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Research Article

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A topology related to implication and upsets on a bounded BCK-algebra

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Abstract: The main purpose of this article is to investigate a topology based on implication and upsets in a bounded BCK-algebra. First, we introduce a special kind of sets associated with implication and upsets in a bounded BCK-algebra, and some basic properties of these sets are derived. Furthermore, a topology related to implication and upsets is constructed by means of the sets. Moreover, we discuss some topological properties of the topology such as compactness and continuity. In particular, it is proved that a Glivenko BCK-algebra with the topology for implication is a left topological BCK-algebra. Also, a bounded product BCK-algebra with such topology for product is a para-topological BCK-algebra. Finally, the relationship between the topology and the quotient topology on a bounded quotient BCK-algebra is revealed.

Keywords: bounded BCK-algebra, upset, semitopological algebra

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1 Introduction

In order to model the set-theoretical difference, BCK-algebras were first introduced by Iséki [1] with a binary operation $*$ and a constant element 0 as a least element. Another motivation comes from classical and non-classical propositional calculi interest modelling logical implications. It was proved by Glivenko that a proposition is classically demonstrable if and only if double negation of this proposition is intuitionistically demonstrable. Glivenko theorems were extended by Cignoli and Torrens [2] to bounded BCK-algebras and BCK-logics. Consequently, Glivenko MTL-algebras were studied [3]. Some useful characterizations for Glivenko theorems in bounded BCK-algebras were derived. Also, it was proved that there is a one-to-one correspondence between i-filters of a bounded BCK-algebra A and i-filters of the set $\mathrm{Reg}(A)$ of all regular elements of A.

Borumand Saeid and Motamed [4] introduced and studied the set $D(F)$ of double complemented elements of a BL-algebra L, for any filter F of L, where $D(F) = \{x \in L | x^{**} \in F\}$. Zahiri and Borzooei (2014) [5] generalized the notion of $D(F)$ in a BL-algebra L and introduced the concept of $D_y(F) = \{x \in L | y^n \Delta x \in F$, for some $n \in \mathbb{N}\}$, where $y\Delta x = (y \to x)^{**}$ and $y^n \Delta x = y\Delta(y^{n-1}\Delta x)$, for an upset F of L and some $y \in L$. By taking the sets $D_y(F)$ as open sets, a topology on a BL-algebra L was constructed, and it was proved that a BL-algebra L with such topology is a semitopological BL-algebra. Following Zahiri and Borzooei [5], Holdon [6] constructed a new kind of topologies on residuated lattices. Moreover, it was stated that any divisible residuated lattice endowed with such topology forms a semitopological residuated lattice. Wu et al. [7] constructed a kind of topologies based on nuclei and upsets in residuated lattices, which is an extension of the two kinds of topologies mentioned earlier.

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Observe that the original definition of the sets $D_y(F)$ is closely related to implications and upsets either in BL-algebras or in residuated lattices. Therefore, we naturally extend the notion of the sets $D_y(F)$ to bounded BCK-algebras and try to construct a topology in a bounded BCK-algebra in a similar way.

In this article, inspired by the aforementioned works, we extend the aforementioned investigations to bounded BCK-algebras. The notion of the sets $D_a^*(F)$ is introduced in a bounded BCK-algebra A, for an upset F of A and some $a \in A$. A topology $\mathcal{T}_a$ taking the sets $D_a^*(F)$ as open sets is presented in a bounded BCK-algebra. Particularly, continuity of $\mathcal{T}_a$ is mainly investigated. Our findings indicate that some results proved by Borumand Saeid and Motamed [4], Zahiri and Borzooei [5], and Holdon [6] can also be extended to the case of bounded BCK-algebras.

This article is organized as follows: in Section 2, we recall some basic notions and results regarding BCK-algebras and topological algebras. In Section 3, the set $D_a^*(F)$ is defined in a bounded BCK-algebra A for an upset F and $a \in A$ and discuss some properties of it. Moreover, we construct a topology $\mathcal{T}_a$ on a bounded BCK-algebra A. Also, we obtain some topological properties of $(A, \mathcal{T}_a)$ such as compactness, connectedness, and continuity. In Section 4, we establish the relationship between such topology and the quotient topology on the quotient BCK-algebra. In addition, we study some properties of the direct product $D_a^*(F) \times D_a^*(G)$, for F and G are i-filters of A.

2 Preliminaries

In this section, we recall some basic definitions and results regarding BCK-algebras and semitopological algebras, which will be used in the following sections.

Definition 2.1. [2,8] A BCK-algebra is an algebra $\mathcal{A} = (A, \to, 1)$ of type (2, 0) satisfying the following, for all $x, y, z \in A$:

(A1) $x \to x = 1$;

(A2) if $x \to y = 1$ and $y \to x = 1$, then $x = y$;

(B) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;

(C) $x \to (y \to z) = y \to (x \to z)$;

(K) $x \to (y \to x) = 1$.

A bounded BCK-algebra is an algebra $\mathcal{A} = (A, \to, 0, 1)$ such that the reduct $(A, \to, 1)$ is a BCK-algebra and 0 is a constant satisfying the equation: $0 \to x = 1$. In what follows, by A, we denote the universe of a BCK-algebra $(A, \to, 1)$.

Definition 2.2. [2,8] A BCK-algebra with the product condition (product BCK-algebra) is a BCK-algebra $\mathcal{A} = (A, \to, 1)$ satisfying the following condition for all $x, y \in A$, $x \odot y$ exists, where $x \odot y = \min\{z \in A | x \le y \to z\}$.

In a product BCK-algebra A, we define: $x^0 = 1$ and $x^n = x^{n-1} \odot x$, for all $n \ge 1$.

The subsequent lemmas provide some main properties of BCK-algebras and product BCK-algebras, which will be used in the following.

Lemma 2.3. [2,8] *Let A be a BCK-algebra. Then, the following hold for all $x, y, z \in A$:*

(1) $x \le y$ *iff* $x \to y = 1$;

(2) $1 \to x = x$;

(3) $x \le y$ *implies* $z \to x \le z \to y$, $y \to z \le x \to z$;

(4) $x \to y \le (y \to z) \to (x \to z)$;

(5) $x \to y \le (z \to x) \to (z \to y)$;

(6) $x \le y \to z$ *iff* $y \le x \to z$;

(7) $x \le (x \to y) \to y$;
(8) $x \to y = ((x \to y) \to y) \to y$.

Lemma 2.4. [2,8] *Let A be a product BCK-algebra. Then, the following hold for all $x, y, z \in A$:*
(1) $x \odot y \le z$ *iff* $x \le y \to z$;
(2) $(x \odot y) \to z = x \to (y \to z)$;
(3) $x \le y$ *implies* $x \odot z \le y \odot z$;
(4) $x \odot (x \to y) \le x, y$.

A deductive system or an implicative filter (i-filter) of a BCK-algebra A is a nonempty subset F of A satisfying the following conditions: $(F1)$ $1 \in F$ and $(F2)$ $x, x \to y \in F$ implies $y \in F$, for all $x, y \in A$. An i-filter F of A is proper, provided that $F \neq A$. We denote by $\mathcal{F}(A)$ the set of i-filters of A. A proper i-filter F of A is called a maximal i-filter if it is not strictly contained in any proper i-filter of A.

For a nonempty subset X of a BCK-algebra A, we denote by $\langle X\rangle$ the i-filter generated by X, and $\langle X\rangle = \{a \in A | x_1 \to (\ldots \to (x_n \to x)\ldots) = 1,$ *for some* $x_1, \ldots, x_n \in X\}$. In particular, for $a \in A$, the i-filter generated by $\{a\}$ will be simply denoted by $\langle a\rangle$ and $\langle a\rangle = \{x \in A | a \to^n x = 1,$ for *some* $n \ge 1\}$, where $a \to^n x$ is defined inductively as $a \to^0 x = x$ and $a \to^{n+1} x = a \to (a \to^n x)$. Let F be an i-filter of a BCK-algebra A and $a \notin F$. $\langle F \cup \{a\}\rangle$ will be denoted by $F(a)$ and $F(a) = \{x \in A | a \to^n (f \to x) = 1,$ for *some* $n \ge 1$ and $f \in F\} = \{x \in A | a \to^n x \in F,$ for *some* $n \ge 1\}$. If F and G are two i-filters of a BCK-algebra A, then $F \vee G = \langle F \cup G\rangle = \{x \in A | f \to (g \to x) = 1,$ for *some* $f \in F, g \in G\}$.

It was proved by Kühr [9] and Borzooei and Bakhshi [10] that $(\mathcal{F}(A), \cap, \vee, \{1\}, A)$ is a complete Heyting algebra.

If F is an i-filter of A, then F can naturally induce a congruence relation on A as follows: $x \sim_F y \Leftrightarrow x \to y, y \to x \in F$. The equivalence class $[x]_F$ is simply denoted by $[x]$ if there is no confusion. By A/F, we denote the quotient BCK-algebra and $A/F = \{[x] | x \in A\}$.

