_1 + \int_{-T}^0 P_{us}^{+}U_1^{-1}(s)\tilde{g}_1 \mathrm{d}s, \tag{3.8}$$

$$\int_{-T}^{T} P_{uu}^{1}U_1^{-1}(s)\tilde{g}_1 \mathrm{d}s + P_{uu}^{1}U_1^{-1}(-T)\eta_1 - P_{uu}^{1}U_1^{-1}(T)\eta_2 = 0. \tag{3.9}$$

From $z_3(0) = z_4(0)$, we can obtain something similar

$$P_{su}^{+}\xi_3 = P_{su}^{+}U_2^{-1}(T)\eta_4 + \int_T^0 P_{su}^{+}U_2^{-1}(s)\tilde{g}_2 \mathrm{d}s, \tag{3.10}$$

$$P_{us}^{-}\xi_4 = P_{us}^{-}U_2^{-1}(-T)\eta_3 + \int_{-T}^0 P_{us}^{-}U_2^{-1}(s)\tilde{g}_2 \mathrm{d}s, \tag{3.11}$$

$$\int_{-T}^{T} P_{uu}^{2}U_2^{-1}(s)\tilde{g}_2 \mathrm{d}s + P_{uu}^{2}U_2^{-1}(-T)\eta_3 - P_{uu}^{2}U_2^{-1}(T)\eta_4 = 0. \tag{3.12}$$

From $z_2(T) = z_3(-T)$ and $z_4(T) = z_1(-T)$, we can obtain

$$\begin{aligned} &U_1(T)(P_{uu}^{1} + P_{su}^{-})U_1^{-1}(T)\eta_2 - U_2(-T)(P_{uu}^{2} + P_{us}^{-})U_2^{-1}(-T)\eta_3 \\ &\quad = U_2(-T)\int_0^{-T}(P_{ss}^{2} + P_{su}^{+})U_2^{-1}(s)\tilde{g}_2 \mathrm{d}s - U_1(T)\int_0^{T}(P_{ss}^{1} + P_{us}^{+})U_1^{-1}(s)\tilde{g}_1 \mathrm{d}s + U_2(-T)P_{su}^{+}\xi_3 - U_1(T)P_{us}^{+}\xi_2 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &U_2(T)(P_{uu}^{2} + P_{su}^{+})U_2^{-1}(T)\eta_4 - U_1(-T)(P_{uu}^{1} + P_{us}^{+})U_1^{-1}(-T)\eta_1 \\ &\quad = U_1(-T)\int_0^{-T}(P_{ss}^{1} + P_{su}^{-})U_1^{-1}(s)\tilde{g}_1 \mathrm{d}s - U_2(T)\int_0^{T}(P_{ss}^{2} + P_{us}^{-})U_2^{-1}(s)\tilde{g}_2 \mathrm{d}s + U_1(-T)P_{su}^{-}\xi_1 - U_2(T)P_{us}^{-}\xi_4. \end{aligned} \tag{3.14}$$

In (3.13), taking T approach infinity, we have

$$\begin{aligned} &C\begin{pmatrix} 0 & 0 \\ 0 & I_u^1 \end{pmatrix}C^{-1}\eta_2 - C\begin{pmatrix} I_s^2 & 0 \\ 0 & 0 \end{pmatrix}C^{-1}\eta_3 \\ &\quad = U_2(-\infty)\int_0^{-\infty}(P_{ss}^{2} + P_{su}^{+})U_2^{-1}(s)\tilde{g}_2 \mathrm{d}s - U_1(\infty)\int_0^{\infty}(P_{ss}^{1} + P_{us}^{+})U_1^{-1}(s)\tilde{g}_1 \mathrm{d}s, \end{aligned} \tag{3.15}$$

where (2.3) and (2.5) are used to ensure the existence of the limit, I_u^1 and I_s^2 are $(n - d_- - s) \times (n - d_- - s)$, and $(d_- + s) \times (d_- + s)$ unit matrixes, respectively. Assume

$$C^{-1}\eta_2 = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \quad C^{-1}\eta_3 = \begin{pmatrix} -u_1 \\ 0 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

From (3.15), we can solve

$$u = C^{-1}(U_2(-\infty)\int_0^{-\infty}(P_{ss}^2 + P_{su}^+)U_2^{-1}(s)\widetilde{g}_2\mathrm{d}s - U_1(\infty)\int_0^{\infty}(P_{ss}^1 + P_{us}^+)U_1^{-1}(s)\widetilde{g}_1\mathrm{d}s).$$

Then, there exists $T_0 > 0$ so that we can solve η_2 and η_3 in (3.13) whenever $T > T_0$. We denote that the solutions are $\eta_2^* = \eta_2^*(\beta, \mu, \alpha_1, T)$ and $\eta_3^* = \eta_3^*(\beta, \mu, \alpha_2, T)$. Applying the same method we can solve η_1 and η_4 in (3.14) whenever $T > T_0$. We denote that the solutions are $\eta_1^* = \eta_1^*(\beta, \mu, \alpha_1, T)$, $\eta_4^* = \eta_4^*(\beta, \mu, \alpha_2, T)$. And η_i^* satisfies $\lim_{T\to\infty} \eta_i^*(0,0, \alpha_j, T) = 0$ for $\alpha_j \in \mathbb{R}$ and $i = 1, \dots, 4, j = 1, 2$.

Substituting $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ for $\eta_1, \eta_2, \eta_3, \eta_4$ in (3.7), (3.8), (3.10), and (3.11), we can obtain $\xi_1, \xi_2, \xi_3, \xi_4$. In addition, if $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ satisfy (3.9) and (3.12), then $z_1(0) = z_2(0)$, $z_2(T) = z_3(-T)$, $z_3(0) = z_4(0)$, $z_4(T) = z_1(-T)$, that is, (3.3) has a periodic solution with $4T$ period consisting of $z_1(t), z_2(t), z_3(t)$, and $z_4(t)$. Hence, we have the following result for equation (1.2).

