Research Article

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Majorization-type inequalities for (m, M, ψ)-convex functions with applications

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Abstract: In 2001, S. S. Dragomir introduced a generalized class of convexity, the so-called (m, M, ψ)-convex functions, which covers many other classes of convexity. In this article, we prove some useful characterizations of this generalized class of convex functions. We obtain majorization-type inequalities for (m, M, ψ)-convex functions, providing also applications to new estimates for some well-known mean inequalities.

Keywords: convex function, strongly convex function, (m, M, ψ)-convex function, majorization inequality, Sherman inequality

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1 Introduction

Recently, various generalized classes of convexity have been studied, and the corresponding inequalities for these classes have been established. Many of these classes of functions can also be observed as special cases of the more general class of convexity introduced by Dragomir in [1]:

Let $m, M \in \mathbb{R}$, $I \subseteq \mathbb{R}$, and $\psi : I \to \mathbb{R}$ be a convex function. A function $\varphi : I \to \mathbb{R}$ is called:
- (m, ψ)-lower convex if the function $\varphi - m\psi$ is convex;
- (M, ψ)-upper convex if the function $M\psi - \varphi$ is convex;
- (m, M, ψ)-convex if it is (m, ψ)-lower convex and (M, ψ)-upper convex.

In accordance with this definition, if φ is (m, M, ψ)-convex, then $\varphi - m\psi$ and $M\psi - \varphi$ are convex and then the function $(M - m)\psi$ is convex, implying that $m \leqslant M$ whenever ψ is not trivial, i.e., is not the zero function.

Let us note that the previous definition can be written in the following way:

Let $m, M \in \mathbb{R}$ and $\varphi, \psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be functions such that ψ is convex. Then, φ is said to be (m, ψ)-lower convex if

$$m[\lambda\psi(x) + (1 - \lambda)\psi(y) - \psi(\lambda x + (1 - \lambda)y)] \leqslant \lambda\varphi(x) + (1 - \lambda)\varphi(y) - \varphi(\lambda x + (1 - \lambda)y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. Furthermore, φ is said to be (M, ψ)-upper convex if

$$\lambda\varphi(x) + (1 - \lambda)\varphi(y) - \varphi(\lambda x + (1 - \lambda)y) \leqslant M[\lambda\psi(x) + (1 - \lambda)\psi(y) - \psi(\lambda x + (1 - \lambda)y)] \tag{1.2}$$

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holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If both inequalities (1.1) and (1.2) hold, then φ is said to be (m, M, ψ)-convex with $m \leqslant M$ whenever ψ is not trivial.

It is easy to prove the following useful lemma.

Lemma 1. *Let $m, M \in \mathbb{R}$, $\varphi, \psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be functions such that ψ is convex. Then,*
- *φ is (m, ψ)-lower convex iff the function $\varphi - m\psi$ is convex;*
- *φ is (M, ψ)-upper convex iff the function $M\psi - \varphi$ is convex;*
- *φ is (m, M, ψ)-convex iff the function $(M - m)\psi$ is convex.*

In the following part, we consider some particular cases of defined generalized classes of convexity.

According to [2], for $\psi = id^2$, where id denotes the identity function, i.e., $id(t) = t$, $t \in I$, function φ is called m-lower convex if the function $\varphi - m \cdot id^2$ is convex and φ is called M-upper convex if the function $M \cdot id^2 - \varphi$ is convex. The same classes of functions were named convexifiable and concavifiable, when considered in [3] and [4].

Note that for $m = 0$ and $M = 0$ in (1.1) and (1.2), we obtain ordinary convexity and concavity, respectively.