In a bounded BCK-algebra A, we define the negation $*$: $x^* = x \to 0$, for all $x \in A$.

Lemma 2.5. [2] *Let A be a bounded BCK-algebra. Then, the following hold for all $x, y \in A$:*
(1) $x \le y$ *implies* $y^* \le x^*$ *and* $x^{**} \le y^{**}$;
(2) $x \le x^{**}, x^* = x^{***}, x^{**} = x^{****}$;
(3) $x \to y^* = y \to x^*$;
(4) $(x \to y^*)^{**} = x \to y^* = x^{**} \to y^* = (x^{**} \to y^*)^{**}$;
(5) $(x \to y^{**})^{**} = x \to y^{**} = x^{**} \to y^{**} = (x^{**} \to y^{**})^{**}$;
(6) $x \to y \le (x \to y)^{**} \le x^{**} \to y^{**}$.

For all $x, y \in A$, $x \to^n y$ is defined inductively as $x \to^0 y = y$ and $x \to^{n+1} y = x \to (x \to^n y)$. Some useful properties of $x \to^n y$ are shown in the following lemma.

Lemma 2.6. *Let A be a bounded BCK-algebra. Then, the following hold for all $x, y, z \in A$:*
(1) $x \le y$ *implies* $z \to^n x \le z \to^n y$, *for some* $n \in \mathbb{N}$;
(2) $x \le y$ *implies* $y \to^n z \le x \to^n z$, *for some* $n \in \mathbb{N}$;
(3) $x \to^m (y \to^n z) = y \to^n (x \to^m z)$, *for some* $m, n \in \mathbb{N}$;
(4) $x \to y \le (z \to^n x) \to (z \to^n y)$, *for some* $n \in \mathbb{N}$;
(5) $(x \to^n y^{**})^{**} = x \to^n y^{**} = x^{**} \to^n y^{**} = (x^{**} \to^n y^{**})^{**}$, *for some* $n \in \mathbb{N}$;
(6) $x \to^m (y \to^n z^{**})^{**} = x \to^m (y \to^n z^{**}) = y \to^n (x \to^m z^{**}) = y \to^n (x \to^m z^{**})^{**}$, *for some* $m, n \in \mathbb{N}$.

Proof.
(1) This follows from Lemma 2.3 (3) by the induction.
(2) This follows from Lemma 2.3 (3) by the induction.
(3) This follows from Definition 2.1 (C) by the induction.
(4) This follows from Lemma 2.3 (5) by the induction.

(5) This follows from Lemma 2.5 (5) by the induction.
(6) This follows from (3) and (5) by the induction. □

The following lemma provides some useful properties of the double negation in a bounded product BCK-algebra, which will be used in the third section.

Lemma 2.7. *Let A be a bounded product BCK-algebra. Then, the following hold for all* $x, y \in A$:
(1) $x^{**} \odot y^{**} \leq (x \odot y)^{**}$;
(2) $(x^{**} \odot y^{**})^{**} = (x^{**} \odot y)^{**} = (x \odot y^{**})^{**} = (x \odot y)^{**}$;
(3) $((x^{**})^2)^{**} = (x^2)^{**}$ *and so* $((x^{**})^n)^{**} = (x^n)^{**}$, *for some* $n \in \mathbb{N}$;
(4) $(x^{**})^n \to y^{**} = ((x^{**})^n)^{**} \to y^{**} = (x^n)^{**} \to y^{**} = x^n \to y^{**}$, *for some* $n \in \mathbb{N}$.

Proof. (1) By Lemmas 2.4 (1) and 2.5 (5), we have $x^{**} \odot y^{**} \leq (x \odot y)^{**}$ iff $x^{**} \leq y^{**} \to (x \odot y)^{**}$ iff $y \leq x^{**} \to (x \odot y)^{**}$ iff $y \leq x \to (x \odot y)^{**}$ iff $x \odot y \leq (x \odot y)^{**}$.

(2) We first prove that $(x^{**} \odot y^{**})^{**} = (x \odot y)^{**}$. Applying (1) and Lemma 2.5 ((1) and (2)), we have $(x^{**} \odot y^{**})^{**} \leq (x \odot y)^{****} = (x \odot y)^{**}$. From $x \leq x^{**}$ and $y \leq y^{**}$, it follows that $x \odot y \leq x^{**} \odot y^{**}$, and so by Lemma 2.5 (1), we obtain $(x \odot y)^{**} \leq (x^{**} \odot y^{**})^{**}$. Hence, $(x^{**} \odot y^{**})^{**} = (x \odot y)^{**}$.

Next, since $x \odot y \leq x^{**} \odot y \leq x^{**} \odot y^{**}$, by Lemma 2.5 (1), we obtain $(x \odot y)^{**} \leq (x^{**} \odot y)^{**} \leq (x^{**} \odot y^{**})^{**}$. So $(x \odot y)^{**} = (x^{**} \odot y)^{**} = (x^{**} \odot y^{**})^{**}$. Similarly, we can prove that $(x \odot y)^{**} = (x \odot y^{**})^{**} = (x^{**} \odot y^{**})^{**}$.

(3) This follows from (2) by the induction.

(4) By (3) and Lemma 2.5 (5), we have $(x^{**})^n \to y^{**} = ((x^{**})^n)^{**} \to y^{**} = (x^n)^{**} \to y^{**} = x^n \to y^{**}$, for some $n \in \mathbb{N}$. □

Let A be a bounded BCK-algebra. Put $\mathrm{Reg}(A) = \{x \in A | x^{**} = x\}$. According to Lemma 2.5 (5), $\mathrm{Reg}(A)$ is closed under the operation $\to$. One can check that $(\mathrm{Reg}(A), \to, 0, 1)$ is a bounded BCK-algebra. Denote $D(A) = \{x \in A | x^{**} = 1\}$ and $D(A)$ is an i-filter of A. Thus, $D(A)$ can induce a congruence relation on A as follows: $\sim_{D(A)}$, $x \sim_{D(A)} y \Leftrightarrow x \to y, y \to x \in D(A)$. By $A/D(A)$, we denote the quotient BCK-algebra.

Theorem 2.8. [2] *Let A be a bounded BCK-algebra. Then, the following are equivalent*:
(1) $\mathrm{Reg}(A/D(A)) = A/D(A)$;
(2) $(x^{**} \to x)^{**} = 1$, *for all* $x \in A$;
(3) $(x \to y)^{**} = x^{**} \to y^{**}$, *for all* $x, y \in A$;
(4) $**$ *is a homomorphism from L onto* $\mathrm{Reg}(A)$ *and* $\mathrm{Reg}(A) \cong A/D(A)$.

Definition 2.9. [2, 8] A bounded BCK-algebra is called Glivenko if it satisfies one of the equivalent conditions from Theorem 2.8.

Let A be a bounded BCK-algebra. For all $x, y \in A$, we define two operations on A: $x \tilde{\vee} y = (x \to y) \to y$ and $x \tilde{\wedge} y = (x^* \tilde{\vee} y^*)^*$. It is stated in [8] that $x \tilde{\vee} y$ is an upper bound of $\{x, y\}$ and $x \tilde{\wedge} y$ is a lower bound of $\{x^{**}, y^{**}\}$.

Definition 2.10. A bounded BCK-algebra X is called De Morgan if the following condition is satisfied for all $x, y \in X$: $(x \tilde{\vee} y)^* = x^* \tilde{\wedge} y^*$.

The following proposition establishes the relationship between Glivenko BCK-algebras and De Morgan BCK-algebras.

Proposition 2.11. *Any Glivenko BCK-algebra is a De Morgan BCK-algebra.*

Proof. Let A be a Glivenko BCK-algebra. Then, $(x \to y)^{**} = x^{**} \to y^{**}$, for all $x, y \in A$. So $(x \to y)^* = (x \to y)^{***} = (x^{**} \to y^{**})^* = (y^* \to x^*)^*$. Thus, $(x \tilde{\vee} y)^* = ((x \to y) \to y)^* = (y^* \to (x \to y)^*)^* = (y^* \to (y^* \to x^*)^*)^* = ((x^{**} \to y^{**}) \to y^{**})^* = (x^{**} \tilde{\vee} y^{**})^* = x^* \tilde{\wedge} y^*$. Therefore, A is a De Morgan BCK-algebra. □

Definition 2.12. [5,6] Let $(X, \leq)$ be an ordered set. Define $\uparrow : \mathcal{P}(X) \to \mathcal{P}(X)$, by $\uparrow Y = \{x \in X | x \geq y$, for some $y \in Y\}$, for any subset Y of X. A subset F of X is called an upset if $\uparrow F = F$. Particularly, by $\uparrow x$, we mean $\uparrow\{x\}$. We denote by $\mathcal{U}(X)$ the set of all upsets of X. An upset F is called finitely generated if there exists $n \in \mathbb{N}$ such that $F = \uparrow\{x_1, x_2, \ldots, x_n\}$, for some $x_1, x_2, \ldots, x_n \in X$.