Lemma 3.3. *Assume $U_1, U_2, P_{uu}^1, P_{uu}^2$ be as in* (2.3) *and* (2.5). *There exists $T_0 > 0$ and if*

$$\int_{-T}^{T} P_{uu}^1 U_1^{-1}(s)\widetilde{g}_1\mathrm{d}s + P_{uu}^1 U_1^{-1}(-T)\eta_1^* - P_{uu}^1 U_1^{-1}(T)\eta_2^* = 0, \tag{3.16}$$

$$\int_{-T}^{T} P_{uu}^2 U_2^{-1}(s)\widetilde{g}_2\mathrm{d}s + P_{uu}^2 U_2^{-1}(-T)\eta_3^* - P_{uu}^2 U_2^{-1}(T)\eta_4^* = 0, \tag{3.17}$$

for $T > T_0$, then equation (1.2) *has a periodic solution with $4T$ period near the unperturbed heteroclinic loop* Γ, *where $\widetilde{g}_1$ and $\widetilde{g}_2$ be as in* (3.4) *and* (3.5).

Next we will give a sufficient condition for the existence of zeros of (3.16) and (3.17). Based on $\widetilde{g}_1$, $\widetilde{g}_2$, and η_i^*, the left hand side of (3.16) and (3.17) depend on $(\beta, \mu, \alpha_i, T)$. To simplify, we define function $\widetilde{H}^i(\beta, \mu, \alpha_i, T) : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^n$ by

$$\widetilde{H}^1(\beta, \mu, \alpha_1, T) = \int_{-T}^{T} P_{uu}^1 U_1^{-1}(s)\widetilde{g}_1\mathrm{d}s + P_{uu}^1 U_1^{-1}(-T)\eta_1^* - P_{uu}^1 U_1^{-1}(T)\eta_2^*,$$

$$\widetilde{H}^2(\beta, \mu, \alpha_1, T) = \int_{-T}^{T} P_{uu}^2 U_2^{-1}(s)\widetilde{g}_2\mathrm{d}s + P_{uu}^2 U_2^{-1}(-T)\eta_3^* - P_{uu}^2 U_2^{-1}(T)\eta_4^*.$$

By the properties of $\widetilde{g}_i$ and η_j^*, $i = 1, 2, j = 1, 2, 3, 4$, we have

$$\widetilde{H}^i(0,0, \alpha_i, T) = 0 \quad \text{and} \quad \lim_{T\to\infty} P_{uu}^i U_i^{-1}(\pm T)\eta_j^* = 0,$$

uniformly with respect to $(\beta, \mu, \alpha_i) \in \overline{B}_1(0, \delta_1) \times \overline{B}_2(0, \delta_2) \times \mathbb{R}$, for $i = 1, 2$, $j = 1, 2, 3, 4$, $\overline{B}_1(0, \delta_1)$, and $\overline{B}_2(0, \delta_2)$ are closed subsets with radius $\delta_1 > 0$ and $\delta_2 > 0$ centered at the origin of $\mathbb{R}^{d_1+d_2-2}$ and $\mathbb{R}^2$, repetitively. For $i = 1, 2$, assume

$$\widetilde{M}^i(\beta, \mu, \alpha_i) = \int_{-\infty}^{\infty} P_{uu}^i U_i^{-1}(s)\widetilde{g}_i(y_i, \beta, \mu, \alpha_i, \infty)\mathrm{d}s, \tag{3.18}$$

and by

$$P_{uu}^i U_i^{-1} = ((u_1^{i\perp})^T, (u_2^{i\perp})^T, \dots, (u_{d_i+(-1)^i s}^{i\perp})^T, 0, \dots, 0)^T,$$

hence

$$\widetilde{M}^i(\beta, \mu, \alpha_i) = (M_1^i(\beta, \mu, \alpha_i), \dots, M_{d_i+(-1)^i s}^i(\beta, \mu, \alpha_i), 0, \dots, 0), \tag{3.19}$$

where

$$M_j^i(\beta,\mu,\alpha_i) = \int_{-\infty}^{\infty} \langle (u_j^i)^{\perp}(s), \widetilde{g}_i(\gamma_i, \beta, \mu, \alpha_i, \infty)(s)\rangle \mathrm{d}s, \tag{3.20}$$

where $j = 1, 2, \ldots, d_i + (-1)^i s$. By Lemma 3.1, we have

$$M_j^i(\beta,\mu,\alpha_i) = \sum_{k=1}^{2} a_{j,k}^i(\alpha_i)\mu_k + \frac{1}{2}\sum_{p=1}^{d_i-1}\sum_{q=1}^{d_i-1} b_{j,pq}^i \beta_p^i \beta_q^i,$$

where

$$a_{j,k}^i(\alpha_i) = \int_{-\infty}^{+\infty} \langle (u_j^i)^{\perp}(s), g_k(\gamma_i(s), 0, s+\alpha_i)\rangle \mathrm{d}s$$

$$b_{j,pq}^i = \int_{-\infty}^{+\infty} \langle (u_j^i)^{\perp}(s), D_{11} f(\gamma_i(s)) u_p^i(s) u_q^i(s)\rangle \mathrm{d}s,$$

$i = 1, 2, j = 1, \ldots, d_i + (-1)^i s, p, q = d_i + (-1)^i s + 1, \ldots, d_i + (-1)^i s + d_i - 1$. Hence,

$$\widetilde{H}^i(\beta,\mu,\alpha_i,T) = \widetilde{M}^i(\beta,\mu,\alpha_i) + \text{hot},$$

for $i = 1, 2$. Define $M^i(\beta,\mu,\alpha_i) : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^{d_i+(-1)^i s}$ by

$$M^i(\beta,\mu,\alpha_i) = (M_1^i(\beta,\mu,\alpha_i), \ldots, M_{d_i+(-1)^i s}^i(\beta,\mu,\alpha_i)).$$

Let

$$H^i(\beta,\mu,\alpha_i,T) = M^i(\beta,\mu,\alpha_i) + \text{hot}.$$

It follows from the definition of the projection P_{uu}^i that

$$\widetilde{H}^i(\beta,\mu,\alpha_i,T) = (H^i(\beta,\mu,\alpha_i,T), 0, \ldots, 0).$$

