Research Article

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Freely quasiconformal and locally weakly quasisymmetric mappings in metric spaces

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Abstract: In this article, we investigate the relationship between freely quasiconformal mappings and locally weakly quasisymmetric mappings in quasiconvex and complete metric spaces. We first prove the equivalence of freely quasiconformal mappings, ring properties, and locally weakly quasisymmetric mappings. Finally, we prove that the composition of two locally weakly quasisymmetric mappings in metric spaces is locally weakly quasisymmetric mapping.

Keywords: free quasiconformality, locally weak quasisymmetry, quasihyperbolic metric, ring property, quasiconvex metric space

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1 Introduction and main results

The quasihyperbolic metric was introduced by Gerhing and Palka in the 1970s [1,2] in the setting of Euclidean spaces $\mathbb{R}^n$. Since its first appearance, the quasihyperbolic metric has become an important tool in geometric function theory, and the theory related to quasihyperbolic metrics has been generalized to metric spaces and Banach spaces [3]. During the past few decades, modern geometric function theory of quasisymmetric and quasiconformal mappings has been studied from several points of view. Quasisymmetric mappings on the real line were first introduced by Beurling and Ahlfors [4]. They found a way to extend each quasisymmetric self-mapping of the real line to a quasiconformal self-mapping of the upper half-planes. This concept was later promoted by Tukia and Väisälä [5], who introduced and studied quasisymmetric mappings between arbitrary metric spaces. In 1990, based on the idea of quasisymmetry, Väisälä developed a “dimension-free” theory of quasiconformal mappings in infinite-dimensional Banach spaces [3,6–9]. In 1998, Heinonen and Koskela [10] showed that these concepts, quasiconformality and quasisymmetry, are quantitatively equivalent in a large class of metric spaces, which includes Euclidean spaces. Since these two concepts are equivalent, mathematicians show much interest in the research of quasisymmetric mappings between suitable metric spaces.

The main goal of this article is to discuss the relation between freely quasiconformal mappings, ring properties, and locally weakly quasisymmetric mappings in metric spaces, and prove that the composition of two locally weakly quasisymmetric mappings in metric spaces is locally weakly quasisymmetric mapping. For a metric space X, we use notation $|x - y|$ to indicate the distance between x and y. Following analogous notations and terminologies of [3,10–22]. We start with the definition of quasisymmetric mappings.

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Definition 1.1. A homeomorphism $f: X \to Y$ is said to be

(i) *η-quasisymmetric* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$, $\eta(0) = 0$, such that

$$|x - a| \le t|x - b| \quad \text{implies} \quad |f(x) - f(a)| \le \eta(t)|f(x) - f(b)|,$$

for each $t > 0$ and for each triple x, a, and b of points in X;

(ii) *weakly (h, H)-quasisymmetric* if there exist constants $h > 0$ and $H \ge 1$ so that

$$|x - a| \le h|x - b| \quad \text{implies} \quad |f(x) - f(a)| \le H|f(x) - f(b)|, \tag{1}$$

for each triple x, a, and b of points in X.

Remark 1.2.

(i) A homeomorphism $f: X \to Y$ is weakly H-quasisymmetric mapping if it satisfies (1) with $h = 1$. The weakly (h, H)-quasisymmetric mapping is a generalization of the weakly H-quasisymmetric mapping since the weakly H-quasisymmetric mapping coincides with the weakly $(1, H)$-quasisymmetric mapping.

(ii) The η-quasisymmetric mapping implies the weakly (h, H)-quasisymmetric mapping with $H = \eta(h)$. Obviously, $\eta(h) \ge h$.

Definition 1.3. Let X and Y be two metric spaces and $G \subsetneq X$, $G' \subsetneq Y$ be two domains (open and connected). Let $h > 0$ and $H \ge 1$ be two constants. A homeomorphism $f: G \to G'$ is said to be *q-locally weakly (h, H)-quasisymmetric* for some $0 < q < 1$, if the map $f|_{B_{x,q}} : B_{x,q} \to f(B_{x,q})$ is weakly (h, H)-quasisymmetric for any point $x \in G$, where $B_{x,q} = B(x, q\delta_G(x))$.

We remark that, in general, the locally weakly quasisymmetric mapping means the locally weakly H-quasisymmetric mapping [12]. Obviously, the locally weakly (h, H)-quasisymmetric mapping is a generalization of the locally weakly H-quasisymmetric mapping. In this article, we always simplify locally weakly (h, H)-quasisymmetric mapping by locally weakly quasisymmetric mapping.

Recall that a metric space is *proper* if all its closed balls are compact. The quasihyperbolic metric of a domain in $\mathbb{R}^n$ was introduced by Gehring and Palka [2]. For the concepts of quasihyperbolic metrics k_G and $k_{G'}$, please see Section 2.

Definition 1.4. Let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains, and let $\varphi : [0, \infty) \to [0, \infty)$ be a homeomorphism with $\varphi(t) \ge t$. We say that a homeomorphism $f: G \to G'$ is

(i) *φ-semisolid* if

$$k_{G'}(f(x), f(y)) \le \varphi(k_G(x, y)),$$

for all $x, y \in G$;

(ii) *φ-solid* if f and f^{-1} are φ-semisolid, i.e.,

$$\varphi^{-1}(k_G(x, y)) \le k_{G'}(f(x), f(y)) \le \varphi(k_G(x, y)),$$

for all $x, y \in G$;

(iii) *fully φ-semisolid* (resp., *fully φ-solid*) if f is φ-semisolid (resp., φ-solid) in every proper subdomain of G, i.e., the homeomorphism $f|_D : D \to f(D)$ is φ-semisolid (resp., φ-solid) for each proper subdomain $D \subsetneq G$. Fully φ-solid mappings are also called *freely φ-quasiconformal mappings*.

In order to state our results, we introduce the following definitions.

Definition 1.5. Let $G \subseteq X$ be a domain in a metric space X. A point $x \in G$ is called *cut point* if $G\backslash\{x\}$ is not connected. A domain G is called *non-cut-point* domain if it has no cut points.

Definition 1.6. Let X be a metric space and $B = B(x, r)$ be a metric open ball in X. Let $0 < \lambda \le 1$ be a constant. We say that B is a *John ball* if for each $y \in B$, there exists a rectifiable curve γ joining x and y in B whose arc length parameterization $\gamma_s : [0, l(\gamma)] \to B$ satisfies: $\gamma_s(0) = y$, $\gamma_s(l(\gamma)) = x$ and $\mathrm{dist}(\gamma_s(u), X \backslash B) \geq \lambda u$ for all $u \in [0, l(\gamma)]$. Here, $l(\gamma)$ is the length of γ, and curve γ_s is the arc length parameterization of γ.