Since ψ is convex function, inequality

$$\lambda\psi(x) + (1 - \lambda)\psi(y) - \psi(\lambda x + (1 - \lambda)y) \geqslant 0$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Then, in case $m > 0$, we have

$$\begin{aligned}\varphi(\lambda x + (1 - \lambda)y) &\leqslant \lambda\varphi(x) + (1 - \lambda)\varphi(y) - m[\lambda\psi(x) + (1 - \lambda)\psi(y) - \psi(\lambda x + (1 - \lambda)y)]\\ &\leqslant \lambda\varphi(x) + (1 - \lambda)\varphi(y),\end{aligned}$$

i.e., (m, ψ)-lower convexity implies ordinary convexity.

In particular, if $m > 0$ and $\psi = id^2$, then we come to the notion of strong convexity. The class of strongly convex functions plays an important role in optimization theory. For more details on this concept, see [5].

For $m < 0$, we are going in the direction of consideration of approximately convex functions.

If $M < 0$, then

$$\begin{aligned}\lambda\varphi(x) + (1 - \lambda)\varphi(y) &\leqslant \lambda\varphi(x) + (1 - \lambda)\varphi(y) - M[\lambda\psi(x) + (1 - \lambda)\psi(y) - \psi(\lambda x + (1 - \lambda)y)]\\ &\leqslant \varphi(\lambda x + (1 - \lambda)y),\end{aligned}$$

i.e., (M, ψ)-upper convexity implies ordinary concavity.

For $M > 0$, (M, ψ)-upper convex functions were investigated in [6] (see also references given therein) and are called delta convex functions. Such functions play an important role in convex analysis. In the special case, when $M > 0$ and $\psi = id^2$, we obtain (M, id^2)-upper convex functions that were considered in [7] and were named approximately concave functions. In [8], the concept of g-convex-dominated functions, where g is a given convex function, was introduced. Namely, a function f is called g-convex-dominated if the functions $g + f$ and $g - f$ are convex. Note that this concept can be obtained as a particular case of (m, M, ψ)-convexity by choosing $m = -1$, $M = 1$, and $\psi = g$.

We cite the following lemmas from [1] for their importance as characterizations of (m, M, ψ)-convex functions. The first one considers the supporting lines of convex functions.

Lemma 2. *Let $\varphi, \psi : I \to \mathbb{R}$ be differentiable functions on* $\operatorname{int} I$, *where* $\operatorname{int} I$ *is the interior of I, and let ψ be convex on* $\operatorname{int} I$.

(*a*) *For $m \in \mathbb{R}$, function φ is (m, ψ)-lower convex iff*

$$m[\psi(y) - \psi(z) - \psi'(z)(y - z)] \leq \varphi(y) - \varphi(z) - \varphi'(z)(y - z). \tag{1.3}$$

(*b*) *For $M \in \mathbb{R}$, function φ is (M, ψ)-upper convex iff*

$$\varphi(y) - \varphi(z) - \varphi'(z)(y - z) \leq M[\psi(y) - \psi(z) - \psi'(z)(y - z)]. \tag{1.4}$$

(*c*) *For $m, M \in \mathbb{R}$, $M \geq m$, function φ is (m, M, ψ)-convex iff both* (1.3) *and* (1.4) *hold.*

The second lemma is a characterization for twice-differentiable functions.

Lemma 3. *Let* $\varphi, \psi : I \to \mathbb{R}$ *be twice-differentiable on* $\operatorname{int}I$, *and let* ψ *be convex on* $\operatorname{int}I$.
(*a*) *For* $m \in \mathbb{R}$, *function* φ *is* (m, ψ)*-lower convex on* $\operatorname{int}I$ *iff*

$$m \cdot \psi''(x) \le \varphi''(x), \quad \text{for all } x \in \operatorname{int}I. \tag{1.5}$$

(*b*) *For* $M \in \mathbb{R}$, *function* φ *is* (M, ψ)*-upper convex on* $\operatorname{int}I$ *iff*

$$\varphi''(x) \le M \cdot \psi''(x), \quad \text{for all } x \in \operatorname{int}I. \tag{1.6}$$

(*c*) *For* $m, M \in \mathbb{R}$, $M \ge m$, *function* φ *is* (m, M, ψ)*-convex iff both* (1.5) *and* (1.6) *hold, i.e.,*

$$m \cdot \psi''(x) \le \varphi''(x) \le M \cdot \psi''(x), \quad \text{for all } x \in \operatorname{int}I.$$

More results related to the class of (m, M, ψ)-convex functions can be found in the study by Dragomir [9].