Hence, $\widetilde{H}^i(\beta,\mu,\alpha_i,T) = 0$ is equivalent to $H^i(\beta,\mu,\alpha_i,T) = 0$. Define a function $H : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^{d_1+d_2}$ by

$$H(\beta,\mu,\alpha,T) = (H^1(\beta,\mu,\alpha_1,T), H^2(\beta,\mu,\alpha_2,T)),$$

where $\alpha = (\alpha_1, \alpha_2)$. So if there exists some (β,μ,α,T) such that $H(\beta,\mu,\alpha,T) = 0$, then (3.16) and (3.17) are valid. By Lemma 3.6, equation (1.2) then has a periodic solution with $4T$ period. Define a function $M : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{d_1+d_2}$ by

$$M(\beta,\mu,\alpha) = (M^1(\beta,\mu,\alpha_1), M^2(\beta,\mu,\alpha_2)),$$

where $\alpha = (\alpha_1, \alpha_2)$. So

$$H(\beta,\mu,\alpha,T) = M(\beta,\mu,\alpha) + \text{hot}. \tag{3.21}$$

Lemma 3.4. *If there exists $(\beta_0,\mu_0,\alpha_0) \in \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2$ such that $M(\beta_0,\mu_0,\alpha_0) = 0$ and $D_{(\beta,\mu)}M(\beta_0,\mu_0,\alpha_0)$ is a nonsingular $(d_1+d_2) \times (d_1+d_2)$ matrix, then there exist constants $T_0 > 0$, $r_0 > 0$, and a differentiable function $\psi^* = (\tilde{\psi}^*, \hat{\psi}^*) : B_2(\alpha_0, r_0) \times (T_0, +\infty) \to \overline{B}_1(r_0)$, where $\overline{B}_1(r_0) \subset \mathbb{R}^{d_1+d_2}$ and $B_2(\alpha_0, r_0) \subset \mathbb{R}^2$, such that $\lim_{T\to\infty}(\tilde{\psi}^*(\alpha,T), \hat{\psi}^*(\alpha,T)) = 0$ and $H(\beta_0 + \tilde{\psi}^*(\alpha,T), \mu_0 + \hat{\psi}^*(\alpha,T), \alpha, T) = 0$ for $(\alpha,T) \in B_2(\alpha_0,r_0) \times (T_0,+\infty)$.*

Proof. Let $\omega_0 = (\beta_0,\mu_0)$, $\omega = (\beta,\mu) - \omega_0$, and $\mathcal{A} := D_{(\beta,\mu)}M(\beta_0,\mu_0,\alpha_0)^{-1}$. Define

$$\begin{cases} \mathcal{M}(\omega,\alpha) = \omega - \mathcal{A}M(\omega_0 + \omega, \alpha), \\ \mathcal{H}(\omega,\alpha,T) = \omega - \mathcal{A}H(\omega_0 + \omega, \alpha, T). \end{cases} \tag{3.22}$$

By the definition of $\mathcal{H}$, we know that the fixed points ψ^* of $\mathcal{H}(\cdot, \alpha, T)$ correspond to $H(\omega_0 + \psi^*, \alpha, T) = 0$. Through direct calculations, we obtain that

$$\|\mathcal{M}(0, \alpha_0)\| = 0, \quad \|D_\omega \mathcal{M}(0, \alpha_0)\| = 0. \tag{3.23}$$

From (3.21) and (3.22), it is easy to check that

$$\begin{aligned}\lim_{T\to\infty} \|\mathcal{H}(0, \alpha_0, T) - \mathcal{M}(0, \alpha_0)\| &= \lim_{T\to\infty} \|\mathcal{A}(H(\omega_0, \alpha_0, T) - M(\omega_0, \alpha_0))\| = 0,\\ \lim_{T\to\infty} \|D_\omega\mathcal{H}(0, \alpha_0, T) - D_\omega\mathcal{M}(0, \alpha_0)\| &= \lim_{T\to\infty} \|\mathcal{A}(D_\omega H(\omega_0, \alpha_0, T) - D_\omega M(\omega_0, \alpha_0))\| = 0.\end{aligned} \tag{3.24}$$

Let $B_1(r) \subset \mathbb{R}^{d_1+d_2}$ be an open ball with radius $r > 0$ centered at the origin and $B_2(\alpha_0, r) \subset \mathbb{R}^2$ be an open ball with radius $r > 0$ centered at α_0. From (3.23), we obtain that there exists $r_0 > 0$ such that

$$\|D_\omega \mathcal{M}(\omega, \alpha)\| \leqslant \frac{1}{4} \quad \text{for } (\omega, \alpha) \in B_1(r_0) \times B_2(\alpha_0, r_0). \tag{3.25}$$

Note that $\mathcal{M}(0, \alpha_0) = 0$. We can obtain from (3.24) that there exists a constant $T_0 > 0$ such that

$$\begin{aligned}&\|\mathcal{H}(0, \alpha_0, T)\| = \|\mathcal{M}(0, \alpha_0) - \mathcal{H}(0, \alpha_0, T)\| \leqslant \frac{r_0}{2},\\ &\|D_\omega\mathcal{H}(\omega, \alpha, T) - D_\omega\mathcal{M}(\omega, \alpha)\| \leqslant \frac{1}{4},\end{aligned} \tag{3.26}$$

for $(\omega, \alpha, T) \in B_1(r_0) \times B_2(\alpha_0, r_0) \times (T_0, \infty)$. Hence, we can obtain that

$$\begin{aligned}\|D_\omega\mathcal{H}(\omega, \alpha, T)\| &= \|D_\omega\mathcal{M}(\omega, \alpha) + D_\omega\mathcal{H}(\omega, \alpha, T) - D_\omega\mathcal{M}(\omega, \alpha)\|\\ &\leqslant \|D_\omega\mathcal{M}(\omega, \alpha)\| + \|D_\omega\mathcal{H}(\omega, \alpha, T) - D_\omega\mathcal{M}(\omega, \alpha)\|\\ &\leqslant \frac{1}{2},\end{aligned} \tag{3.27}$$

for $(\omega, \alpha, T) \in B_1(r_0) \times B_2(\alpha_0, r_0) \times (T_0, \infty)$.

For any $(\omega, \alpha, T) \in B_1(r_0) \times B_2(\alpha_0, r_0) \times (T_0, \infty)$, define a map $\psi_1 : [0, 1] \to \mathbb{R}^{d_1+d_2}$ by $\psi_1(s) = \mathcal{H}(s\omega, \alpha, T)$. We obtain

$$\begin{aligned}\|\mathcal{H}(\omega, \alpha, T)\| = \|\psi_1(1)\| &= \left\| \psi_1(0) + \int_0^1 \dot\psi_1(s)\mathrm{d}s \right\|\\ &\leq \|\mathcal{H}(0, \alpha, T)\| + \int_0^1 \|D_\omega\mathcal{H}(\omega, \alpha, T)\|\|\omega\|\mathrm{d}s\\ &\leq \frac{r_0}{2} + \int_0^1 \frac{1}{2} r_0 \mathrm{d}s = r_0,\end{aligned}$$

which implies that $\mathcal{H}(\cdot, \alpha, T)$ maps $\overline{B}_1(r_0)$ into itself.