Definition 1.7. Let X be a metric space and $G \subsetneq X$ be a domain. We called G a *λ-John-ball domain* if for every metric open ball, $B(x, r) \subsetneq G$ is a *John ball.*

Väisälä [3] obtained several characterizations of locally quasisymmetric in Banach spaces; Huang et al. [14] investigated the relationship between semisolid and locally weakly quasisymmetric in quasiconvex and complete metric spaces. Under suitable geometric conditions (Section 2), in this article, we shall prove a more general result (Theorem 1.8) for metric spaces.

Theorem 1.8. *Let X and Y be two c-quasiconvex, complete metric spaces, and let $G \subsetneq X$ and $G' \subsetneq Y$ be two λ-John-ball and non-cut-point domains. Suppose that $f : G \to G'$ is a homeomorphism, for some $0 < q < 1$. Then, the following are quantitatively equivalent*:
(*i*) *f and f^{-1} are q-locally weakly (h, H)-quasisymmetric mappings*;
(*ii*) *f and f^{-1} have the (M, α, β)-ring properties*;
(*iii*) *f is freely φ-quasiconformal mapping, where the constants q, h, H, M, α, β, and φ depend on each other and constants c and λ.*

As an application of Theorem 1.8, we show that the composite mapping of two locally weakly quasisymmetric mappings in a large class of metric spaces is a locally weakly quasisymmetric mapping.

Theorem 1.9. *Let X, Y, and Z be three c-quasiconvex, complete metric spaces. Suppose that $G_1 \subsetneq X, G_2 \subsetneq Y$, and $G_3 \subsetneq Z$ are three λ-John-ball and non-cut-point domains. If $f : G_1 \to G_2$ and $f^{-1} : G_2 \to G_1$ are q_1-locally weakly (h_1, H_1)-quasisymmetric mappings, $g : G_2 \to G_3$ and $g^{-1} : G_3 \to G_2$ are q_2-locally weakly (h_2, H_2)-quasisymmetric mappings, then the compositions $g \circ f$ and $(g \circ f)^{-1}$ are q-locally weakly (h, H)-quasisymmetric mappings, where q, h, and H depend on $q_1, h_1, H_1, q_2, h_2, H_2, c$, and λ.*

This article is organized as follows. In Section 2, we will introduce some necessary notations and concepts, recall some known results, and prove a series of basic and useful results. The goal of Section 3 is to show the equivalence between (i), (ii), and (iii) in Theorem 1.8, and in Section 4, we shall prove that the composition of two locally weakly quasisymmetric mappings in metric spaces is locally weakly quasisymmetric mapping.

2 Preliminaries

Let X be a metric space. We always denote the metric open and close balls with center $x \in X$ and radius $r > 0$ by

$$B(x, r) = \{y \in X : |y - x| < r\} \quad \text{and} \quad \overline{B}(x, r) = \{y \in X : |y - x| \leq r\},$$

and denote the metric sphere by

$$S(x, r) = \{y \in X : |y - x| = r\}.$$

If $B = B(x, r)$ (or $\overline{B} = \overline{B}(x, r)$, $S = S(x, r)$), then $\lambda B = B(x, \lambda r)$ (or $\lambda\overline{B} = \overline{B}(x, \lambda r)$, $\lambda S = S(x, \lambda r)$) for any $\lambda > 0$. For a set A in X, we always use $\overline{A}$ (resp., ∂A) to denote the closure (resp., the boundary) of A. The diameter of a set $E \subseteq X$ is the quantity

$$\mathrm{diam}(E) = \sup\{|x - y| : x, y \in E\},$$

and the distance between $E, F \subseteq X$ is

$$\mathrm{dist}(E, F) = \inf\{|x - y| : x \in E, y \in F\}.$$

2.1 Quasihyperbolic metrics

In this subsection, we give some basic properties of quasihyperbolic metric (see [3,6,16,23–26], and references therein for more details).

Let $\gamma \subset X$ be a rectifiable curve of length $l(\gamma)$ with endpoints a and b. If $l(\gamma) < \infty$, then the curve γ is said to be a *rectifiable curve*. Suppose that $\gamma : [a, b] \to X$ is a rectifiable curve. For each $t \in [a, b]$, we denote $s_\gamma(t) = l(\gamma\mid_{[a,t]})$, where the function $s_\gamma : [a, b] \to [0, l(\gamma)]$ is called the *length function* of γ. For any rectifiable curve $\gamma : [a, b] \to X$, there exists a unique curve $\gamma_s : [0, l(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. The curve γ_s is called the *arc length parameterization* of γ. This parameterization is characterized by the relation $l(\gamma_s\mid_{[t_1,t_2]}) = t_2 - t_1$ for all $0 \le t_1 < t_2 \le l(\gamma)$.

Definition 2.1. Let X be a metric space and $G \subsetneq X$ be a non-empty open set, and let γ be a rectifiable curve in a domain $G \subsetneq X$. The *quasihyperbolic length* of γ in G is

$$l_{qh}(\gamma) = \int_\gamma \frac{ds}{\delta_G(x)},$$

where $\delta_G(x)$ denotes the distance from x to $X\backslash G$, i.e., $\delta_G(x) = \operatorname{dist}(x, X\backslash G)$.

The *quasihyperbolic distance* between x and y in G is defined by

$$k_G(x, y) = \inf_\gamma l_{qh}(\gamma),$$

where γ runs over all rectifiable curves in G joining x and y. If there is no rectifiable curve in G joining x and y, then we define $k_G(x, y) = +\infty$.

A non-empty open set G of a metric space X is said to be *rectifiably connected* if for any two points $x, y \in G$, there exists a rectifiable curve in G joining x and y. If $G \subsetneq X$ is a rectifiably connected open set, it is clear that $k_G(x, y) < \infty$ for any two points $x, y \in G$. Thus, it is easy to verify that k_G is a metric in G, and we call it the *quasihyperbolic metric* of G, then (G, k_G) is a quasihyperbolic metric space.

For $c \ge 1$, a metric space X is *c-quasiconvex* if each pair of points $x, y \in X$ can be joined by a curve γ with length $l(\gamma) \le c|x - y|$.