Definition 2.13. [11,12] Let $(X, \star)$ be an algebra of type 2 and $\mathcal{T}$ be a topology on X. Then, $(X, \star, \mathcal{T})$ is called:
(1) A left (right) topological algebra, if for all $a \in X$, the map $\star : X \to X$ defined by $x \mapsto a \star x$ $(x \mapsto x \star a)$ is continuous;
(2) a semitopological algebra, or the operation $\star$ is separately continuous, if $(X, \star, \mathcal{T})$ is a left and right topological algebra;
(3) a para-$\star$-topological algebra if $\star$, as a mapping of $X \times X$ to X, is continuous when $X \times X$ is endowed with the product topology, or equivalently, if for any $x, y \in X$ and any open neighbourhood W of $x \star y$, there exist two open neighbourhoods U and V of x and y, respectively, such that $U \star V \subseteq W$.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X. We say that $\mathcal{T}'$ is finer than $\mathcal{T}$, provided that $\mathcal{T} \subseteq \mathcal{T}'$.

Theorem 2.14. [5] *Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases for two topologies $\mathcal{T}$ and $\mathcal{T}'$ on a given set X, respectively. Then, the following are equivalent*:
(1) *$\mathcal{T}'$ is finer that $\mathcal{T}$;*
(2) *for any $x \in X$ and any basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

From now on, A is always a bounded BCK-algebra in the following sections and a is an arbitrary element of A, unless otherwise specified.

3 Topology on bounded BCK-algebras

In this section, for any upset F of A, we introduce the set $D_a^*(F)$ and study some properties of it. It is shown that $\mathcal{T}_a = \{D_a^*(F) | F \in \mathcal{U}(A)\}$ forms a topology on A. We investigate some topological properties of $(A, \mathcal{T}_a)$ such as compactness and continuity. It is proved that $(A, \{\to\}, \mathcal{T}_a)$ is a left topological BCK-algebra whenever A is a Glivenko BCK-algebra. $(A, \{\odot\}, \mathcal{T}_a)$ is a para-$\odot$-topological BCK-algebra when A is a bounded product BCK-algebra.

Definition 3.1. Let F be an upset of A. Define:

$$D_a^*(F) = \{x \in A | a \to^n x^{**} \in F, \text{ for some } n \in \mathbb{N}\}.$$

Clearly, $D_a^*(\emptyset) = \emptyset$ and $D_a^*(A) = A$.

Example 3.2. Consider the bounded BCK-algebra $A = \{0, a, b, c, d, 1\}$ from [13] and the operation $\rightarrow$ on A is given in the following table:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

The double negation is shown as follows:

	0	a	b	c	d	1
**	0	a	b	c	d	1

Obviously, $F = \{c, 1\}$ is an upset. One can check that $D_a^*(F) = \{x \in A | a \rightarrow^0 x^{**} \in F\} = \{x \in A | x^{**} \in F\} = \{c, 1\}$, $U_a^*(F) = \{x \in A | a \rightarrow^1 x^{**} \in F\} = \{a, c, d, 1\}$ and $V_a^*(F) = \{x \in A | a \rightarrow^n x^{**} \in F, \textit{for } n \geq 2\} = A$.

Remark 3.3. (1) Zahiri and Borzooei [5] introduced the concept of $D_y(F)$, for any upset F of a BL-algebra L, where

$$D_y(F) = \{x \in L | y^n \Delta x \in F, \text{for some } n \in \mathbb{N}\},$$

$y\Delta x = (y \rightarrow x)^{**}$ and $y^n \Delta x = y\Delta(y^{n-1}\Delta x)$, for all $x, y \in A$.

Using the Glivenko property in the case of a BL-algebra L, it was proved that $y^n \Delta x = (y^{**})^n \rightarrow x^{**}$, for $x, y \in L$. Hence, $D_y(F) = \{x \in L | (y^{**})^n \rightarrow x^{**} \in F, \text{for some } n \in \mathbb{N}\}$. And it is shown that any BL-algebra L with such topology is a semitopological BL-algebra.

According to Lemma 2.7(4), we obtain that $(y^{**})^n \rightarrow x^{**} = y^n \rightarrow x^{**}$. Thus,

$$D_y(F) = \{x \in L | y^n \rightarrow x^{**} \in F, \text{for some } n \in \mathbb{N}\}.$$

(2) Holdon [6] generalized the aforementioned concept to residuated lattices and defined the set $D_a(X) = \{x \in L | a^n \ominus x \in X, \text{ for some } n \in \mathbb{N}\}$ in a residuated lattice L, for any upset X of L, where $a \ominus x = a^{**} \rightarrow x^{**}$ and $a^n \ominus x = a \ominus (a^{n-1} \ominus x)$.

From Lemma 2.7 (4), it follows that $D_a(X) = \{x \in L | (a^{**})^n \rightarrow x^{**} \in X, \text{for some } n \in \mathbb{N}\} = \{x \in L | a^n \rightarrow x^{**} \in X, \text{for some } n \in \mathbb{N}\}$.

(3) But in the case of a bounded BCK-algebra A, the set $D_a(F)$ introduced by Zahiri et al. is different from the set $D_a(F)$ given by Holdon. In general, the former is contained in the latter. In fact, from Lemma 2.5 (5) it follows inductively that $a^n\Delta x = a \rightarrow^{n-1} (a \rightarrow x)^{**}$, for $a, x \in A$. Applying Lemma 2.5 (6), $a \rightarrow^{n-1} (a \rightarrow x)^{**} \leq a \rightarrow^{n-1} (a^{**} \rightarrow x^{**}) = a \rightarrow^n x^{**}$. Thus, $a^n\Delta x \leq a^n \ominus x$, which means that $\{x \in A | a^n\Delta x \in F, \text{ for some } n \in \mathbb{N}\} \subseteq \{x \in A | a^n \ominus x \in F, \text{ for some } n \in \mathbb{N}\}$, for an upset F of A.

Proposition 3.4. *Let F and G be upsets of A. Then, the following hold:*

(1) *$D_a^*(F)$ is an upset of A;*
(2) *$a \in D_a^*(F)$, $F \subseteq D_a^*(F)$;*
(3) *if $F \subseteq G$, then $D_a^*(F) \subseteq D_a^*(G)$;*
(4) *$D_a^*(D_a^*(F)) = D_a^*(F)$;*
(5) *if F is an i-filter of A, then so is $D_a^*(F)$ and $F(a) \subseteq D_a^*(F)$;*

(6) $a \le b$ *implies* $D_b^*(F) \subseteq D_a^*(F)$;
(7) *if* $\{F_\alpha | \alpha \in \Gamma\}$ *is a family of upsets of* A, *then* $D_a^*(\bigcup_{\alpha\in\Gamma} F_\alpha) = \bigcup_{\alpha\in\Gamma} D_a^*(F_\alpha)$;
(8) *if* $\{F_1, F_2, \dots, F_n\}$ *is a finite set of upsets of* A, *then* $D_a^*(\bigcap_{i=1}^n F_i) = \bigcap_{i=1}^n D_a^*(F_i)$;
(9) $D_a^*(D_b^*(F)) = D_b^*(D_a^*(F))$;
(10) *if* F *and* G *are i-filters of* A, *then* $\langle a \cup F \cup G\rangle \subseteq D_a^*(F \vee G)$;
(11) *if* F *and* G *are i-filters of* A, *then* $D_a^*(F) \cap D_a^*(G) = D_a^*(F \cap G)$ *and* $D_a^*(F \vee G) = D_a^*(D_a^*(F) \vee D_a^*(G))$;
(12) $(D_a^*(\mathcal{F}(A)), \cap, \sqcup, D_a^*(\{1\}), A)$ *is a complete Heyting algebra, where* $D_a^*(F) \cap D_a^*(G) = D_a^*(F \cap G)$ *and* $\sqcup_{i\in I} D_a^*(G_i) = D_a^*(\vee_{i\in I} G_i)$, *for* $F \in \mathcal{F}(A)$ *and* $\{G_i | i \in I\} \subseteq \mathcal{F}(A)$.

Proof. (1) Assume that $t \in \uparrow D_a^*(F)$. Then, there is $x \in D_a^*(F)$ such that $x \le t$ and so $a \to^n x^{**} \in F$, for some $n \in \mathbb{N}$. From $x \le t$, by Lemma 2.5 (1), we have $x^{**} \le t^{**}$. Since F is an upset, by Lemma 2.6 (1), we obtain $a \to^n x^{**} \le a \to^n t^{**} \in F$, i.e., $t \in D_a^*(F)$. Thus, $D_a^*(F)$ is an upset of A.