For $\omega_1, \omega_2 \in \overline{B}_1(r_0)$, $(\alpha, T) \in B_2(\alpha_0, r_0) \times (T_0, \infty)$, define a map $\psi_2 : [0, 1] \to \mathbb{R}^{d_1+d_2}$ by $\psi_2(s) = \mathcal{H}(s\omega_1 + (1 - s)\omega_2, \alpha, T)$. Clearly, $\psi_2 \in C^2$. Then, there exists $s_0 \in (0, 1)$ such that

$$\begin{aligned}&\|\mathcal{H}(\omega_1, \alpha, T) - \mathcal{H}(\omega_2, \alpha, T)\|\\ &= \|\psi_2(1) - \psi_2(0)\| = \left\| \int_0^1 \dot\psi_2(s_0)\mathrm{d}s \right\|\\ &\leqslant \|D_\omega\mathcal{H}(\omega, \alpha, T)\| \cdot \|\omega_1 - \omega_2\|\\ &\leqslant \frac{1}{2}\|\omega_1 - \omega_2\|.\end{aligned}$$

Hence, for any $(\alpha, T) \in B_2(\alpha_0, r_0) \times (T_0, \infty)$, $\mathcal{H}(\cdot, \alpha, T)$ is a uniformly contraction map in $\overline{B}_1(r_0)$. By the contraction mapping principle, there exists a C^2 function $\psi^*(\alpha, T) : B_2(\alpha_0, r_0) \times (T_0, \infty) \to \overline{B}_1(r_0)$ such that $\lim_{T\to\infty}\psi^*(\alpha, T) = 0$ and

$$\mathcal{H}(\psi^*(\alpha, T), \alpha, T) = \psi^*(\alpha, T).$$

By the definition of $\mathcal{H}$, this equality is equivalent to

$$H(\omega_0 + \psi^*(\alpha, T), \alpha, T) = H(\beta_0 + \tilde{\psi}^*(\alpha, T), \mu_0 + \hat{\psi}^*(\alpha, T), \alpha, T) = 0,$$

where $\psi^* = (\tilde{\psi}^*, \hat{\psi}^*) \in \mathbb{R}^{d_1+d_2-1} \times \mathbb{R}^2$. The proof is complete. □

By transformations (3.1) and (3.2), we know that for $(\alpha, T) \in B_2(\alpha_0, r_0) \times (T_0, \infty)$, equation (1.2) has a periodic solution $y(t)$ with period 4T near the heteroclinic loop Γ as follows:

$$y(t) = \begin{cases} x_1(t), & t \in [\alpha_{01} - T, T + \alpha_{01}], \\ x_2(t), & t \in [\alpha_{02} - T, T + \alpha_{02}], \end{cases}$$

where $x_1(t)$ and $x_2(t)$ are

$$x_1(t) = \begin{cases} z_1(\omega_0 + \psi^*, \alpha_{01}, T) + \frac{1}{2} b(\beta_0 + \tilde{\psi}^*, T) \\ \quad + \sum_{i=1}^{d_1-1} (\beta_{0,i}^1 + \tilde{\psi}_{1,i}^*) u_{d_1-s+i}^1(t - \alpha_{01}) + \gamma_1(t - \alpha_{01}), & t \in [\alpha_{01} - T, \alpha_{01}] \\ z_2(\omega_0 + \psi^*, \alpha_{01}, T) + \frac{1}{2} b(\beta_0 + \tilde{\psi}^*, T) \\ \quad + \sum_{i=1}^{d_1-1} (\beta_{0,i}^1 + \tilde{\psi}_{1,i}^*) u_{d_1-s+i}^1(t - \alpha_{01}) + \gamma_1(t - \alpha_{01}), & t \in [\alpha_{01}, T + \alpha_{01}], \end{cases}$$

$$x_2(t) = \begin{cases} z_3(\omega_0 + \psi^*, \alpha_{02}, T) + \frac{1}{2T} b(\beta_0 + \tilde{\psi}^*, T)(t - \alpha_{02}) \\ \quad + \sum_{i=1}^{d_2-1} (\beta_{0,i}^2 + \tilde{\psi}_{2,i}^*) u_{d_2+s+i}^2(t - \alpha_{02}) + \gamma_2(t - \alpha_{02}), & t \in [\alpha_{02} - T, \alpha_{02}] \\ z_4(\omega_0 + \psi^*, \alpha_{02}, T) + \frac{1}{2T} b(\beta_0 + \tilde{\psi}^*, T)(t - \alpha_{02}) \\ \quad + \sum_{i=1}^{d_2-1} (\beta_{0,i}^2 + \tilde{\psi}_{2,i}^*) u_{d_2+s+i}^2(t - \alpha_{02}) + \gamma_2(t - \alpha_{02}), & t \in [\alpha_{02}, T + \alpha_{02}], \end{cases}$$

where $\tilde{\psi}^* = (\tilde{\psi}_1^*, \tilde{\psi}_2^*) \in \mathbb{R}^{d_1-1} \times \mathbb{R}^{d_2-1}$, $\alpha_0 = (\alpha_{01}, \alpha_{02})$.

Next we seek homoclinic solution near the unperturbed heteroclinic loop Γ.

3.2 Homoclinic solution bifurcated from heteroclinic loop Γ

In this section, we consider the homoclinic solution bifurcated from heteroclinic loop Γ. The homoclinic solution can be asymptotic to hyperbolic equilibrium P_- or asymptotic to hyperbolic equilibrium P_+ near the unperturbed heteroclinic loop Γ. No matter which equilibrium, the construct method of the homoclinic solutions is similar. Hence, in these situations, one of those classes is chosen as the proof. The construction of the homoclinic solutions is just a modification of the constructed above periodic solutions. For the completeness of the study, we will give it briefly. Next we will find a homoclinic solution asymptotic to P_- of equation (1.2). On the other hand, the method can be referred in [25].