Lemma 2.2. (See, [25], lemma 2.5) *Let X be a c-quasiconvex metric space, and let $G \subsetneq X$ be a domain. For any $x, y \in B(z, q\delta_G(z))$ with $0 < q \le \frac{1}{2c+1}$, there exists a rectifiable curve $\gamma \subseteq B(z, (2c + 1)q\delta_G(z))$ joining x and y such that $l(\gamma) \le c|x - y|$.*

Lemma 2.3. (See, [16], Theorem 2.7) *Let X be a c-quasiconvex metric space, and let $G \subsetneq X$ be a domain. Then,*

(*i*) *for each $x, y \in G$,*

$$|x - y| \le (e^{k_G(x,y)} - 1)\delta_G(x);$$

(*ii*) *if $z \in G$, $0 < t < 1$, and $x, y \in \overline{B}\left(z, \frac{t\delta_G(z)}{4c}\right)$, then*

$$\frac{1}{1 + 2t}\frac{|x - y|}{\delta_G(z)} \le k_G(x, y) \le \frac{c}{1 - t}\frac{|x - y|}{\delta_G(z)}.$$

Lemma 2.4. (See, [15], Theorem 2.6) *Let X be a c-quasiconvex metric space and $G \subsetneq X$ be a domain. Suppose that $x, y \in G$ and either $|x - y| \le \frac{\delta_G(x)}{8c}$ or $k_G(x, y) \le \frac{1}{4}$. Then,*

$$\frac{1}{2}\frac{|x - y|}{\delta_G(x)} \le k_G(x, y) \le 2c\frac{|x - y|}{\delta_G(x)}.$$

Lemma 2.5. (See, [15], Theorem 2.7) *Let X be a c-quasiconvex metric space and $G \subsetneq X$ be a domain. Then,*
(*i*) $l_{k_G}(\gamma) = l_{qh}(\gamma)$, *where* $l_{k_G}(\gamma)$ *is the length of* γ *in the metric space* (G, k_G);
(*ii*) *the metric space* $(G, k_G(\cdot))$ *is a 2-quasiconvex metric space.*

2.2 Ring properties, relative homeomorphisms, and uniformly continuous

Definition 2.6. Let $x \in G$, $M > 0$, and $1 < \alpha \le \beta$ be two positive real numbers. We say that a homeomorphism $f: G \to G'$ has (M, α, β)-ring property if

$$\sup_{0<r<r_{x,\beta}} \left\{ \frac{\operatorname{diam}(f(\overline{B}))}{\operatorname{dist}(f(\overline{B}), G' \backslash f(\alpha B))} \right\} \le M,$$

where

$$r_{x,\beta} = \frac{\delta_G(x)}{\beta}, \quad B = B(x, r), \quad \text{and} \quad \alpha B = B(x, \alpha r).$$

Definition 2.7. Let $0 < t_0 \le 1$, and let $\theta : [0, t_0) \to [0, \infty)$ be an embedding with $\theta(0) = 0$. Suppose that $G \subsetneq X$ and $G' \subsetneq Y$ be two domains. We say that a homeomorphism $f: G \to G'$ is
(i) (θ, t_0)-*relative* if there is a constant $t_0 \in (0,1]$ and a homeomorphism $\theta : [0, t_0) \to [0, \infty)$ such that

$$\frac{|f(x) - f(y)|}{\delta_{G'}(f(x))} \le \theta\left(\frac{|x - y|}{\delta_G(x)}\right),$$

whenever $x, y \in G$ and $|x - y| < t_0\delta_G(x)$; in particular, if $t_0 = 1$, then we say that f is θ-*relative*;
(ii) *fully* (θ, t_0)-*relative* (resp., *fully* θ-*relative*) if f is (θ, t_0)-relative (resp., θ-relative) in every subdomain of G.

Definition 2.8. Let X and Y be metric spaces. A map $f: X \to Y$ is a *uniformly continuous* if there is $t_0 \in (0, \infty]$ and a homeomorphism $\varphi : [0, t_0) \to [0, \infty)$ such that

$$|f(x) - f(y)| \le \varphi(|x - y|),$$

for all $x, y \in X$ with $|x - y| < t_0$. The function φ is a modulus of continuity of f, and we say that f is (φ, t_0)-*uniformly continuous*. If $t_0 = \infty$, we briefly say that f is φ-*uniformly continuous*.

Theorem 2.9. *Let X and Y be two c-quasiconvex and complete metric spaces, and let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains. Suppose that $f: G \to G'$ is a homeomorphism. Then, the following conditions are quantitatively equivalent*:
(*i*) f *is* (φ, t_0)-*uniformly continuous in the quasihyperbolic metric*;
(*ii*) f *is* φ'-*semisolid mapping*,

where φ and φ' depend on each other and constant t_0.

Proof. Obviously, we only need to show the implication from (i) to (ii), since the implication from (ii) to (i) is a direct consequence of the definitions of uniformly continuous and semisolid mapping. To prove this implication from (i) to (ii), we assume that f is (φ, t_0)-uniformly continuous in the quasihyperbolic metric, i.e.,

$$k_{G'}(f(x), f(y)) \le \varphi(k_G(x, y)), \tag{2}$$

for all $x, y \in G$ with $k_G(x, y) \le t_0$.

Since X is a c-quasiconvex complete metric space, by Observation 2.6 of [16], we have G rectifiably connected. By Lemma 2.5, we know that (G, k_G) is 2-quasiconvex. Therefore, for any given $x, y \in G$, there is a path $\gamma \subset G$ joining x and y with

$$l_{k_G}(\gamma) \le 2k_G(x, y).$$

Let $m \geq 0$ be the unique integer satisfying

$$mt_0 < l_{k_G}(\gamma) \leq (m+1)t_0. \tag{3}$$

Let $\gamma_s^k : [0, l_{k_G}(\gamma)] \to G$ be the arc length parameterization of γ with metric k_G. Denote

$$\tau_j = \frac{j}{m+1} l_{k_G}(\gamma) \quad \text{and} \quad x_j = \gamma_s^k(\tau_j), \tag{4}$$

for all $1 \leq j \leq m+1$. Thus, from (3) and (4), we know that

$$k_G(x_{j-1}, x_j) \leq l_{k_G}(\gamma_s^k |_{[\tau_{j-1}, \tau_j]}) = \tau_j - \tau_{j-1} = \frac{l_{k_G}(\gamma)}{m+1} \leq t_0.$$