The connection of convex functions with the concept of vector majorization is briefly presented in the sequel.

For $n, k \in \mathbb{N}$, we denote with $C_{nk}([0, \infty))$ the space of all $n \times k$ column stochastic matrices, i.e., matrices with nonnegative entries and column sums equal to 1.

Let $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, \ldots, y_k) \in I^k$ be the vectors associated with nonnegative weights $\mathbf{a} = (a_1, \ldots, a_n) \in [0, \infty)^n$ and $\mathbf{b} = (b_1, \ldots, b_k) \in [0, \infty)^k$, such that

$$a_i = \sum_{j=1}^{k} b_j s_{ij}, \quad i = 1, \ldots, n, \quad \text{and} \quad y_j = \sum_{i=1}^{n} x_i s_{ij}, \quad j = 1, \ldots, k, \tag{1.7}$$

hold for $\mathbf{S} = (s_{ij}) \in C_{nk}([0, \infty))$.

Conditions (1.7) are given in the matrix form as follows:

$$\mathbf{a} = \mathbf{b}\mathbf{S}^T \quad \text{and} \quad \mathbf{y} = \mathbf{x}\mathbf{S}, \tag{1.8}$$

where $\mathbf{S}^T$ is the transpose matrix of $\mathbf{S}$.

For pairs $(\mathbf{y}, \mathbf{b})$ and $(\mathbf{x}, \mathbf{a})$, which satisfy (1.8) for some $\mathbf{S} \in C_{nk}([0, \infty))$, we use notation

$$(\mathbf{y}, \mathbf{b}) \prec (\mathbf{x}, \mathbf{a})$$

and say that the pair $(\mathbf{y}, \mathbf{b})$ is weighted majorized with $(\mathbf{x}, \mathbf{a})$.

Under the previous conditions (1.8), Sherman [10] proved that inequality

$$\sum_{j=1}^{k} b_j f(y_j) \leqslant \sum_{i=1}^{n} a_i f(x_i) \tag{1.9}$$

holds for every convex function $f : I \to \mathbb{R}$. Recently, Sherman's inequality (1.9) has attracted a lot of attention and many related results have been proved (see, e.g., [11–15]).

Classical majorization is a special case of the previous consideration. Namely, if we put $n = k$ and take $\mathbf{a} = \mathbf{b} = \mathbf{e} = (1, \ldots, 1) \in [0, \infty)^n$, then (1.8) becomes

$$\mathbf{e} = \mathbf{e}\mathbf{S}^T \quad \text{and} \quad \mathbf{y} = \mathbf{x}\mathbf{S}.$$

The first condition $\mathbf{e} = \mathbf{e}\mathbf{S}^T$ implies that $\mathbf{S}$ is also row stochastic, i.e., a matrix with row sums equal to 1. Moreover, $\mathbf{S}$ is doubly stochastic matrix. Then, the second condition $\mathbf{y} = \mathbf{x}\mathbf{S}$ implies a classical majorization with notation

$$\mathbf{y} \prec \mathbf{x}.$$

As an easy consequence of Sherman's inequality (1.9), we obtain that the Hardy-Littlewood-Pólya inequality

$$\sum_{i=1}^{n} f(y_i) \leqslant \sum_{i=1}^{n} f(x_i)$$

holds for every convex function $f : I \to \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in I^n$ such that $\mathbf{y} \prec \mathbf{x}$ [16].

This article is organized as follows. In Section 2, we prove useful propositions with important characterizations of (m, ψ)-lower convex and (M, ψ)-upper convex functions. Using results from these propositions, we derive majorization-type inequalities for (m, ψ)-lower convex, (M, ψ)-upper convex, and (m, M, ψ)-convex functions. In Section 4, we present the applications of the obtained majorization inequalities by deriving new lower and upper bound estimations for some well-known mean inequalities.