Define a function $b : \mathbb{R}^{d_1+d_2-2} \times (0, +\infty) \to \mathbb{R}^n$ by

$$b(\beta, t) = \gamma_2(-t) - \gamma_1(t) + \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(-t) - \sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t),$$

where $\beta = (\beta_1^1, \ldots, \beta_{d_1-1}^1, \beta_1^2, \ldots, \beta_{d_2-1}^2)$. Note that

$$|b(\beta, t)| = O(e^{-Mt}),$$

uniformly with respect to β from any bounded subset of $\mathbb{R}^{d_1+d_2-2}$. For $i = 1$ or $i = 2$, we suppose $x_i(t)$ is a solution of equation (1.2). Fix $T \in \mathbb{N}$ and take the change in variable

$$x_1(t+\alpha_1) = \gamma_1(t) + y_1(t) + \sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t) + \frac{1}{2} b(\beta, T), \tag{3.28}$$

$$x_2(t+\alpha_2) = \gamma_2(t) + y_2(t) + \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(t) - \frac{1}{2} b(\beta, T). \tag{3.29}$$

Then, $y_i(t)$ satisfies the following equation:

$$\dot{y}_i = Df(\gamma_i) y_i + \widetilde{g}_i(y_i, \beta, \mu, \alpha_i), \tag{3.30}$$

with

$$\begin{aligned}\widetilde{g}_1(y_1, \beta, \mu, \alpha_1, T)(t) = {} & f\Big(\gamma_1(t) + y_1(t) + \sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t) + \frac{1}{2} b(\beta, T)\Big) - f(\gamma_1(t)) - Df(\gamma_1(t)) y_1(t) \\ & - \sum_{i=1}^{d_1-1} \beta_i^1 Df(\gamma_1(t)) u_{d_1-s+i}^1(t) + \sum_{j=1}^{2} \mu_j g_j\Big(\gamma_1(t) + y_1(t) + \sum_{i=1}^{d_1-1} \beta_i^1 u_{d_1-s+i}^1(t) \\ & + \frac{1}{2} b(\beta, T), \mu, t + \alpha_1\Big)\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\widetilde{g}_2(y_2, \beta, \mu, \alpha_2, T)(t) = {} & f\Big(\gamma_2(t) + y_2(t) + \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(t) - \frac{1}{2} b(\beta, T)\Big) - f(\gamma_2(t)) \\ & - Df(\gamma_2(t)) y_2(t) - \sum_{i=1}^{d_2-1} \beta_i^2 Df(\gamma_2(t)) u_{d_2+s+i}^2(t) \\ & + \sum_{j=1}^{2} \mu_j g_j\Big(\gamma_2(t) + y_2(t) + \sum_{i=1}^{d_2-1} \beta_i^2 u_{d_2+s+i}^2(t) - \frac{1}{2} b(\beta, T), \mu, t + \alpha_2\Big).\end{aligned} \tag{3.32}$$

With regard to $\widetilde{g}_i(y_i, \beta, \mu, \alpha_i, T)$, a simple computation yields the same result as Lemma 3.1, so it is omitted.

By the definition of the function $b_i(\beta, t)$, if $y_1(T) = y_2(-T)$, then $x_1(T + \alpha_1) = x_2(-T + \alpha_2)$. From (3.28) and (3.29), if $\lim_{t\to-\infty} y_1(t) = 0$ and $\lim_{t\to+\infty} y_2(t) = 0$, then $\lim_{t\to-\infty} x_1(t + \alpha_1) = \lim_{t\to+\infty} x_2(t + \alpha_2) = P_-$. Therefore, to seek such solutions, define the following Banach spaces:

$$\mathbb{Z}_+ = \{z \in C^1(\mathbb{R}^+, \mathbb{R}^n) : \sup_{t\in\mathbb{R}^+} |z(t)| e^{Mt} < \infty\},$$
$$\mathbb{Z}_- = \{z \in C^1(\mathbb{R}^-, \mathbb{R}^n) : \sup_{t\in\mathbb{R}^-} |z(t)| e^{-Mt} < \infty\},$$

with the norm $\|z_\pm\| = \sup_{t\in\mathbb{R}^\pm} |z(t)| e^{M|t|}$, for $z_\pm \in \mathbb{Z}_\pm$. And M be the same as in (2.3) and (2.5).

For any $\eta_2, \eta_3 \in \mathbb{R}^n$ and $\xi_i \in \mathbb{R}^n$, $i = 1, 2, 3, 4$, consider the functions $z_1 \in \mathbb{Z}_-$, $z_2 \in \mathbb{Y}_T$, $z_3 \in \mathbb{X}_T$, $z_4 \in \mathbb{Z}_+$ by

$$\begin{aligned}z_1(t) = {} & U_1(t) P_{su}^- \xi_1 + U_1(t) \int_0^t (P_{ss}^1 + P_{su}^-) U_1^{-1}(s) \widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s) ds \\ & + U_1(t) \int_{-\infty}^t (P_{uu}^1 + P_{us}^+) U_1^{-1}(s) \widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s) ds,\end{aligned}$$

$$z_2(t) = U_1(t)P^+_{us}\xi_2 + U_1(t)\int_0^t (P^1_{ss} + P^+_{us})U_1^{-1}(s)\widetilde{g}_1(z_2, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s$$
$$+ U_1(t)(P^1_{uu} + P^-_{su})U_1^{-1}(T)\eta_2 + U_1(t)\int_T^t (P^1_{uu} + P^-_{su})U_1^{-1}(s)\widetilde{g}_1(z_2, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s,$$
$$z_3(t) = U_2(t)P^+_{su}\xi_3 + U_2(t)\int_0^t (P^2_{ss} + P^+_{su})U_2^{-1}(s)\widetilde{g}_2(z_3, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s$$
$$+ U_2(t)(P^2_{uu} + P^-_{us})U_2^{-1}(-T)\eta_3 + U_2(t)\int_{-T}^t (P^2_{uu} + P^-_{us})U_2^{-1}(s)\widetilde{g}_2(z_3, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s,$$
$$z_4(t) = U_2(t)P^-_{us}\xi_4 + U_2(t)\int_0^t (P^2_{ss} + P^-_{us})U_2^{-1}(s)\widetilde{g}_2(z_4, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s$$
$$+ U_2(t)\int_\infty^t (P^2_{uu} + P^+_{su})U_2^{-1}(s)\widetilde{g}_2(z_4, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s.$$