Since f is (φ, t_0)-uniformly continuous in the quasihyperbolic metric, by (2), it follows that

$$k_{G'}(f(x_{j-1}), f(x_j)) \leq \varphi(k_G(x_{j-1}, x_j)) \leq \varphi(t_0),$$

for all $1 \leq j \leq m+1$. Hence, we deduce that

$$k_{G'}(f(x), f(y)) \leq \sum_{j=1}^{m+1} k_{G'}(f(x_{j-1}), f(x_j)) \leq (m+1)\varphi(t_0). \tag{5}$$

For all $x, y \in G$, remembering that $mt_0 < l_{k_G}(\gamma) \leq 2k_G(x, y)$, (5) implies that

$$k_{G'}(f(x), f(y)) \leq \frac{2\varphi(t_0)}{t_0} k_G(x, y) + \varphi(t_0). \tag{6}$$

If $k_G(x, y) > t_0$, according to (6), we obtain that

$$k_{G'}(f(x), f(y)) \leq \frac{2\varphi(t_0)}{t_0} k_G(x, y) + \varphi(t_0).$$

If $\frac{t_0}{2} < k_G(x, y) \leq t_0$, by the definition of uniformly continuous in the quasihyperbolic metric, we deduce that

$$k_{G'}(f(x), f(y)) \leq \varphi(k_G(x, y)) \leq \varphi(t_0) \leq \frac{4\varphi(t_0)}{t_0}\left(k_G(x, y) - \frac{t_0}{2}\right) + \varphi(t_0).$$

If $0 < k(x, y) \leq \frac{t_0}{2}$, applying the definition of uniformly continuous in the quasihyperbolic metric again, we know that

$$k_{G'}(f(x), f(y)) \leq \varphi(k_G(x, y)) \leq \varphi(k_G(x, y)) + \frac{2k_G(x, y)}{t_0}\left(\varphi(t_0) - \varphi\left(\frac{t_0}{2}\right)\right).$$

Therefore, we obtain that

$$k_{G'}(f(x), f(y)) \leq \varphi'(k_G(x, y)),$$

for all $x, y \in G$. Here,

$$\varphi'(t) = \begin{cases} \dfrac{2\varphi(t_0)}{t_0} t + \varphi(t_0), & \text{for } t_0 \leq t; \\ \dfrac{4\varphi(t_0)}{t_0}\left(t - \dfrac{t_0}{2}\right) + \varphi(t_0), & \text{for } \dfrac{t_0}{2} \leq t \leq t_0; \\ \varphi(t) + \dfrac{2t}{t_0}\left(\varphi(t_0) - \varphi\left(\dfrac{t_0}{2}\right)\right), & \text{for } 0 < t \leq \dfrac{t_0}{2}. \end{cases}$$

Hence, Theorem 2.9 is proved. □

3 Proof of Theorem 1.8

In what follows, we always assume that X and Y are two c-quasiconvex and complete metric spaces, and that $G \subsetneq X$ and $G' \subsetneq Y$ are two domains. Furthermore, we suppose that f and f^{-1} are q-locally weakly (h, H)-quasisymmetric mappings for some $0 < q < 1$.

Now, we are ready to prove Theorem 1.8. We verify the implications indicated by the following routes:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)}.$$

By Lemmas 5.3 and 5.4 of [14], the implication (ii) ⇒ (iii) is true in all metric spaces. It suffices to prove that (i) ⇒ (ii) and (iii) ⇒ (i).

The proof is based on a refinement of the method due to Väisälä [3] and Huang et al. [14]. The proof is given in the following two subsections.

3.1 Proof of the implication from (i) to (ii) in Theorem 1.8

Assume f and f^{-1} are q-locally weakly (h, H)-quasisymmetric mappings for some $0 < q < 1$. Let

$$r_{x,\beta} = \frac{\delta_G(x)}{\beta}, \quad B = B(x, r) \quad \text{and} \quad \alpha B = B(x, \alpha r).$$

We shall show that f and f^{-1} have the (M, α, β)-ring properties, i.e.,

$$\sup_{0<r<r_{x,\beta}} \left\{ \frac{\operatorname{diam}(f(\overline{B}))}{\operatorname{dist}(f(\overline{B}), G' \backslash f(\alpha B))} \right\} \le M,$$

where

$$M = 2H^2(H+1), \quad \alpha = \max\left\{\frac{3}{h^2}, 3\right\} \quad \text{and} \quad \beta = \max\left\{\frac{3}{qh^2}, \frac{3}{q}\right\}.$$

In what follows, we only need to verify the f has the (M, α, β)-ring property. The proof of the (M, α, β)-ring property of f^{-1} follows in a similar manner.

In order to prove that f has the (M, α, β)-ring property, we divide the discussions into two cases:

Case 3.1.1. $0 < h < 1$.

For $x \in G$, $0 < q < 1$, let $B = B(x, r)$ and $\alpha = 3/(h^2)$, where $0 < r \le \frac{q}{3/h^2}\delta_G(x)$. Then,

$$\overline{B}(x, r) \subseteq \overline{B}(x, \alpha r) \subseteq B(x, q\delta_G(x)).$$

For all $a, b \in \overline{B}(x, r)$, we assume that

$$y \in \partial\left(\frac{1}{h}\overline{B}(x, r)\right) \quad \text{and} \quad z \in G \backslash \left(\frac{3}{h^2}B(x, r)\right).$$

It is clear that

$$\max\{|a - x|, |b - x|\} \le h|x - y|.$$

Since f is q-locally weakly (h, H)-quasisymmetric mapping for some $0 < q < 1$, it follows from the definition that

$$|f(a) - f(x)| \le H|f(x) - f(y)| \quad \text{and} \quad |f(b) - f(x)| \le H|f(x) - f(y)|.$$

It follows from the aforementioned fact that

$$|f(a) - f(b)| \le |f(a) - f(x)| + |f(b) - f(x)| \le 2H|f(x) - f(y)|. \tag{7}$$

Using $z \in G\backslash((3/(h^2))B(x,r))$ and $y \in \partial((1/h)\overline{B}(x,r))$, we obtain

$$|z-y| \geq |z-x| - |x-y| \geq \frac{3r}{h^2} - \frac{r}{h} \geq \frac{2r}{h^2} > \frac{1}{h}|x-y|.$$

From the definition of q-locally weakly (h,H)-quasisymmetric mapping, we have

$$|f(x)-f(y)| \leq H|f(z)-f(y)|. \tag{8}$$

Together with inequalities (7) and (8), it follows that

$$|f(a)-f(b)| \leq 2H^2|f(z)-f(y)|. \tag{9}$$

Let u be any point in $\overline{B}(x,r)$, we obtain that

$$|u-y| \leq \operatorname{diam}\left(\frac{1}{h}\overline{B}(x,r)\right) = \frac{2r}{h} \tag{10}$$

and

$$|u-z| \geq |z-x| - |x-u| \geq \frac{3r}{h^2} - r \geq \frac{2r}{h^2}. \tag{11}$$