(2) From $a \to a^{**} = 1 \in F$, we have $a \in D_a^*(F)$. Let $x \in F$. Since F is an upset and $x \le a \to^n x^{**}$, for any $n \in \mathbb{N}$, we have $a \to^n x^{**} \in F$ and so $x \in D_a^*(F)$. Hence, $F \subseteq D_a^*(F)$.

(3) Let $F \subseteq G$ and $x \in D_a^*(F)$. Then, there is $n \in \mathbb{N}$ such that $a \to^n x^{**} \in F \subseteq G$, so $x \in D_a^*(G)$.

(4) By (2), we have $D_a^*(F) \subseteq D_a^*(D_a^*(F))$. Let $x \in D_a^*(D_a^*(F))$. Then, there exists $n \in \mathbb{N}$ such that $a \to^n x^{**} \in D_a^*(F)$, so $a \to^m (a \to^n x^{**})^{**} \in F$, for some $m \in \mathbb{N}$. Applying Lemma 2.6 (5), $a \to^{m+n} x^{**} \in F$, i.e., $x \in D_a^*(F)$.

(5) If F is an i-filter of A, then F is a nonempty upset. Clearly, $1 \in D_a^*(F)$. Let $x, x \to y \in D_a^*(F)$. Then, there are $m, n \in \mathbb{N}$ such that $a \to^m x^{**} \in F$ and $a \to^n (x \to y)^{**} \in F$. By Lemmas 2.6 ((3) and (4)) and 2.5 (6), we obtain successively:

$$\begin{aligned}(a \to^m x^{**}) \to (a \to^{m+n} y^{**}) &= a \to^n [(a \to^m x^{**}) \to (a \to^m y^{**})]\\ &\ge a \to^n (x^{**} \to y^{**})\\ &\ge a \to^n (x \to y)^{**}.\end{aligned}$$

Since F is an i-filter of A, it follows that $a \to^{m+n} y^{**} \in F$, i.e., $y \in D_a^*(F)$. Thus, $D_a^*(F)$ is an i-filter of A. Since $a \to^n x \le a \to^n x^{**}$, for some $n \in \mathbb{N}$, we have $F(a) \subseteq D_a^*(F)$.

(6) Let $a \le b$ and $x \in D_b^*(F)$. Then, there are $n \in \mathbb{N}$ such that $b \to^n x^{**} \in F$. From $a \le b$, by Lemma 2.6 (2), we have $b \to^n x^{**} \le a \to^n x^{**} \in F$, i.e., $x \in D_a^*(F)$. Thus, $D_b^*(F) \subseteq D_a^*(F)$.

(7) For $x \in L$, we obtain successively:

$$\begin{aligned}x \in D_a^*(\bigcup_{\alpha\in\Gamma} F_\alpha) &\Leftrightarrow a \to^n x^{**} \in \bigcup_{\alpha\in\Gamma} F_\alpha, && \text{for some } n \in \mathbb{N}\\ &\Leftrightarrow a \to^n x^{**} \in F_\alpha, && \text{for some } n \in \mathbb{N} \text{ and } \alpha \in \Gamma\\ &\Leftrightarrow x \in \bigcup_{\alpha\in\Gamma} D_a^*(F_\alpha).\end{aligned}$$

Thus, $D_a^*(\bigcup_{\alpha\in\Gamma} F_\alpha) = \bigcup_{\alpha\in\Gamma} D_a^*(F_\alpha)$.

(8) By (3), $D_a^*(\bigcap_{i=1}^n F_i) \subseteq \bigcap_{i=1}^n D_a^*(F_i)$. Let $x \in \bigcap_{i=1}^n D_a^*(F_i)$. Then, there are $m_1, m_2, \dots, m_n \in \mathbb{N}$ such that $a \to^{m_i} x^{**} \in F_i$, for all $i \in \{1,2,\dots,n\}$. Put $m = \max\{m_1, m_2, \dots, m_n\}$. Then, $a \to^{m_i} x^{**} \le a \to^m x^{**}$. It follows that $a \to^m x^{**} \in F_i$, so $a \to^m x^{**} \in \bigcap_{i=1}^n F_i$, i.e., $x \in D_a^*(\bigcap_{i=1}^n F_i)$. Thus, $D_a^*(\bigcap_{i=1}^n F_i) \subseteq D_a^*(\bigcap_{i=1}^n F_i)$. Therefore, $\bigcap_{i=1}^n D_a^*(F_i) = D_a^*(\bigcap_{i=1}^n F_i)$.

(9) Let $x \in D_a^*(D_b^*(F))$. Then, there is $n \in \mathbb{N}$ such that $a \to^n x^{**} \in D_b^*(F)$ and so $b \to^m (a \to^n x^{**})^{**} \in F$, for some $m \in \mathbb{N}$. By Lemma 2.6 (6), $b \to^m (a \to^n x^{**})^{**} = a \to^n (b \to^m x^{**})^{**}$, so $a \to^n (b \to^m x^{**})^{**} \in F$. This means that $x \in D_b^*(D_a^*(F))$, so $D_a^*(D_b^*(F)) \subseteq D_b^*(D_a^*(F))$. Similarly, we can prove that $D_b^*(D_a^*(F)) \subseteq D_a^*(D_b^*(F))$. We deduce that $D_a^*(D_b^*(F)) = D_b^*(D_a^*(F))$.

(10) Let $x \in \langle a \cup F \cup G\rangle$. Then, there are $n \in \mathbb{N}$, $f \in F$ and $g \in G$ such that $a \to^n (f \to (g \to x)) = 1$. Since $F \vee G$ is an i-filter generated by $F \cup G$, we have $a \to^n x^{**} \in F \vee G$ and so $x \in D_a^*(F \vee G)$.

(11) If F and G are i-filters of A, then $F \cap G$ is an i-filter of A. From (5) and (8), it follows that $D_a^*(F) \cap D_a^*(G) = D_a^*(F \cap G)$.

Now, we prove that $D_a^*(F \vee G) = D_a^*(D_a^*(F) \vee D_a^*(G))$. Since $F, G \subseteq F \vee G$, by (3), we have $D_a^*(F), D_a^*(G) \subseteq D_a^*(F \vee G)$, so $D_a^*(F) \vee D_a^*(G) \subseteq D_a^*(F \vee G)$. Applying (3) and (4), we have $D_a^*(D_a^*(F) \vee D_a^*(G)) \subseteq D_a^*(D_a^*(F \vee G)) = D_a^*(F \vee G)$. Conversely, by (2), we have $F \subseteq D_a^*(F)$ and $G \subseteq D_a^*(G)$, so $F \vee G \subseteq D_a^*(F) \vee D_a^*(G)$. From (3), it follows that $D_a^*(F \vee G) \subseteq D_a^*(D_a^*(F) \vee D_a^*(G))$. We conclude that $D_a^*(F \vee G) = D_a^*(D_a^*(F) \vee D_a^*(G))$.

(12) Let $\{G_i | i \in I\}$ be a family of i-filters of A. Since $\mathcal{F}(A)$ is a complete Heyting algebra, by (11), we have $D_a^*(F) \cap (\sqcup_{i\in I} D_a^*(G_i)) = D_a^*(F) \cap (D_a^*(\vee_{i\in I} G_i)) = D_a^*(F \cap (\vee_{i\in I} G_i)) = D_a^*(\vee_{i\in I}(F \cap G_i)) = \sqcup_{i\in I} D_a^*(F \cap G_i) = \sqcup_{i\in I}(D_a^*(F) \cap D_a^*(G_i))$. Thus, $D_a^*(\mathcal{F}(A))$ is a complete Heyting algebra. □

Corollary 3.5. *$D_a^* : \mathcal{U}(A) \to \mathcal{U}(A)$ is a closure operator and $D_a^* = D_{a^{**}}^*$.*

Proof. From Proposition 3.4 (1), (2), (3), and (4), it follows that $D_a^* : \mathcal{U}(A) \to \mathcal{U}(A)$ is a closure operator. From Lemma 2.6 (5) it follows that $D_a^* = D_{a^{**}}^*$. □

If F is an i-filter of A and $a = 1$, then $D_1^*(F) = \{x \in A | x^{**} \in F\}$, which is just the set of double negations of a filter in a BL-algebra presented by Borumand Saeid and Motamed [4]. Next, we derive a condition such that $D_a^*(F) = D_1^*(F)$.