2 More about (m, M, ψ)-convexity

The equivalent statements of the following proposition combined with the characterizations of (m, M, ψ)-convex functions given in Section 1 form a new direction established in proving our main results.

In the sequel, I denotes a real interval.

Proposition 1. *Let $\varphi, \psi : I \to \mathbb{R}$ be differentiable functions, and suppose that ψ is convex. Let $m \in \mathbb{R}$. The following statements are equivalent:*

(*i*) *Function φ is (m, ψ)-lower convex, i.e.,*

$$m[\psi(y) - \psi(x) - \psi'(x)(y - x)] \leq \varphi(y) - \varphi(x) - \varphi'(x)(y - x), \tag{2.1}$$

for all $x, y \in I$.

(*ii*) *For $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and a nonnegative n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ such that $\sum_{i=1}^n a_i = 1$ with $\bar{x} = \sum_{i=1}^n a_i x_i$, it holds*

$$\varphi(\bar{x}) \leq \sum_{i=1}^n a_i \varphi(x_i) - m\left(\sum_{i=1}^n a_i \psi(x_i) - \psi(\bar{x})\right). \tag{2.2}$$

(*iii*) *Inequality*

$$\varphi(\lambda y_1 + (1 - \lambda)y_2) \leq \lambda\varphi(y_1) + (1 - \lambda)\varphi(y_2) - m[\lambda\psi(y_1) + (1 - \lambda)\psi(y_2) - \psi(\lambda y_1 + (1 - \lambda)y_2)] \tag{2.3}$$

holds for all $y_1, y_2 \in I$ and $\lambda \in [0,1]$.

(*iv*) *For all $x, y_1, y_2 \in I$, such that $y_1 < x < y_2$, it holds*

$$\varphi(x) - \frac{y_2 - x}{y_2 - y_1}\varphi(y_1) - \frac{x - y_1}{y_2 - y_1}\varphi(y_2) \leq m\left(\psi(x) - \frac{y_2 - x}{y_2 - y_1}\psi(y_1) - \frac{x - y_1}{y_2 - y_1}\psi(y_2)\right), \tag{2.4}$$

or equivalently,

$$\frac{\varphi(y_1) - \varphi(x) - m(\psi(y_1) - \psi(x))}{y_1 - x} \leq \frac{\varphi(y_2) - \varphi(x) - m(\psi(y_2) - \psi(x))}{y_2 - x}. \tag{2.5}$$

Proof. (*i*) ⇒ (*ii*) Let φ be (m, ψ)-lower convex, i.e., (2.1) holds for all $x, y \in I$. If we choose $x = \bar{x}$ and $y = x_i$, $i \in \{1, \ldots n\}$ in (2.1), then we obtain

$$m[\psi(x_i) - \psi(\bar{x}) - \psi'(\bar{x})(x_i - z)] \leq \varphi(x_i) - \varphi(\bar{x}) - \varphi'(\bar{x})(x_i - \bar{x}).$$

Now, multiplying with a_i and summing over i, $i = 1, \ldots, n$, we obtain

$$\sum_{i=1}^n a_i\varphi(x_i) - \varphi(\bar{x}) - \sum_{i=1}^n a_i\varphi'(\bar{x})(x_i - \bar{x}) \geq \sum_{i=1}^n a_i m[\psi(x_i) - \psi(\bar{x}) - \psi'(\bar{x})(x_i - \bar{x})]. \tag{2.6}$$

Since

$$\sum_{i=1}^n a_i\varphi'(\bar{x})(x_i - \bar{x}) = 0 \quad \text{and} \quad \sum_{i=1}^n a_i\psi'(\bar{x})(x_i - \bar{x}) = 0,$$

inequality (2.2) follows from (2.6).