Using inequalities (10) and (11), we now obtain

$$|u-y| \leq h|u-z|.$$

Therefore, by the definition of q-locally weakly (h,H)-quasisymmetric mapping again, we have

$$|f(u)-f(y)| \leq H|f(u)-f(z)|.$$

Thus,

$$|f(z)-f(y)| \leq |f(z)-f(u)| + |f(u)-f(y)| \leq (H+1)|f(u)-f(z)|. \tag{12}$$

Combining inequalities (9) and (12), we have thus proved that

$$|f(a)-f(b)| \leq 2H^2(H+1)|f(u)-f(z)|.$$

Since u and z are arbitrary, it follows from Remark 2.7 that

$$\sup_{0<r<r_{x,\beta}}\left\{\frac{\operatorname{diam}(f(\overline{B}))}{\operatorname{dist}(f(\overline{B}), G'\backslash f((3/(h^2))B))}\right\} \leq 2H^2(H+1),$$

where $\beta = 3/(qh^2)$.

Case 3.1.2. $h \geq 1$.

In this case, let $\alpha = 3$ and $0 < r \leq (q/3)\delta_G(x)$, for all $x \in G$, then

$$\overline{B}(x,r) \subseteq \overline{B}(x,\alpha r) \subseteq B(x, q\delta_G(x)).$$

Suppose that a and b are any two points in $\overline{B}(x,r)$. Let

$$y \in \partial(\overline{B}(x,r)) \quad \text{and} \quad z \in G\backslash(3B(x,r)),$$

then we have

$$\max\{|a-x|, |b-x|\} \leq |x-y| \leq h|x-y|.$$

From the assumption of $z \in G\backslash(3B(x,r))$, it is clear that

$$|x-y| \leq |z-y| \leq h|z-y|.$$

For any $u \in \overline{B}(x,r)$, we obtain

$$|y-u| \leq \operatorname{diam}(B(x,r)) = 2r \tag{13}$$

and

$$|z-u| \geq |z-x| - |x-u| \geq 3r - r = 2r. \tag{14}$$

By combining (13) and (14), we have

$$|y - u| \le |z - u| \le h|z - u|.$$

Since f is q-locally weakly (h, H)-quasisymmetric mapping for some $0 < q < 1$, using a similar argument as in case 3.1.1, we obtain that

$$\sup_{0<r<r_{x,\beta}} \left\{ \frac{\operatorname{diam}(f(\overline{B}))}{\operatorname{dist}(f(\overline{B}), G' \backslash f(3B))} \right\} \le 2H^2(H+1),$$

where $\beta = 3/q$.

Thus, as discussion earlier, it follows that f has the (M, α, β)-ring property. Here,

$$M = 2H^2(H+1), \quad \alpha = \max\left\{\frac{3}{h^2}, 3\right\} \quad \text{and} \quad \beta = \max\left\{\frac{3}{qh^2}, \frac{3}{q}\right\}.$$

Hence, the proof of implication from (i) to (ii) is complete.

3.2 Proof of the implication from (iii) to (i) in Theorem 1.8

In this subsection, in order to prove the implication from (iii) to (i) in Theorem 1.8, we need the following lemma.

Lemma 3.1. *Under the assumptions of* Theorem 1.8, *suppose that the homeomorphism $f : G \to G'$ is freely φ-quasiconformal mapping. Then, f and f^{-1} are fully θ-relative with θ depending only on λ, φ, and c.*

Proof. Let $D \subseteq G$ be any subdomain of G and $D' = f(D)$. Assume $f : G \to G'$ is freely φ-quasiconformal mapping, i.e., for all $x, y \in D$,

$$\varphi^{-1}(k_D(x, y)) \le k_{D'}(f(x), f(y)) \le \varphi(k_D(x, y)).$$

Here, $\varphi : [0, \infty) \to [0, \infty)$ is a homeomorphism with $\varphi(t) \ge t$.

In what follows, we only need to verify that the f is fully θ-relative. The proof of fully θ-relative of f^{-1} follows in a similar manner.

To prove that f is fully θ-relative with θ depending only on λ, φ, and c, we suppose that $x, y \in D$ with $|x - y| = t\delta_D(x)$, $0 \le t < 1$. We have the following claim.

Claim 3.1. For any $0 \le t < 1$, if $x, y \in D$ with $|x - y| = t\delta_D(x)$, then

$$k_D(x, y) \le \theta_0(t),$$

where $\theta_0 : [0,1) \to [0, \infty)$ is a homeomorphism.

By Lemma 2.4, if $0 < t \le \frac{1}{8c}$, then

$$k_D(x, y) \le 2c\frac{|x - y|}{\delta_D(x)} = 2ct.$$

So we can assume that $\frac{1}{8c} < t < 1$ in the following. Since X is a c-quasiconvex metric space, by ([16], Observation 2.6), we know that $D \subseteq G$ is rectifiably connected. Since $G \subsetneq X$ is a λ-John-ball domain, we have every metric open ball $B_x = B(x, \delta_D(x))$ in $D \subseteq G$ is a John ball. Let $0 < \lambda \le 1$, from the definition of John ball, it is clear that

$$\operatorname{dist}(\gamma_s(l(\gamma)), X \backslash B_x) \ge \lambda l(\gamma).$$

Because $\partial B_x = \{y \in X : |y - x| = \delta_D(x)\}$ and $B_x \subseteq D$, it follows that

$$\delta_D(x) = \operatorname{dist}(x, X \backslash B_x) \le \operatorname{dist}(x, X \backslash D) = \delta_D(x).$$

Therefore, by $\delta_{B_x}(x) = \operatorname{dist}(x, X\backslash B_x)$, we obtain that $\delta_{B_x}(x) = \delta_D(x)$. Furthermore,

$$l(\gamma) \le \frac{\operatorname{dist}(\gamma_s(l(\gamma)), X\backslash B_x)}{\lambda} = \frac{\operatorname{dist}(x, X\backslash B_x)}{\lambda} = \frac{\delta_D(x)}{\lambda}. \tag{15}$$

Denote

$$u_1 := \sup\left\{u \in [0, l(\gamma)] : \gamma_s|_{[0,u]} \subseteq \overline{B}\left(y, \frac{1-t}{2}\delta_D(x)\right)\right\}.$$