Proposition 3.6. *Let F be an i-filter of A. Then, $D_a^*(F) = D_1^*(F)$ if and only if $a^{**} \in F$.*

Proof. Let $D_a^*(F) = D_1^*(F)$. From Proposition 3.4 (2), it follows that $a \in D_1^*(F)$, so $a^{**} \in F$. Conversely, assume that $a^{**} \in F$. By Proposition 3.4 (6), we have $D_1^*(F) \subseteq D_a^*(F)$. Let $x \in D_a^*(F)$. Then, there exists $n \in \mathbb{N}$ such that $a \to^n x^{**} \in F$. Applying Lemma 2.6 (5), we have $a \to^n x^{**} = a^{**} \to^n x^{**} \in F$. Since F is an i-filter of A and $a^{**} \in F$, we have $x^{**} \in F$, i.e., $x \in D_1^*(F)$. Thus, $D_a^*(F) \subseteq D_1^*(F)$. We conclude that $D_a^*(F) = D_1^*(F)$. □

Let M be a proper i-filter of a bounded product BCK-algebra A. It is proved in [8] that M is maximal iff for any $x \notin M$, there is $n \geq 1$ such that $(x^n)^* \in M$. In a bounded product BCK-algebra A, for a maximal i-filter F of A, the conditions under what $D_a^*(F) = F$ and $D_a^*(F) = A$ are given in the following proposition.

Proposition 3.7. *Let A be a bounded product BCK-algebra and F be an i-filter of A. Then, the following hold*:
(1) *if F is a maximal i-filter of A, then $D_a^*(F) = F$ if and only if $a^{**} \in F$;*
(2) *$D_a^*(F) = A$ if and only if $(a^n)^* \in F$, for some $n \in \mathbb{N}$;*
(3) *if F is a maximal i-filter of A, then $D_a^*(F) = A$ if and only if $a \notin F$.*

Proof. (1) Let F be a maximal i-filter of A and $a^{**} \in F$. By Proposition 3.4 (2) and (5), we have $F \subseteq D_a^*(F)$. We now prove that $F = D_a^*(F)$. If $0 \in D_a^*(F)$, then there is $n \in \mathbb{N}$ such that $a^n \to 0 \in F$. By Lemma 2.7 (4) $a^n \to 0 = (a^{**})^n \to 0 = (a^{**})^{n-1} \to a^* \in F$. Since F is maximal and $a^{**} \in F$, we have $a^* \in F$, so $0 = a^{**} \odot a^* \in F$, which is a contradiction. Thus, $D_a^*(F) \neq A$ and so $D_a^*(F) = F$. Conversely, assume that $D_a^*(F) = F$. By Proposition 3.4 (2), we have $a \in F$. From $a \leq a^{**}$, it follows that $a^{**} \in F$.

(2) $D_a^*(F) = A$ iff $0 \in D_a^*(F)$ iff for some $n \in \mathbb{N}$ such that $a^n \to 0^{**} \in F$ iff for some $n \in \mathbb{N}$ such that $a^n \to 0 \in F$ iff for some $n \in \mathbb{N}$ such that $(a^n)^* \in F$.

(3) Let F be a maximal i-filter of A and $D_a^*(F) = A$. If $a \in F$, then from $a \leq a^{**}$, it follows $a^{**} \in F$. By (1), we have $F = D_a^*(F) = A$, which is a contradiction. Thus, $a \notin F$. Conversely, consider $a \notin F$. Since F is a maximal i-filter of A, there exists $n \in \mathbb{N}$ such that $(a^n)^* \in F$. By (2), we have $D_a^*(F) = A$. □

Example 3.8. Let $A = \{0, a, b, c, 1\}$ be a chain. The operations $\odot$, $\to$ on A are given in the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

One can check that $(A, \odot, \to, 0, 1)$ is a bounded product BCK-algebra and the double negation is given in the following table:

	0	a	b	c	1
$**$	0	a	b	1	1

Obviously, $F = \{b, c, 1\}$ is a maximal i-filter of A and $c^{**} = 1 \in F$ and $a \notin F$. Routine calculation shows that $D_c^*(F) = \{x \in A | c \to^n x^{**} \in F, n \geq 0\} = F$ and $D_a^*(F) = \{x \in A | a \to^n x^{**} \in F, n \geq 1\} = A$.

Proposition 3.9. *Let F be an i-filter of A and $a \in F$. Then, the following are equivalent*:

(1) $D_a^*(F) = F$;

(2) $x^{**} \in F$ *implies* $x \in F$.

Proof. (1) ⇒ (2) Suppose that $D_a^*(F) = F$ and $x^{**} \in F$. By Proposition 3.4 (6), we have $x \in D_1^*(F) \subseteq D_a^*(F)$, so $x \in D_a^*(F)$. By (1), we obtain $x \in F$. (2) ⇒ (1) Assume that condition (2) holds. By Proposition 3.4 (2), we have $F \subseteq D_a^*(F)$. Conversely, let $x \in D_a^*(F)$. Then, there is $n \in \mathbb{N}$ such that $a \to^n x^{**} \in F$. Since $a \in F$, we have $x^{**} \in F$. By the hypothesis, we obtain $x \in F$, so $D_a^*(F) \subseteq F$. We conclude that $D_a^*(F) = F$. □

Theorem 3.10. *$\mathcal{T}_a = \{D_a^*(F) | F \in \mathcal{U}(A)\}$ forms a topology on A, and $(A, \mathcal{T}_a)$ is a topological space.*

Proof. Obviously, $D_a^*(\varnothing) = \varnothing$ and $D_a^*(A) = A$. From Proposition 3.4 (7) and (8), it follows that the set $\mathcal{T}_a = \{D_a^*(F) | F \in \mathcal{U}(A)\}$ is a topology on A and $(A, \mathcal{T}_a)$ is a topological space. □

Example 3.11. (1) Consider the bounded BCK-algebra $A = \{0, a, b, c, 1\}$ with $0 < a < b, c < 1$ and b and c incomparable. The operation $\to$ on A is given in the following table:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

The double negation is shown in the following table:

	0	a	b	c	1
$**$	0	1	1	1	1

Clearly, $\mathcal{U}(A) = \{\varnothing, \{1\}, \uparrow b, \uparrow c, \{b, c, 1\}, \uparrow a, A\}$. Put $D_c^*(U) = \{x \in A | c \to^n x^{**} \in U, n \geq 0\}$, for any $U \in \mathcal{U}(A)$. One can easily check that $D_c^*(\{1\}) = D_c^*(\uparrow b) = D_c^*(\uparrow c) = D_c^*(\{b, c, 1\}) = D_c^*(\uparrow a) = \{a, b, c, 1\}$ and $D_c^*(A) = A$. Thus, the topology $\mathcal{T}_c = \{\varnothing, \{a, b, c, 1\}, A\}$.

(2) Let $A = \{0, a, b, c, d, e, 1\}$ be a chain from [14]. The operation $\to$ on A is given in the following table:

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	1	1	1	1	1
b	d	e	1	c	1	1	1
c	c	d	d	1	1	1	1
d	b	d	d	d	1	1	1
e	0	b	b	c	d	1	1
1	0	a	b	c	d	e	1

The double negation is shown in the following table:

	0	a	b	c	d	e	1
$**$	0	b	b	c	d	1	1

One can easily check that A is a Glivenko BCK-algebra. Set $U_1 = \varnothing$, $U_2 = \uparrow 1 = \{1\}$, $U_3 = \uparrow e = \{e, 1\}$, $U_4 = \uparrow d = \{d, e, 1\}$, $U_5 = \uparrow c = \{c, d, e, 1\}$, $U_6 = \uparrow b = \{b, c, d, e, 1\}$, $U_7 = \uparrow a = \{a, b, c, d, e, 1\}$, and $U_8 = A$. Then, $\mathcal{U}(A) = \{U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8\}$. Put $D_e^*(U) = \{x \in A | e \to^n x^{**} \in U, n \geq 0\}$, for any $U \in \mathcal{U}(A)$. One can check that $D_e^*(U_1) = U_1$, $D_e^*(U_2) = D_e^*(U_3) = U_3$, $D_e^*(U_4) = U_4$, $D_e^*(U_5) = U_5$, $D_e^*(U_6) = D_e^*(U_7) = U_7$, and $D_e^*(U_8) = U_8$. Thus, $\mathcal{T}_e = \{\varnothing, \uparrow e, \uparrow d, \uparrow c, \uparrow a, A\}$.