$(ii) \Rightarrow (iii)$ Let (2.2) hold for $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and a nonnegative n-tuple $\boldsymbol{a} = (a_1, \ldots, a_n)$ such that $\sum_{i=1}^n a_i = 1$ with $\bar{x} = \sum_{i=1}^n a_i x_i$.

If we in a special case choose $n = 2$, $a_1 = \lambda \in [0,1]$, $a_2 = 1 - \lambda$, and for $x_1 = y_1$ and $x_2 = y_2$, then from (2.2), we obtain (2.3).

$(iii) \Rightarrow (iv)$ Let (2.3) hold for all $y_1, y_2 \in I$ and $\lambda \in [0,1]$.

Let $x, y_1, y_2 \geq 0$ be such that $y_1 < x < y_2$. Then, there is $\lambda \in [0,1]$ such that $x = \lambda y_1 + (1-\lambda) y_2$. Furthermore, we have $\lambda = \frac{y_2 - x}{y_2 - y_1}$ and $1 - \lambda = \frac{x - y_1}{y_2 - y_1}$. If we use such terms in (2.3), we obtain

$$\varphi(x) \leq \frac{y_2 - x}{y_2 - y_1}\varphi(y_1) + \frac{x - y_1}{y_2 - y_1}\varphi(y_2) - m\left(\frac{y_2 - x}{y_2 - y_1}\psi(y_1) + \frac{x - y_1}{y_2 - y_1}\psi(y_2) - \psi(x)\right). \tag{2.7}$$

If we multiply (2.7) with $y_2 - y_1 > 0$, then we have

$$\begin{aligned}(y_2 - x + x - y_1)\varphi(x) \leq (y_2 - x)\varphi(y_1) + (x - y_1)\varphi(y_2) - m[(y_2 - x)\psi(y_1) + (x - y_1)\psi(y_2)\\ - (y_2 - x + x - y_1)\psi(x)].\end{aligned}$$

Dividing with $(x - y_1)(y_2 - x) > 0$, we obtain

$$\frac{\varphi(y_1) - \varphi(x) - m(\psi(y_1) - \psi(x))}{y_1 - x} \leq \frac{\varphi(y_2) - \varphi(x) - m(\psi(y_2) - \psi(x))}{y_2 - x},$$

what we need to prove.

$(iv) \Rightarrow (i)$ Let (2.5) hold. It is equivalent to

$$\frac{g(y_1) - g(x)}{y_1 - x} \leq \frac{g(y_2) - g(x)}{y_2 - x},$$

where g is a convex function defined by $g = \varphi - m\psi$. Taking limits $y_1 \to x_-$ and $y_2 \to x_+$, we have

$$\frac{g(y_1) - g(x)}{y_1 - x} \leq g'_-(x) \leq g'_+(x) \leq \frac{g(y_2) - g(x)}{y_2 - x}.$$

Differentiability at x implies $g'_-(x) = g'_+(x) = g'(x)$, i.e.,

$$\frac{g(y_1) - g(x)}{y_1 - x} \leq g'(x) \leq \frac{g(y_2) - g(x)}{y_2 - x}, \quad y_1 < x < y_2. \tag{2.8}$$

The left-hand side of (2.8), with the substitution $y_1 \coloneqq y$, yields (2.1) in the case $y < x$. Analogously, the right-hand side of (2.8), with the substitution $y_2 \coloneqq y$, reduces to (2.1) in the case $x < y$. The case $x = y$ is obvious. This completes the proof. □

Analogous statements are also valid for (M, ψ)-upper convex functions, which makes the contents of the following proposition.

Proposition 2. *Let $\varphi, \psi : I \to \mathbb{R}$ be differentiable functions, and suppose that ψ is convex. Let $M \in \mathbb{R}$. The following statements are equivalent*:

(i) *Function φ is (M, ψ)-upper convex, i.e.,*

$$\varphi(y) - \varphi(x) - \varphi'(x)(y - x) \leq M[\psi(y) - \psi(x) - \psi'(x)(y - x)],$$

for all $x, y \in I$.