Obviously, if $u_1 = l(\gamma)$, then we have $\gamma \subseteq \overline{B}\left(y, \frac{1-t}{2}\delta_D(x)\right)$. We observe, for all $u \in [0, l(\gamma)]$,

$$\delta_D(\gamma_s(u)) = \operatorname{dist}(\gamma_s(u), X\backslash D) \ge \operatorname{dist}(\gamma_s(u), X\backslash B_x) \ge \frac{1-t}{2}\delta_D(x). \tag{16}$$

By the definition of quasihyperbolic metric $k_D(x, y)$ and (15) and (16), it follows that

$$k_D(x, y) \le \int_0^{l(\gamma)} \frac{\mathrm{d}u}{\delta_D(\gamma_s(u))} \le \frac{l(\gamma)}{((1-t)/2)\delta_D(x)} \le \frac{2}{\lambda(1-t)}. \tag{17}$$

Now, we assume that $u_1 < l(\gamma)$. Denote

$$u_2 := \inf\{u \in [0, l(\gamma)] : \gamma_s|_{[u,l(\gamma)]} \subseteq \overline{B}(x, t\delta_D(x))\}.$$

Then, for any $u \in [u_2, l(\gamma)]$, we have

$$\delta_D(\gamma_s(u)) \ge \delta_D(x) - |x - \gamma_s(u)| \ge (1-t)\delta_D(x). \tag{18}$$

If $u_2 \le u_1$, by combining (15), (16), and (18), we have

$$\begin{aligned} k_D(x, y) &\le \int_0^{l(\gamma)} \frac{\mathrm{d}u}{\delta_D(\gamma_s(u))} \le \int_0^{u_1} \frac{\mathrm{d}u}{\delta_D(\gamma_s(u))} + \int_{u_1}^{l(\gamma)} \frac{\mathrm{d}u}{\delta_D(\gamma_s(u))} \\ &\le \frac{u_1}{((1-t)/2)\delta_D(x)} + \frac{l(\gamma) - u_1}{(1-t)\delta_D(x)} \le \frac{2l(\gamma)}{(1-t)\delta_D(x)} \le \frac{2}{\lambda(1-t)}. \end{aligned} \tag{19}$$

If $u_2 > u_1$, then for any $u \in [u_1, u_2]$, by the definitions of u_1 and John ball, we have

$$\begin{aligned} \delta_D(\gamma_s(u)) &= \operatorname{dist}(\gamma_s(u), X\backslash D) \ge \operatorname{dist}(\gamma_s(u), X\backslash B_x) \\ &\ge \lambda \cdot u \ge \lambda \cdot u_1 \ge \lambda \cdot |\gamma_s(u_1) - y| \ge \lambda \cdot \frac{1-t}{2}\delta_D(x). \end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned} k_D(x, y) &\le \int_0^{l(\gamma)} \frac{\mathrm{d}u}{\delta_D(\gamma_s(u))} = \int_0^{u_1} \frac{\mathrm{d}u}{\delta_D(\gamma_s(u))} + \int_{u_1}^{u_2} \frac{\mathrm{d}u}{\delta_D(\gamma_s(u))} + \int_{u_2}^{l(\gamma)} \frac{\mathrm{d}u}{\delta_D(\gamma_s(u))} \\ &\le \frac{u_1}{\frac{1-t}{2}\delta_D(x)} + \frac{u_2 - u_1}{\lambda \cdot \frac{1-t}{2}\delta_D(x)} + \frac{l(\gamma) - u_2}{(1-t)\delta_D(x)} \\ &= \frac{(2\lambda - 2)u_1 + (2-\lambda)u_2 + \lambda \cdot l(\gamma)}{\lambda \cdot (1-t)\delta_D(x)} \quad (\text{since } u_1 < u_2 \le l(\gamma)) \\ &\le \frac{(2-\lambda)u_2 + \lambda \cdot l(\gamma)}{\lambda \cdot (1-t)\delta_D(x)} \quad (\text{since } 0 < \lambda \le 1) \\ &\le \frac{2+\lambda}{\lambda^2(1-t)}. \end{aligned} \tag{21}$$

Therefore, remembering that $\frac{1}{8c} < t < 1$, (17), (19), and (21) imply that

$$k_D(x, y) \le \frac{2+\lambda}{\lambda^2(1-t)}.$$

We consider function $\theta_0(t)$ in the following:

$$\theta_0(t)=\begin{cases}\dfrac{64(2+\lambda)c^2t}{\lambda^2(8c-1)}, & \text{for } 0<t\le\dfrac{1}{8c},\\ \dfrac{2+\lambda}{\lambda^2(1-t)}, & \text{for } \dfrac{1}{8c}<t<1.\end{cases}$$

Then, it follows that

$$k_D(x,y)\le\theta_0(t),$$

for any $x,y\in D$ and $0\le t<1$. Hence, the statements in Claim 3.1 are proved.

We are now turning to the proof of Lemma 3.1.

The proof of Lemma 3.1. For $x,y\in D$ with $|x-y|=t\delta_D(x)$. By applying the inequality (i) of Lemma 2.3, we have

$$\frac{|f(x)-f(y)|}{\delta_{D'}(f(x))}\le e^{k_{D'}(f(x),f(y))}-1=\psi(k_{D'}(f(x),f(y))). \tag{22}$$

Here, $\psi:[0,\infty)\to[0,\infty)$ is a homeomorphism by $\psi(t)=e^t-1$.

Since f is freely φ-quasiconformal mapping, from the conclusion of Claim 3.1 and inequality (22), it follows that

$$\frac{|f(x)-f(y)|}{\delta_{D'}(f(x))}\le\psi(k_{D'}(f(x),f(y)))\le\psi(\varphi(k_D(x,y)))\le\psi\left(\varphi\left(\theta_0\left(\frac{|x-y|}{\delta_D(x)}\right)\right)\right),$$

which implies that the map f is θ-relative, where $\theta(t)=\psi(\varphi(\theta_0(t)))$ is a homeomorphism from $[0,1)$ to $[0,\infty)$ and θ depends only on λ, φ, and c. This completes the proof of Lemma 3.1. □

The proof of Theorem 1.8. Suppose that the homeomorphism $f:G\to G'$ is freely φ-quasiconformal mapping; by Lemma 3.1, f and f^{-1} are fully θ-relative, with θ depending only on λ, φ, and c.

In what follows, we only need to verify the f is q-locally weakly (h,H)-quasisymmetric mapping. The proof of q-locally weakly (h,H)-quasisymmetric mapping of f^{-1} follows in a similar manner.