(3) Consider the bounded product BCK-algebra $A = \{0, a, b, c, d, 1\}$ with $0 < a < b < d < 1$, $0 < c < d < 1$ and the operations $\odot, \to$ given in the following tables:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	0	a	a
b	0	a	a	0	a	b
c	0	0	0	c	c	c
d	0	a	a	c	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	c	1	1
b	c	d	1	c	1	1
c	b	b	b	1	1	1
d	0	b	b	c	1	1
1	0	a	b	c	d	1

The double negation is shown in the following table:

	0	a	b	c	d	1
**	0	b	b	c	1	1

Set $U_1 = \emptyset$, $U_2 = \uparrow 1 = \{1\}$, $U_3 = \uparrow d = \{d, 1\}$, $U_4 = \uparrow b = \{b, d, 1\}$, $U_5 = \uparrow c = \{c, d, 1\}$, $U_6 = \uparrow a = \{a, b, d, 1\}$, $U_7 = \{b, c, d, 1\}$, $U_8 = \{a, b, c, d, 1\}$ and $U_9 = A$. Then, $\mathcal{U}(A) = \{U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9\}$. Take $D_d^*(U) = \{x \in A | d \to^n x^{**} \in U, n \geq 0\}$, for any $U \in \mathcal{U}(A)$. One can check that $D_d^*(U_1) = \emptyset$, $D_d^*(U_2) = D_d^*(U_3) = U_3$, $D_d^*(U_4) = D_d^*(U_6) = U_6$, $D_d^*(U_5) = U_5$, and $D_d^*(U_7) = D_d^*(U_8) = U_8$ and $D_d^*(A) = A$. Thus, the topology is shown as follows: $\mathscr{T}_d = \{U_1, U_3, U_5, U_6, U_8, U_9\}$, i.e.,

$$\mathscr{T}_d = \{\emptyset, \{d, 1\}, \{c, d, 1\}, \{a, b, d, 1\}, \{a, b, c, d, 1\}, A\}.$$

Proposition 3.12. *The set $\mathcal{B}_a = \{D_a^*(\uparrow x) | x \in A\}$ is a base for the topology $\mathcal{T}_a$ on A.*

Proof. Let O be an open subset of $(A, \mathcal{T}_a)$. Then, there is $U \in \mathcal{U}(A)$ such that $O = D_a^*(U)$. Since U is an upset of A, we have $U = \cup\{\uparrow x | x \in U\}$. By Proposition 3.4 (7), $D_a^*(U) = \cup\{D_a^*(\uparrow x) | x \in U\}$. Thus, $\mathcal{B}_a$ is a base for the topology $\mathcal{T}_a$ on A. □

Proposition 3.13. *If $a, b \in A$ such that $a \leq b$, then the topology $\mathcal{T}_b$ is finer than the topology $\mathcal{T}_a$.*

Proof. Let $\mathcal{B}_a$ and $\mathcal{B}_b$ be the bases for the topologies $\mathcal{T}_a$ and $\mathcal{T}_b$, respectively. If $D_a^*(\uparrow x)$ is an element of the basis $\mathcal{B}_a$ and $z \in D_a^*(\uparrow x)$, for $z \in L$, then there is $n \in \mathbb{N}$ such that $a \to^n z^{**} \in \uparrow x$, i.e., $x \leq a \to^n z^{**}$. From Proposition 3.4 (6), it follows that $D_b^*(\uparrow z) \subseteq D_a^*(\uparrow z)$. We will prove that $D_a^*(\uparrow z) \subseteq D_a^*(\uparrow x)$. Let $t \in D_a^*(\uparrow z)$. Then, there is $m \in \mathbb{N}$ such that $a \to^m t^{**} \in \uparrow z$, so $z \leq a \to^m t^{**}$. By Lemmas 2.5 (1) and 2.6 (5), we have $z^{**} \leq (a \to^m t^{**})^{**} = a \to^m t^{**}$. By Lemma 2.6 (1), $a \to^n z^{**} \leq a \to^n (a \to^m t^{**}) = a \to^{m+n} t^{**}$. Since $x \leq a \to^n z^{**}$, we obtain $x \leq a \to^{m+n} t^{**}$, i.e., $t \in D_a^*(\uparrow x)$. Hence, $D_a^*(\uparrow z) \subseteq D_a^*(\uparrow x)$ and so $D_b^*(\uparrow z) \subseteq D_a^*(\uparrow z) \subseteq D_a^*(\uparrow x)$. This shows that $z \in D_b^*(\uparrow z) \subseteq D_a^*(\uparrow x)$. By Theorem 2.14, the topology $\mathcal{T}_b$ is finer than the topology $\mathcal{T}_a$. □

Proposition 3.14. *Let F be a nonempty subset of A. Then, F is a compact subset of $(A, \mathcal{T}_a)$ if and only if $F \subseteq D_a^*(\uparrow\{x_{i_1}, x_{i_2} \dots ,x_{i_n}\})$, for some $x_{i_1}, x_{i_2} \dots ,x_{i_n} \in F$.*

Proof. Let $F \subseteq D_a^*(\uparrow\{x_{i_1}, x_{i_2} \dots ,x_{i_n}\})$, for some $x_{i_1}, x_{i_2} \dots ,x_{i_n} \in F$. If $\{D_a^*(F_\alpha) | \alpha \in \Gamma\}$ is a family of open subsets of A such that $F \subseteq \cup_{\alpha\in\Gamma} D_a^*(F_\alpha)$, then there is $k_j \in \Gamma$ such that $x_{i_j} \in D_a^*(F_{k_j})$, for $j \in \{1,2, \dots n\}$ and so $D_a^*(\uparrow x_{i_j}) \subseteq D_a^*(F_{k_j})$. Hence $F \subseteq D_a^*(\uparrow\{x_{i_1}, x_{i_2} \dots ,x_{i_n}\}) = D_a^*(\uparrow x_{i_1}) \cup D_a^*(\uparrow x_{i_2}) \cup \dots \cup D_a^*(\uparrow x_{i_n}) \subseteq D_a^*(F_{k_1}) \cup D_a^*(F_{k_2}) \cup \dots \cup D_a^*(F_{k_n})$. This means that F is a compact subset of $(A, \mathcal{T}_a)$.

Conversely, suppose that F is a compact subset of A. Since $F \subseteq \cup_{x\in F} (\uparrow x)$, by Proposition 3.4 (2) and (7) we have $F \subseteq \cup_{x\in F} D_a^*(\uparrow x)$. Hence, $\{D_a^*(\uparrow x) | x \in F\}$ is a family of open subsets of A such that $F \subseteq \cup_{x\in F} D_a^*(\uparrow x)$. By the hypothesis, there are $x_1, x_2 \dots ,x_n \in F$ such that $F \subseteq D_a^*(\uparrow x_1) \cup D_a^*(\uparrow x_2) \cup \dots \cup D_a^*(\uparrow x_n) = D_a^*(\uparrow\{x_1, x_2, \dots x_n\})$. □

Proposition 3.15. *The topological space $(A, \mathcal{T}_a)$ is connected.*

Proof. Let X be a nonempty both closed and open subset of A. Then, there exists an upset F of A such that $X = D_a^*(F)$. If $0 \in X$ and by Proposition 3.4 (1), $X = A$. Let $0 \in A \backslash X$. Since X is closed, $A \backslash X$ is open, so by Proposition 3.4(1), $A \backslash X$ is an upset. From $0 \in A \backslash X$, we have $A \backslash X = X$. Hence, $X = \emptyset$, which is a contradiction. We conclude that $\{\emptyset, A\}$ is the set of all subset of A, which are both closed and open. Thus, A is connected. □

Theorem 3.16. *Let A be a Glivenko BCK-algebra. Then, $(A, \{\to\}, \mathcal{T}_a)$ is a left topological BCK-algebra.*

Proof. Consider the map $I_b : A \to A$, defined by $I_b(x) = b \to x$, for all $x \in A$. We prove that $I_b^{-1}(D_a^*(\uparrow x)) \in \mathcal{T}_a$:

$$\begin{aligned} I_b^{-1}(D_a^*(\uparrow x)) &= \{t \in A | I_b(t) \in D_a^*(\uparrow x)\} \\ &= \{t \in A | a \to^n (b \to t)^{**} \in \uparrow x, \text{ for some } n \in \mathbb{N}\} \\ &= \{t \in A | x \leq a \to^n (b \to t)^{**}, \text{ for some } n \in \mathbb{N}\}, \end{aligned}$$

Set $B = \{t \in A | x \leq a \to^n (b \to t)^{**}, \text{ for some } n \in \mathbb{N}\}$. We will prove that B is an upset. If $t \in B$ and $t \leq u$, then there is $n \in \mathbb{N}$ such that $x \leq a \to^n (b \to t)^{**}$. By Lemmas 2.3 (3), 2.5 (1), and 2.6 (1), we have $a \to^n (b \to t)^{**} \leq a \to^n (b \to u)^{**}$ and so $x \leq a \to^n (b \to u)^{**}$, i.e., $u \in B$.