Lemma 3.5. *Given η_2, η_3, and ξ_i, the functions z_1, z_2, z_3, z_4 are solutions of equation* (3.30) *for $i = 1, 2, 3, 4$.*

We only need to show that the definitions of z_1 and z_4 are reasonable. And the others have the same proof as Lemma 3.2, so it is omitted. It is needed to verify the infinite integral is convergence in the definition of z_1 and z_4. By (2.3) and (2.5), we know

$$|U_1(t)(P^+_{us} + P^1_{uu})U_1^{-1}(s)| \leqslant K_0 e^{2M(s-t)}, \quad s \leqslant t < 0,$$
$$|U_2(t)(P^+_{su} + P^2_{uu})U_2^{-1}(s)| \leqslant K_0 e^{2M(t-s)}, \quad 0 < t \leqslant s,$$

and $\widetilde{g}_i$ is bounded in suitable regions. So the infinite integral

$$U_1(t)\int_{-\infty}^t (P^1_{uu} + P^+_{us})U_1^{-1}(s)\widetilde{g}_1(z_1, \beta, \mu, \alpha_1, T)(s)\mathrm{d}s,$$
$$U_2(t)\int_{\infty}^t (P^2_{uu} + P^+_{su})U_2^{-1}(s)\widetilde{g}_2(z_4, \beta, \mu, \alpha_2, T)(s)\mathrm{d}s,$$

are convergent. Hence the definition of z_1, z_4 are reasonable.

Here η_2, η_3, and ξ_i are arbitrary. If we can seek some η_2, η_3, and ξ_i such that $z_1(0) = z_2(0)$, $z_3(0) = z_4(0)$, then $z_1(t)$ and $z_2(t)$ stick together at $t = 0$ and form a solution $y_1(t)$ of equation (3.30) in $(-\infty, T]$ near the heteroclinic orbit γ_1, $z_3(t)$ and $z_4(t)$ stick together and form a solution $y_2(t)$ of equation (3.30) in $[-T, \infty)$ near the heteroclinic orbit γ_2. By the definition of $b_i(\beta, t)$, (3.28), and (3.29), we have $x_1(T + \alpha_1) = x_2(-T + \alpha_2)$. Hence, equation (1.2) has a homoclinic solution asymptotic to P_- consisting of $x_1(t)$ and $x_2(t)$ near the unperturbed heteroclinic loop Γ. From the above description, we know that the homoclinic solution consists of $z_1(t)$, $z_2(t)$, $z_3(t)$, and $z_4(t)$

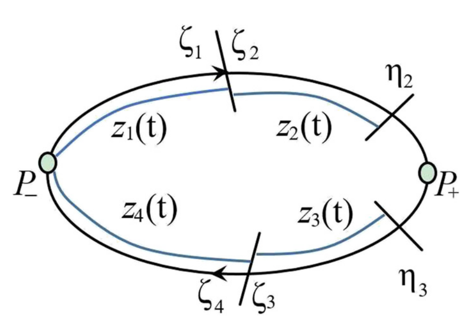

Figure 5: $Z_1(t)$,...,$Z_4(t)$ near the the heteroclinic loop Γ.

near the heteroclinic loop Γ (Figure 5). The construction of homoclinic solution asymptotic to P_+ near the unperturbed heteroclinic loop Γ is similar, so it is omitted. We can decompose $z_1(0) = z_2(0)$ into the following three equations:

$$P_{su}^{-}\xi_1 = P_{su}^{-}U_1^{-1}(T)\eta_2 - \int_0^T P_{su}^{-}U_1^{-1}(s)\widetilde{g}_1 \mathrm{d}s, \tag{3.33}$$

$$P_{us}^{+}\xi_2 = \int_{-\infty}^0 P_{us}^{+}U_1^{-1}(s)\widetilde{g}_1 \mathrm{d}s, \tag{3.34}$$

$$\int_{-\infty}^T P_{uu}^{1}U_1^{-1}(s)\widetilde{g}_1 \mathrm{d}s = P_{uu}^{1}U_1^{-1}(T)\eta_2. \tag{3.35}$$

From $z_3(0) = z_4(0)$, we can obtain something similar

$$P_{su}^{+}\xi_3 = \int_{+\infty}^0 P_{su}^{+}U_2^{-1}(s)\widetilde{g}_2 \mathrm{d}s, \tag{3.36}$$

$$P_{us}^{-}\xi_4 = P_{us}^{-}U_2^{-1}(-T)\eta_3 + \int_{-T}^0 P_{us}^{-}U_2^{-1}(s)\widetilde{g}_2 \mathrm{d}s, \tag{3.37}$$

$$P_{uu}^{2}U_2^{-1}(-T)\eta_3 + \int_{-T}^{+\infty} P_{uu}^{2}U_2^{-1}(s)\widetilde{g}_2 \mathrm{d}s = 0. \tag{3.38}$$

From $z_2(T) = z_3(-T)$, we can obtain

$$\begin{aligned} &U_1(T)(P_{uu}^1 + P_{su}^-)U_1^{-1}(T)\eta_2 - U_2(-T)(P_{uu}^2 + P_{us}^-)U_2^{-1}(-T)\eta_3 \\ =\ & U_2(-T)\int_0^{-T}(P_{ss}^2 + P_{su}^+)U_2^{-1}(s)\widetilde{g}_2 \mathrm{d}s - U_1(T)\int_0^T(P_{ss}^1 + P_{us}^+)U_1^{-1}(s)\widetilde{g}_1 \mathrm{d}s \\ &+ U_2(-T)P_{su}^+\xi_3 - U_1(T)P_{us}^+\xi_2, \end{aligned} \tag{3.39}$$

In (3.39), taking the T approach infinity, we have

$$C\begin{pmatrix} 0 & 0 \\ 0 & I_u^1 \end{pmatrix}C^{-1}\eta_2 - C\begin{pmatrix} I_s^2 & 0 \\ 0 & 0 \end{pmatrix}C^{-1}\eta_3 = U_2(-\infty)\int_0^{-\infty}(P_{ss}^2 + P_{su}^+)U_2^{-1}(s)\widetilde{g}_2 \mathrm{d}s - U_1(\infty)\int_0^{\infty}(P_{ss}^1 + P_{us}^+)U_1^{-1}(s)\widetilde{g}_1 \mathrm{d}s, \tag{3.40}$$

where (2.3) and (2.5) are used to make sure the existence of the limit, and I_u^1 and I_s^2 are $(n - d_- - s) \times (n - d_- - s)$, $(d_- + s) \times (d_- + s)$ unit matrixes, respectively. Assume

$$C^{-1}\eta_2 = \begin{pmatrix} 0 \\ k_2 \end{pmatrix}, \quad C^{-1}\eta_3 = \begin{pmatrix} -k_1 \\ 0 \end{pmatrix}, \quad k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