Let $z\in G$, $0<q<1$. For $x,a,b\in B(z,q\delta_G(z))$ with $|a-x|=h|b-x|$ and in order to prove that f is q-locally weakly (h,H)-quasisymmetric mapping, we need only to show that

$$|f(a)-f(x)|\le H|f(b)-f(x)|, \tag{23}$$

where $H\ge1$.

We divide the discussions into three cases:

Case 3.2.1. $0<h\le\frac{2c}{2c+1}$.

Set $D=G\backslash\{b\}$, then $f(D)=G'\backslash\{f(b)\}$, and let $D'=f(D)$. Since G is a non-cut-point domain, D is a sub-domain. So the assumption on the fully θ-relative of f implies $f|_D$ is θ-relative. Note that $\theta:[0,1)\to[0,\infty)$ is a homeomorphism. Without loss of generality, we assume that

$$\theta\left(\frac{2c}{2c+1}\right)\ge1.$$

Assume that $x,a,b\in B(z,q\delta_G(z))$ with $|a-x|=h|b-x|$. For any $0<q<\frac{1}{3}$, it is clear that

$$\delta_G(x)\ge\delta_G(z)-|x-z|\ge\delta_G(z)-q\delta_G(z)>2q\delta_G(z)\ge|x-b|.$$

Therefore, we have

$$|a-x|=h|b-x|\le h\delta_D(x).$$

From the definition of θ-relative, it follows that

$$|f(a)-f(x)|\le\theta(h)\delta_{D'}(f(x))\le\theta(h)|f(b)-f(x)|\le H_1|f(b)-f(x)|, \tag{24}$$

where

$$H_1 = \theta\left(\frac{2c}{2c+1}\right).$$

Case 3.2.2. $\frac{2c}{2c+1} < h \leq 1$.

Let $0 < q \leq \frac{1}{4c+3}$. For each point $x, a, b \in B(z, q\delta_G(z))$ with $|a-x| = h|b-x|$, since X is c-quasiconvex metric space, by Lemma 2.2, there is a rectifiable curve $\gamma : [u, v] \to G$ joining x and a with $\gamma \subseteq B(z, (2c+1)q\delta_G(z))$ such that $l(\gamma) \leq c|x-a|$.

Define inductively the successive points $x = z_0, z_1, \ldots, z_{n-1}, z_n = a$ of γ as follows. Let $t_0 = u$,

$$t_j = \sup_t \left\{ t \in [u, v] \ : \ |\gamma(t) - z_{j-1}| \leq \left(\frac{2c}{2c+1}\right)^j |x-b|, 1 \leq j \leq n \right\},$$

and $z_j = \gamma(t_j)$, $0 \leq j \leq n$.

Furthermore, since $\frac{2c}{2c+1} < h \leq 1$, for $1 \leq j \leq n-1$, we have $n \geq 2$,

$$|z_{j-1} - z_j| = \left(\frac{2c}{2c+1}\right)^j |x-b| \quad \text{and} \quad |z_{n-1} - z_n| \leq \left(\frac{2c}{2c+1}\right)^n |x-b|.$$

Therefore, for $1 \leq j \leq n-1$, it is clear that

$$l(\gamma\,|_{[z_{j-1}, z_j]}) \geq |z_{j-1} - z_j| = \left(\frac{2c}{2c+1}\right)^j |x-b|. \tag{25}$$

By summing (25) over $1 \leq j \leq n-1$, we have

$$\sum_{j=1}^{n-1} \left(\frac{2c}{2c+1}\right)^j |x-b| \leq l(\gamma) \leq c|x-a| = ch|x-b| \leq c|x-b|.$$

Hence, we obtain that $n \leq k$ with

$$k \coloneqq \frac{\ln 2}{\ln(2c+1) - \ln(2c)} + 1. \tag{26}$$

For each $1 \leq j \leq n-1$, since $x, b \in B(z, q\delta_G(z))$, we see that

$$|z_j - z_{j-1}| = \left(\frac{2c}{2c+1}\right)^j |x-b| \leq |x-b| \leq 2q\delta_G(z) \tag{27}$$

and

$$\sum_{i=0}^{j-1} |z_i - z_{i+1}| = \sum_{i=0}^{j-1} \left(\frac{2c}{2c+1}\right)^{i+1} |x-b| \leq 2c|x-b| \leq 4cq\delta_G(z). \tag{28}$$

Combining (27) and (28), we have

$$\begin{aligned} \delta_G(z_j) - |z_j - z_{j-1}| &\geq \delta_G(z_0) - \sum_{i=0}^{j-1} |z_i - z_{i+1}| - |z_j - z_{j-1}| \\ &\geq \delta_G(z) - |z_0 - z| - \sum_{i=0}^{j-1} |z_i - z_{i+1}| - |z_j - z_{j-1}| \\ &\geq \delta_G(z) - q\delta_G(z) - 4cq\delta_G(z) - 2q\delta_G(z) \\ &= (1 - (4c+3)q)\delta_G(z) \\ &\geq 0. \end{aligned} \tag{29}$$

Set $D_j = G \backslash \{z_j\}$. Since G is a non-point-cut domain, D_j is a subdomain. So $f(D_j) = G' \backslash \{f(z_j)\}$ is also a subdomain. Let $D'_j = f(D_j)$ for $1 \leq j \leq n-1$; according to (29), we obtain

$$|z_j - z_{j+1}| \leq \frac{2c}{2c+1} |z_{j-1} - z_j| = \frac{2c}{2c+1} \delta_{D_{j-1}}(z_j) < \delta_{D_{j-1}}(z_j).$$

Since $f|_{D_j}$ is θ-relative and $|z_1 - z_0| = \frac{2c}{2c+1}|x - b|$, by Case 3.2.1, it follows that

$$\begin{aligned} |f(z_1) - f(z_0)| &\le \theta\left(\frac{2c}{2c+1}\right)|f(b) - f(x)| = H_1|f(b) - f(x)|; \\ |f(z_2) - f(z_1)| &\le H_1\delta_{D_0'}(f(z_1)) \le H_1|f(z_1) - f(z_0)| \le H_1^2|f(b) - f(x)|; \\ &\cdots \\ |f(z_n) - f(z_{n-1})| &\le H_1\delta_{D_{n-2}'}(f(z_{n-1})) \le H_1^n|f(b) - f(x)|. \end{aligned} \tag{30}$$

Summation of aforementioned formulas gives

$$|f(x) - f(a)| \le (H_1 + H_1^2 + \ldots + H_1^n)|f(b) - f(x)| \le nH_1^n|f(b) - f(x)|.$$

Combining this estimate with (26), it follows immediately that

$$|f(x) - f(a)| \le H_2|f(b) - f(x)|, \tag{31}$$

where

$$H_2 = k\theta\left(\frac{2c}{2c+1}\right)^k \quad \text{and} \quad k = \frac{\ln 2}{\ln(2c+1) - \ln(2c)} + 1.$$

Case 3.2.3. $h > 1$.