We now show that $D_a^*(B) = B$. By Proposition 3.4 (2), $B \subseteq D_a^*(B)$. Conversely, suppose that $t \in D_a^*(B)$. Then, there is $m \in \mathbb{N}$ such that $a \to^m t^{**} \in B$, so there is $n \in \mathbb{N}$ such that $x \leq a \to^n (b \to (a \to^m t^{**}))^{**}$. By Lemmas 2.6 (5) and 2.5 (5), we have $a \to^n (b \to (a \to^m t^{**}))^{**} = a \to^n (b \to (a \to^m t^{**})^{**})^{**} = a \to^n (b \to (a \to^m t^{**})^{**}) = a \to^n (b \to (a \to^m t^{**})) = b \to (a \to^n (a \to^m t^{**})) = a \to^{m+n} (b \to t^{**})$. Since A is Glivenko, we obtain $a \to^{m+n} (b \to t^{**}) = a \to^{m+n} (b \to t)^{**}$, so $x \leq a \to^{m+n} (b \to t)^{**}$, which implies that $t \in B$. Hence, $D_a^*(B) \subseteq B$, i.e., I_b a continuous map. We conclude that $(A, \{\to\}, \mathcal{T}_a)$ is a left topological BCK-algebra. □

The following theorem shows that a bounded product BCK-algebra A with the topology $\mathcal{T}_a$ such that the operation $\odot$ is continuous, so the topology $\mathcal{T}_a$ is a para-$\odot$-topology, which is similar to the result introduced by Yang et al. [15] in the context of residuated lattices.

Theorem 3.17. *Let A be a bounded product BCK-algebra. Then, $(A, \{\odot\}, \mathcal{T}_a)$ is a para-$\odot$-topological BCK-algebra.*

Proof. It is enough to prove that $D_a^*(\uparrow x) \odot D_a^*(\uparrow y) \subseteq D_a^*(\uparrow(x \odot y))$, for $x, y \in A$. Let $u \in D_a^*(\uparrow x)$ and $v \in D_a^*(\uparrow y)$. Then, there are $m, n \in \mathbb{N}$ such that $x \leq a^m \to u^{**}$ and $y \leq a^n \to v^{**}$, so $x \odot y \leq (a^m \to u^{**}) \odot (a^n \to v^{**})$. From Lemma 2.4 (4), it follows that $a^{m+n} \odot (a^m \to u^{**}) \odot (a^n \to v^{**}) = a^m \odot (a^m \to u^{**}) \odot a^n \odot (a^n \to v^{**}) \leq u^{**} \odot v^{**}$. By Lemmas 2.4 (1) and 2.7 (1), we have $(a^m \to u^{**}) \odot (a^n \to v^{**}) \leq a^{m+n} \to u^{**} \odot v^{**} \leq a^{m+n} \to (u \odot v)^{**}$. Thus, $x \odot y \leq a^{m+n} \to (u \odot v)^{**}$, which implies that $u \odot v \in D_a^*(\uparrow(x \odot y))$. Therefore, $D_a^*(\uparrow x) \odot D_a^*(\uparrow y) \subseteq D_a^*(\uparrow(x \odot y))$. □

Corollary 3.18. *$(A, \{\odot\}, \mathcal{T}_a)$ is a semitopological bounded product BCK-algebra.*

Proof. This is an immediate consequence of Theorem 3.17. □

4 On two topologies on quotient BCK-algebras

In this section, it is proved that a homomorphism between bounded BCK-algebras is continuous with respect to the topology $\mathcal{T}_a$. The relationship between the topology $\mathcal{T}_{[a]}$ and the quotient topology $\widetilde{\mathcal{T}_a}$ on a quotient BCK-algebra is revealed. Some properties of the direct product $D_a^*(F) \times D_a^*(G)$ are shown, for i-filters F and G of a bounded BCK-algebra A.

Proposition 4.1. *Let A and B be bounded BCK-algebras. If $f : A \to B$ is a homomorphism and F is an i-filter of B, then,*

(1) $f^{-1}(D_{f(a)}^*(F)) = D_a^*(f^{-1}(F))$;

(2) $D_a^*(Ker(f)) = f^{-1}(D_{f(a)}^*(\{1\}))$;

(3) *f is a continuous map from $(A, \mathcal{T}_a)$ to $(B, \mathcal{T}_{f(a)})$.*

Proof. (1) Since f is a homomorphism, $f^{-1}(F)$ is an i-filter of A and

$$\begin{aligned} f^{-1}(D^*_{f(a)}(F)) &= \{x \in A | f(x) \in D^*_{f(a)}(F)\},\\ &= \{x \in A | f(a) \to^n (f(x))^{**} \in F, \text{for some } n \in \mathbb{N}\},\\ &= \{x \in A | f(a) \to^n f(x^{**}) \in F, \text{for some } n \in \mathbb{N}\},\\ &= \{x \in A | f(a \to^n x^{**}) \in F, \text{for some } n \in \mathbb{N}\},\\ &= \{x \in A | a \to^n x^{**} \in f^{-1}(F), \text{for some } n \in \mathbb{N}\},\\ &= D^*_a(f^{-1}(F)). \end{aligned}$$

(2) For $x \in A$, we have

$$\begin{aligned} x \in D^*_a(\mathrm{Ker}(f)) &\Leftrightarrow a \to^n x^{**} \in \mathrm{Ker}(f), \text{for some } n \in \mathbb{N},\\ &\Leftrightarrow f(a \to^n x^{**}) = 1, \text{for some } n \in \mathbb{N},\\ &\Leftrightarrow f(a) \to^n f(x^{**}) = 1, \text{for some } n \in \mathbb{N},\\ &\Leftrightarrow f(x) \in D^*_{f(a)}(\{1\}). \end{aligned}$$

(3) By (1), we have $f^{-1}(D^*_{f(a)}(F)) = D^*_a(f^{-1}(F))$, so $f^{-1}(D^*_{f(a)}(F)) \in \mathcal{T}_a$. Thus, f is a continuous map from $(A, \mathcal{T}_a)$ to $(B, \mathcal{T}_{f(a)})$. □

Let $F \in \mathcal{F}(A)$. If $\pi : A \to A/F$ is the canonical homomorphism defined by $\pi(x) = [x]$, then π is a continuous map. It is clear that $(A/F, \mathcal{T}_{[a]})$ is a topological space, where $\mathcal{T}_{[a]} = \{D^*_{[a]}(U/F) | F \subseteq U \in \mathcal{U}(A)\}$ is a topology on A/F and the set $\{D^*_{[a]}(\uparrow[x]) | x \in A\}$ is a base for the topology $\mathcal{T}_{[a]}$ on A/F. Moreover, the set $\widetilde{\mathcal{T}}_a = \{[D^*_a(U)] | U \in \mathcal{U}(A)\}$ is the quotient topology on A/F and $(A/F, \widetilde{\mathcal{T}}_a)$ is the quotient topological space, where the set $\{[D^*_a(\uparrow x)] | x \in A\}$ is a base for the topology $\widetilde{\mathcal{T}}_a$.

Proposition 4.2. *Let F be an i-filter of A. Then,*
(1) $[D^*_a(\uparrow x)] \subseteq D^*_{[a]}(\uparrow[x])$, *for all* $x \in A$;
(2) *the quotient topology* $\widetilde{\mathcal{T}}_a$ *is finer than the topology* $\mathcal{T}_{[a]}$.

Proof. (1) Let $[y] \in [D^*_a(\uparrow x)] = \{[z] | z \in D^*_a(\uparrow x)\}$, for any $y \in A$. Then, there is $z \in D^*_a(\uparrow x)$ such that $[y] = [z]$, so there is $n \in \mathbb{N}$ such that $x \to (a \to^n z^{**}) = 1$. This implies that $[x] \to ([a] \to^n [z]^{**}) = [1]$. From $[y] = [z]$, we have $[x] \to ([a] \to^n [y]^{**}) = [1]$. Thus, $[y] \in D^*_{[a]}(\uparrow[x])$.

(2) According to Proposition 3.12, the set $\{D^*_{[a]}(\uparrow[x]) | [x] \in A/F\}$ is the base for the topology $\mathcal{T}_{[a]}$ on A/F. Moreover, $\{[D^*_a(\uparrow x)] | x \in A\}$ is the base for the topology $\widetilde{\mathcal{T}}_a$. Thus, (2) is a direct consequence of (1). □

Proposition 4.3. *Let A be a Glivenko BCK-algebra and F be an i-filter of A. Then,*
(1) $[D^*_a(\uparrow x)] = D^*_{[a]}(\uparrow[x])$, *for all* $x \in A$;
(2) $\mathcal{T}_{[a]} = \widetilde{\mathcal{T}}_a$.