From (3.40), we can solve

$$k = C^{-1}\Big(U_2(-\infty)\int_0^{-\infty}(P_{ss}^2 + P_{su}^+)U_2^{-1}(s)\widetilde{g}_2 \mathrm{d}s - U_1(\infty)\int_0^{\infty}(P_{ss}^1 + P_{us}^+)U_1^{-1}(s)\widetilde{g}_1 \mathrm{d}s\Big).$$

Then, there exists $T_0 > 0$ so that we can solve η_2 and η_3 in (3.40) whenever $T > T_0$. We denote that the solutions are $\eta_2^* = \eta_2^*(\beta, \mu, \alpha_1, T)$ and $\eta_3^* = \eta_3^*(\beta, \mu, \alpha_2, T)$. And η_j^* satisfies $\lim_{T\to\infty}\eta_j^*(0, 0, \alpha_i, T) = 0$ for $\alpha_i \in \mathbb{R}$ and $i = 1, 2, j = 2, 3$.

Substituting η_1^*, η_2^* for η_1, η_2 in (3.33), (3.34), (3.36), and (3.37), we can obtain $\xi_1, \xi_2, \xi_3, \xi_4$. Furthermore if η_1^*, η_2^* satisfy (3.35) and (3.38), then $z_1(0) = z_2(0)$, $z_2(T) = z_3(-T)$, $z_3(0) = z_4(0)$, that is, (3.30) has a homoclinic solution consisting of $z_1(t)$, $z_2(t)$, $z_3(t)$, and $z_4(t)$. Hence, we have the following result for equation (1.2).

Lemma 3.6. *Assume U_1, U_2, P^1_{uu}, P^2_{uu} be as in* (2.3) *and* (2.5). *There exists T_0 and if*

$$\int_{-\infty}^{T} P^1_{uu}U_1^{-1}(s)\tilde{g}_1 ds - P^1_{uu}U_1^{-1}(T)\eta_1^* = 0, \tag{3.41}$$

$$\int_{-T}^{+\infty} P^2_{uu}U_2^{-1}(s)\tilde{g}_2 ds - P^2_{uu}U_2^{-1}(-T)\eta_2^* = 0, \tag{3.42}$$

for $T > T_0$, then equation (1.2) *has a homoclinic solution near the unperturbed heteroclinic loop* Γ, *where $\tilde{g}_1$ and $\tilde{g}_2$ are as in* (3.31) *and* (3.32).

Next, in a similar way, we give a sufficient condition for the existence of zeros of (3.41) and (3.42) and obtain the same bifurcation function as (3.21). Hence, the zeros of the bifurcation function correspond to the existence of a homoclinic solution for the perturbed equation. So, under the conditions of Lemma 3.4, we know that for $(\alpha, T) \in B_2(\alpha_0, r_0) \times (T_0, \infty)$, equation (1.2) has a homoclinic solution $\gamma(t)$ near the heteroclinic loop Γ as follows:

$$\gamma(t) = \begin{cases} x_1(t) = \begin{cases} z_1(\omega_0 + \psi^*, \alpha_{01}, T) + \sum\limits_{i=1}^{d_1-1} (\beta^1_{0,i} + \tilde{\psi}^*_{1,i}) u^1_{d_1-s+i}(t - \alpha_{01}) \\ \quad + \gamma_1(t - \alpha_{01}) + \frac{1}{2} b_1(\beta_0 + \tilde{\psi}^*, T), \quad t \in (-\infty, \alpha_{01}] \\ z_2(\omega_0 + \psi^*, \alpha_{01}, T) + \sum\limits_{i=1}^{d_1-1} (\beta^1_{0,i} + \tilde{\psi}^*_{1,i}) u^1_{d_1-s+i}(t - \alpha_{01}) \\ \quad + \gamma_1(t - \alpha_{01}) + \frac{1}{2} b_1(\beta_0 + \tilde{\psi}^*, T), \quad t \in [\alpha_{01}, T + \alpha_{01}] \end{cases} \\ x_2(t) = \begin{cases} z_3(\omega_0 + \psi^*, \alpha_{02}, T) + \sum\limits_{i=1}^{d_2-1} (\beta^2_{0,i} + \tilde{\psi}^*_{2,i}) u^2_{d_2+s+i}(t - \alpha_{02}) \\ \quad + \gamma_2(t - \alpha_{02}) - \frac{1}{2} b_2(\beta_0 + \tilde{\psi}^*, T), \quad t \in [\alpha_{02} - T, \alpha_{02}] \\ z_4(\omega_0 + \psi^*, \alpha_{02}, T) + \sum\limits_{i=1}^{d_2-1} (\beta^2_{0,i} + \tilde{\psi}^*_{2,i}) u^2_{d_2+s+i}(t - \alpha_{02}) \\ \quad + \gamma_2(t - \alpha_{02}) - \frac{1}{2} b_2(\beta_0 + \tilde{\psi}^*, T), \quad t \in [\alpha_{02}, \infty) \end{cases} \end{cases}$$

where $\tilde{\psi}^* = (\tilde{\psi}_1^*, \tilde{\psi}_2^*) \in \mathbb{R}^{d_1-1} \times \mathbb{R}^{d_2-1}$, $\alpha_0 = (\alpha_{01}, \alpha_{02})$.

In summary, we have demonstrated the perturbed equation (1.2) has periodic or homoclinic solution $\gamma(t)$ near the unperturbed heteroclinic loop Γ.

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References

[1] A. J. Homburg and B. Sandstede, *Homoclinic and heteroclinic bifurcations in vector fields*, Chapter 8 in Handbook of Dynamical Systems, vol. 3, Elsevier, Amsterdam, 2010.

[2] C. Bonatti, L. J. Daz, and M. Viana, *Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective*, Springer-Verlag, Berlin, 2004.