For any $x, a, b \in B\left(z, \frac{1}{(2c+1)(4c+3)}\delta_G(z)\right)$. Since X is a c-quasiconvex metric space, by Lemma 2.2, there is a rectifiable curve $\gamma_1 : [u, v] \to G$ joining x and a with $\gamma_1 \subseteq B\left(z, \frac{1}{4c+3}\delta_G(z)\right)$ such that $l(\gamma_1) \le c|x - a|$.

Define inductively the successive points $x = p_0, p_1, \ldots, p_{m-1}, p_m = a$ of γ_1 as follows. Let $t_0 = u$,

$$t_i = \sup_t\{t \in [u, v] \ : \ |\gamma_1(t) - p_{i-1}| \le |x - b|, 1 \le i \le m\},$$

and $p_i = \gamma_1(t_i)$, $0 \le i \le m$.

Moreover, since $h > 1$, for $1 \le i \le m - 1$, we have $m \ge 2$,

$$|p_{i-1} - p_i| = |x - b| \quad \text{and} \quad |p_{m-1} - p_m| \le |x - b|.$$

Note that for $1 \le i \le m - 1$,

$$l(\gamma_1|_{[p_{i-1}, p_i]}) \ge |p_{i-1} - p_i| = |x - b|. \tag{32}$$

By summing (32) over $1 \le i \le m - 1$, we have

$$(m - 1)|x - b| \le \sum_{i=1}^{m-1} l(\gamma_1|_{[p_{i-1}, p_i]}) \le l(\gamma_1) \le c|x - a| = ch|x - b|. \tag{33}$$

According to (32) and (33), it follows that

$$m \le 1 + ch. \tag{34}$$

Note that $x = p_0, p_1, \ldots, p_m = a$ and b are contained in $B\left(z, \frac{1}{4c+3}\delta_G(z)\right)$. Using a similar argument as in Case 3.2.2, we obtain

$$|f(a) - f(x)| \le \sum_{i=1}^{m} |f(p_i) - f(p_{i-1})| \le mH_2|f(x) - f(b)|. \tag{35}$$

Combining (34) and (35), we have the desired estimate

$$|f(a) - f(x)| \le H_3|f(x) - f(b)|, \tag{36}$$

where

$$H_3 = (1 + ch)H_2.$$

Therefore, Case 3.2.3 is completed.

Thus, in terms of (24), (31), and (36), for any $x, a, b \in B\left(z, \frac{1}{(2c+1)(4c+3)}\delta_G(z)\right)$ with $|x - a| = h|b - x|$, we obtain that

$$|f(a) - f(x)| \leq H|f(x) - f(b)|.$$

Here,

$$H = \max\{H_1, H_2, H_3\} = (1 + ch)\left(k\theta\left(\frac{2c}{2c+1}\right)^k\right),$$

where

$$k = \frac{\ln 2}{\ln(2c+1) - \ln(2c)} + 1.$$

Therefore, f is q-locally weakly (h, H)-quasisymmetric mapping with h and H depending only on θ and c. Here,

$$q = \frac{1}{(2c+1)(4c+3)}.$$

Hence, the proof of implication from (iii) to (i) is complete.

4 Proof of Theorem 1.9

In this section, we assume that X, Y, and Z are three c-quasiconvex, complete metric spaces and that $G_1 \subsetneq X$, $G_2 \subsetneq Y$, and $G_3 \subsetneq Z$ are three λ-John-ball and non-cut-point domains. Furthermore, we suppose that $f : G_1 \to G_2$ and $f^{-1} : G_2 \to G_1$ are q_1-locally weakly (h_1, H_1)-quasisymmetric mappings; $g : G_2 \to G_3$ and $g^{-1} : G_3 \to G_2$ are q_2-locally weakly (h_2, H_2)-quasisymmetric mappings.

Lemma 4.1. *Under the assumptions of* Theorem 1.9, *if $f : G_1 \to G_2$ is a freely φ_1-quasiconformal mapping and $g : G_2 \to G_3$ is a freely φ_2-quasiconformal mapping, then, the composition $g \circ f : G_1 \to G_3$ is a freely $\varphi_2 \circ \varphi_1$-quasiconformal mapping.*

Proof. Let $D \subseteq G_1$ be any subdomain of G_1 and $D' = f(D) \subseteq G_2$, $D'' = g \circ f(D) \subseteq G_3$. In what follows, we only need to verify that $g \circ f : G_1 \to G_3$ is fully semisolid mapping, i.e.,

$$k_{D''}(g \circ f(x), g \circ f(y)) \leq \varphi_2 \circ \varphi_1(k_D(x, y)).$$

The proof of $(g \circ f)^{-1}$ is fully semisolid mapping that follows in a similar manner.

For any domain $D \subseteq G_1$, by the assumption, we see that $f : D \to D'$ is φ_1-semisolid mapping and $g : D' \to D''$ is φ_2-semisoild mapping, which shows that

$$k_{D'}(f(x), f(y)) \leq \varphi_1(k_D(x, y))$$

and

$$k_{D''}(g \circ f(x), g \circ f(y)) \leq \varphi_2(k_{D'}(f(x), f(y))),$$

for all $x, y \in D$. Therefore, we obtain that

$$k_{D''}(g \circ f(x), g \circ f(y)) \leq \varphi_2 \circ \varphi_1(k_D(x, y)).$$

Hence, the proof of Lemma 4.1 is complete. □

The proof of Theorem 1.9. Under the assumptions of Theorem 1.9, we see that f and g are freely φ_i-quasiconformal mappings with some homeomorphisms φ_i, $i = 1, 2$. By Lemma 4.1, the composition $g \circ f : G_1 \to G_3$ is freely $\varphi_2 \circ \varphi_1$-quasiconformal mapping. It follows from Theorem 1.8 that there exist constants q, h, and H such that $g \circ f$ and $(g \circ f)^{-1}$ are q-locally weakly (h, H)-quasisymmetric mappings. Hence, Theorem 1.9 is proved.

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