Proof. (1) According to Proposition 4.2, we have $[D^*_a(\uparrow x)] \subseteq D^*_{[a]}(\uparrow[x])$. Now, we prove that $D^*_{[a]}(\uparrow[x]) \subseteq [D^*_a(\uparrow x)]$. Let $[y] \in D^*_{[a]}(\uparrow[x])$, for any $y \in A$. Since

$$\begin{aligned} D^*_{[a]}(\uparrow[x]) &= \{[y] \in A/F | [x] \le [a] \to^n [y]^{**}, \text{for some } n \in \mathbb{N}\},\\ &= \{[y] \in A/F | [x] \le [a] \to^n [y^{**}], \text{for some } n \in \mathbb{N}\},\\ &= \{[y] \in A/F | x \to (a \to^n y^{**}) \in F, \text{for some } n \in \mathbb{N}\}, \end{aligned}$$

then $x \to (a \to^n y^{**}) \in F$, for some $n \in \mathbb{N}$. Put $t = x \to (a \to^n y^{**})$. By Definition 2.1 (C), Lemma 2.5 (5), and the hypothesis, we have:

$$\begin{aligned} 1 = t \to (x \to (a \to^n y^{**})) &= x \to (a \to^n (t \to y^{**}))\\ &= x \to (a \to^n (t^{**} \to y^{**}))\\ &= x \to (a \to^n (t \to y)^{**}), \end{aligned}$$

Thus, $t \to y \in D_a^*(\uparrow x)$. From $t \in F$, we have $[t] = [1]$ and so $[t \to y] = [t] \to [y] = [1] \to [y] = [y]$, which implies that $[y] \in [D_a^*(\uparrow x)]$. Hence, $D_{[a]}^*(\uparrow[x]) \subseteq [D_a^*(\uparrow x)]$.

(2) is an immediate consequence of (1). □

Example 4.4. Consider the bounded BCK-algebra $A = \{0, a, b, c, d, 1\}$ from Example 3.11 (3). One can check that A is a Glivenko BCK-algebra. Take $F = \{d, 1\}$. Then, F is an i-filter of A. One can easily verify that

$$A/F = \{\{0\}, \{a, b\}, \{c\}, \{d, 1\}\} = \{[0], [a], [c], [1]\}.$$

The operation $\to$ on A/F is calculated by the following table:

$\to$	$[0]$	$[a]$	$[c]$	$[1]$
$[0]$	$[1]$	$[1]$	$[1]$	$[1]$
$[a]$	$[c]$	$[1]$	$[c]$	$[1]$
$[c]$	$[a]$	$[a]$	$[1]$	$[1]$
$[1]$	$[0]$	$[a]$	$[c]$	$[1]$

The double negation on A/F is shown in the following table:

	$[0]$	$[a]$	$[c]$	$[1]$
$**$	$[0]$	$[a]$	$[c]$	$[1]$

Take $D_a^*(U) = \{x \in A | a \to^n x^{**} \in U, n \geq 1\}$, for any $U \in \mathcal{U}(A)$. Routine calculation shows that $D_a^*(\uparrow 1) = D_a^*(\uparrow d) = D_a^*(\uparrow b) = D_a^*(\uparrow a) = \{a, b, d, 1\}$ and $D_a^*(\uparrow c) = D_a^*(\{b, c, d, 1\}) = D_a^*(\{a, b, c, d, 1\}) = D_a^*(A) = A$. Hence, $[D_a^*(\uparrow 1)] = [D_a^*(\uparrow d)] = [D_a^*(\uparrow b)] = [D_a^*(\uparrow a)] = [\{a, b, d, 1\}] = \{[a], [1]\} = D_{[a]}^*(\uparrow[a]) = D_{[a]}^*(\uparrow[1])$ and $[D_a^*(\uparrow c)] = [D_a^*(\{b, c, d, 1\})] = [D_a^*(\{a, b, c, d, 1\})] = [D_a^*(A)] = A/F = \{[0], [a], [c], [1]\} = D_{[a]}^*(\uparrow[c])$. Therefore, the topology $\mathcal{T}_{[a]}$ is equal to the quotient topology $\widetilde{\mathcal{T}_a}$ on the quotient BCK-algebra A/F. Moreover, the topology $\mathcal{T}_{[a]} = \{\varnothing, \uparrow[a], A/F\} = \{\varnothing, \{[a], [1]\}, A/F\}$.

Proposition 4.5. *Let F and G be i-filters of A and $F \subseteq G$. Then, $D_{[a]}^*(G/F) = D_a^*(G)/F$.*

Proof.

$$\begin{aligned} D_{[a]}^*(G/F) &= \{[x] \in A/F | [a] \to^n [x]^{**} \in G/F, \text{for some } n \in \mathbb{N}\}, \\ &= \{[x] \in A/F | [a \to^n x^{**}] \in G/F, \text{for some } n \in \mathbb{N}\}, \\ &= \{[x] \in A/F | a \to^n x^{**} \in G, \text{for some } n \in \mathbb{N}\}, \\ &= \{[x] \in A/F | x \in D_a^*(G), \text{for some } n \in \mathbb{N}\}, \\ &= D_a^*(G)/F. \end{aligned}$$

□

Lemma 4.6. *Let A and B be BCK-algebras. Then, H is an i-filter of $A \times B$ iff there exist $F \in \mathcal{F}(A)$ and $G \in \mathcal{F}(B)$ such that $H = F \times G$.*

Proof. Let H be an i-filter of $A \times B$. Consider $F = \{x \in A | (x, y) \in H, \text{for some } y \in B\}$ and $G = \{y \in A | (x, y) \in H, \text{for some } x \in A\}$. We prove that F and G are i-filters of A and B, respectively. From $(1,1) \in H$, it follows that $1 \in F$. Let $x, x \to y \in F$. Then, there are $u, v \in B$ such that $(x, u) \in H$ and $(x \to y, v) = (x, 1) \to (y, v) \in H$. Hence, $(y, v) \in H$, so $y \in F$. Thus, F is an i-filter of A. Similarly, we prove that G is an i-filter of B.

Conversely, let $H = F \times G$, for some $F \in \mathcal{F}(A)$ and $G \in \mathcal{F}(B)$. Clearly, $(1,1) \in F \times G = H$. Let $(x, y) \in H$ and $(x, y) \to (u, v) \in H$. Then, $(x \to u, y \to v) \in H$, so $x \to u \in F$ and $y \to v \in G$. This means that $u \in F$ and $v \in G$, so $(u, v) \in H$. □

Lemma 4.7. *Let F and G be i-filters of A. Then,* $D_a^*(F) \times D_a^*(G) = D_a^*(F \times G)$.

Proof.

$$\begin{aligned} &D_a^*(F) \times D_a^*(G),\\ &= \{(x, y) \in A \times A | x \in D_a^*(F), y \in D_a^*(G)\},\\ &= \{(x, y) \in A \times A | a \to^m x^{**} \in F, a \to^n y^{**} \in G, \text{for some } m, n \in \mathbb{N}\},\\ &= \{(x, y) \in A \times A | a \to^k x^{**} \in F, a \to^k y^{**} \in G, k \geq \max\{m, n\}\},\\ &= D_a^*(F \times G). \end{aligned}$$

□

Theorem 4.8. *Let F and G be i-filters of A. Then, there is a homeomorphism from* $(A \times A)/D_a^*(F \times G)$ *to* $(A/D_a^*(F)) \times (A/D_a^*(G))$.

Proof. Define the map $\Phi : (A \times A) \to (A/D_a^*(F)) \times (A/D_a^*(G)), (x, y) \mapsto ([x]_{D_a^*(F)}, [y]_{D_a^*(G)})$. Obviously, Φ is onto and $(x, y) \in \ker(\Phi)$ iff $[x]_{D_a^*(F)} = [1]_{D_a^*(F)}$ and $[y]_{D_a^*(G)} = [1]_{D_a^*(G)}$. Hence, $\ker(\Phi) = D_a^*(F) \times D_a^*(G)$. It follows from Lemma 4.7 that $D_a^*(F) \times D_a^*(G) = D_a^*(F \times G)$.

The map $\Psi : (A \times A)/D_a^*(F \times G) \to (A/D_a^*(F)) \times (A/D_a^*(G))$, defined by $\Psi([(x, y)]_{D_a^*(F \times G)}) = ([x]_{D_a^*(F)}, [y]_{D_a^*(G)})$, then by the first isomorphism theorem, Ψ is an isomorphism. Suppose that W be an open subset of $(A/D_a^*(F)) \times (A/D_a^*(G))$. Then, there exist open subsets $U, V \in \mathcal{T}_a$ such that $W = (U/D_a^*(F)) \times (V/D_a^*(G))$. Clearly, $\Psi^{-1}(W) = (U \times V)/D_a^*(F \times G)$ is an open subset of $(A \times A)/D_a^*(F \times G)$. Thus, Ψ is a continuous map. Similarly, Ψ^{-1} is a continuous map. We conclude that Ψ is a homeomorphism. □

5 Conclusion

In this article, we extend the notion of the set of double complemented elements to bounded BCK-algebras only related to implication operations. Moreover, we construct a topology on a bounded BCK-algebra and investigate some topological properties of a bounded BCK-algebra with the topology such as compactness and continuity. Finally, we give a condition under what such topology is equal to the quotient topology on the quotient BCK-algebra.

In our further work, we will study an extension of filters based on nuclei in product BCK-algebras, which is inspired by Borumand Saeid and Motamed [4], Han and Zhao [16], and Zhao and Zhou [17]. Moreover, we will investigate more general algebraic version of Glivenko theorems based on nuclei in product BCK-algebras.

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