[3] K. H. Alfsen, and J. Frøyland, *Systematics of the Lorenz model at* $\sigma = 10$, Phys. Scr. **31** (1985), 15–20, DOI: https://doi.org/10.1088/0031-8949/31/1/003.

[4] W. Zhang, B. Krauskopf, and V. Kirk, *How to find a codimension-one heteroclinic cycle between two periodic orbits*, Discrete Contin. Dyn. Syst. **32** (2012), 2825–2851, DOI: https://doi.org/10.3934/dcds.2012.32.2825.

[5] S. N. Chow, J. K. Hale, and J. Mallet-Parret, *An example of bifurcation to homoclinic orbits*, J. Differential Equations **37** (1980), 551–573, DOI: https://doi.org/10.1016/0022-0396(80)90104-7.

[6] X. B. Lin, *Lin's method*, Scholarpedia **3** (2008), no. 9, 6972, DOI: https://dx.doi.org/10.4249/scholarpedia.6972.

[7] J. Knobloch, *Lin's method for discrete dynamical systems*, J. Difference Equ. Appl. **6** (2000), no. 5, 577–623, DOI: https://doi.org/10.1080/10236190008808247.

[8] H. P. Krishnan, *Uniqueness of rapidly oscillating periodic solutions to a singularly perturbed differential-delay equation*, Electron. J. Differential Equations 2000 (2000), no. 56, 1–18.

[9] B. E. Oldeman, R. A. Champneys, and B. Krauskopf, *Homoclinic branch switching: A numerical implementation of Lin's method*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), no. 10, 2977–2999, DOI: https://doi.org/10.1142/S0218127403008326.

[10] C. Zhu, *From homoclinics to quasi-periodic solutions for ordinary differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 5, 1091–1114, DOI: https://doi.org/10.1017/S0308210515000189.

[11] C. Zhu and B. Long, *The periodic solution bifurcated from homoclinic orbit for a coupled ordinary differential equations*, Math. Methods Appl. Sci. **40** (2017), no. 8, 2834–2846, DOI: https://doi.org/10.1002/mma.4200.

[12] C. Zhu and W. Zhang, *Multiple chaos arising from single-parametric perturbation of a degenerate homoclinic orbit*, J. Differential Equations **268** (2020), no. 10, 5672–5703, DOI: https://doi.org/10.1016/j.jde.2019.11.024.

[13] S. N. Chow, B. Deng, and D. Terman, *The bifurcation of homoclinic and periodic orbits from two heteroclinic orbits*, SIAM J. Math. Anal. **21** (1990), no. 1, 179–204, DOI: https://doi.org/10.1137/0521010.

[14] J. D. M. Rademacher, *Homoclinic orbits near heteroclinic cycles with one equilibrium and one periodic orbit*, J. Differential Equations **218** (2005), no. 2, 390–443, DOI: https://doi.org/10.1016/j.jde.2005.03.016.

[15] Y. Jin, D. Zhang, N. Wang, and D. Zhu, *Bifurcations of twisted of fine heteroclinic Loop for high-dimensional systems*, J. Appl. Anal. Comput. **13** (2023), no. 5, 2906–2921, DOI: https://doi.org/10.11948/20230052.

[16] D. M. Zhu and Y. Sun, *Homoclinic and periodic orbits arising near the heteroclinic cycle connecting saddle-focus and saddle under reversible condition*, Acta Math. Sin. (Engl. Ser.) **23** (2007), no. 8, 1495–1504, DOI: https://doi.org/10.1007/s10114-005-0746-7.

[17] I. S. Labouriau and A. A. P. Rodrigues, *Dense heteroclinic tangencies near a Bykov cycle*, J. Differential Equations **259** (2015), no. 11, 5875–5902, DOI: https://doi.org/10.1016/j.jde.2015.07.017.

[18] J. Knobloch, J. S. W. Lamb, and K. N. Webster, *Using Lin's method to solve Bykov's problems*, J. Differential Equations **257** (2014), no. 8, 2984–3047, DOI: https://doi.org/10.1016/j.jde.2014.06.006.

[19] B. Long and S. S. Xu, *Persistence of the heteroclinic loop under periodic perturbation*, Electron. Res. Arch. **31** (2023), no. 2, 1089–1105, DOI: https://doi.org/10.3934/era.2023054.

[20] F. Chen, A. Oksasoglu, and Q. Wang, *Heteroclinic tangles in time-periodic equations*, J. Differential Equations **254** (2013), no. 3, 1137–1171, DOI: https://doi.org/10.1016/j.jde.2012.10.010.

[21] I. S. Labouriau and A. A. P. Rodrigues, *Bifurcations from an attracting heteroclinic cycle under periodic forcing*, J. Differential Equations **269** (2020), no. 5, 4137–4174, DOI: https://doi.org/10.1016/j.jde.2020.03.024.

[22] I. S. Labouriau and A. A. P. Rodrigues, *Periodic forcing of a heteroclinic network*, J. Differential Equations **35** (2023), 2951–2969, DOI: https://doi.org/10.1007/s10884-021-10054-w.

[23] X. Sun, *Exact bound on the number of limit cycles arising from a periodic annulus bounded by a symmetric heteroclinic loop*, J. Appl. Anal. Comput. **10** (2020), no. 1, 378–390, DOI: https://doi.org/10.11948/20190294.

[24] Y. Xiong and G. Hu, *Heteroclinic loop bifurcations by perturbing a class of Z2-equivariant quadratic switching Hamiltonian systems with nilpotent singular points*, J. Math. Anal. Appl. **532** (2024), no. 2, 127977, DOI: https://doi.org/10.1016/j.jmaa.2023.127977.

[25] B. Long and Y. Yang, *Applying Lin's method to constructing heteroclinic orbits near the heteroclinic chain*, Math. Methods Appl. Sci. **47** (2024), no. 12, 10235–10255, DOI: https://doi.org/10.1002/mma.10118.

[26] W. Coppel, *Dichotomies and Stability Theory*, Springer-Verlag, Berlin, 1978.

[27] K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differential Equations **55** (1984), no. 2, 225–256, DOI: https://doi.org/10.1016/0022-0396(84)90082-2.

[28] M. Feckan and J. Gruendler, *Bifurcation from homoclinic to periodic solutions in singular ordinary differential equations*, J. Math. Anal. Appl. **246** (2000), no. 1, 245–264, DOI: https://doi.org/10.1006/jmaa.2000.6791.