(ii) *For $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and a nonnegative n-tuple $\boldsymbol{a} = (a_1, \ldots, a_n)$ such that $\sum_{i=1}^n a_i = 1$* with $\bar{x} = \sum_{i=1}^n a_i x_i$, *it holds*

$$\sum_{i=1}^n a_i \varphi(x_i) - M\left(\sum_{i=1}^n a_i \psi(x_i) - \psi(\bar{x})\right) \leq \varphi(\bar{x}). \tag{2.9}$$

(*iii*) *Inequality*

$$\lambda\varphi(y_1) + (1-\lambda)\varphi(y_2) - M[\lambda\psi(y_1) + (1-\lambda)\psi(y_2) - \psi(\lambda y_1 + (1-\lambda)y_2)] \le \varphi(\lambda y_1 + (1-\lambda)y_2)$$

holds for all $y_1, y_2 \in I$, *and* $\lambda \in [0,1]$.

(*iv*) *For all* $x, y_1, y_2 \in I$, *such that* $y_1 < x < y_2$, *it holds*

$$\varphi(x) - \frac{y_2 - x}{y_2 - y_1}\varphi(y_1) - \frac{x - y_1}{y_2 - y_1}\varphi(y_2) \ge M\left[\psi(x) - \frac{y_2 - x}{y_2 - y_1}\psi(y_1) - \frac{x - y_1}{y_2 - y_1}\psi(y_2)\right], \tag{2.10}$$

or equivalently,

$$\frac{M(\psi(y_1) - \psi(x)) - \varphi(y_1) + \varphi(x)}{y_1 - x} \le \frac{M(\psi(y_2) - \psi(x)) - \varphi(y_2) + \varphi(x)}{y_2 - x}.$$

Proof. Here, we consider (M, ψ)-upper convex function φ and proceed completely analogously as in the previous proof. Hence, we omit the details. □

Remark 1. Inequality (2.2) is Jensen's inequality for (m, ψ)-lower convex functions, while inequality (2.9) is Jensen's inequality for (M, ψ)-upper convex functions. If φ is (m, M, ψ)-convex function and $m \le M$, then both (2.2) and (2.9) hold, i.e., we obtain Jensen's inequality for (m, M, ψ)-convex functions

$$m\left(\sum_{i=1}^{n} a_i\psi(x_i) - \psi(\bar{x})\right) \le \sum_{i=1}^{n} a_i\varphi(x_i) - \varphi(\bar{x}) \le M\left(\sum_{i=1}^{n} a_i\psi(x_i) - \psi(\bar{x})\right). \tag{2.11}$$

The Jensen-type inequalities for positive linear functionals for (m, M, ψ)-convex functions were proved in [9].

Furthermore, in the trivial case of ψ being a zero function, i.e., $\psi(t) = 0$, for all $t \in I$, function φ is convex in the usual sense and (2.2) reduces to Jensen's inequality for convex functions:

$$\varphi(\bar{x}) \le \sum_{i=1}^{n} a_i\varphi(x_i). \tag{2.12}$$

Note that if $m = c > 0$ and $\psi = id^2$, then (2.2) becomes Jensen's inequality for strongly convex functions with modulus c, i.e., we obtain

$$\varphi(\bar{x}) \le \sum_{i=1}^{n} a_i\varphi(x_i) - c\sum_{i=1}^{n} a_i(x_i - \bar{x})^2.$$

For $M = c > 0$ and $\psi = id^2$, inequality (2.9) assumes the form

$$\sum_{i=1}^{n} a_i\varphi(x_i) \le \varphi(\bar{x}) + c\sum_{i=1}^{n} a_i(x_i - \bar{x})^2$$

of the Jensen inequality for approximately concave functions. Some related results for approximately concave functions can be found in the study by Merentes and Nikodem [7] (see [7, Theorem 4, Corollary 6]).

3 Majorization-type inequalities for (m, M, ψ)-convex functions

In order to present majorization inequalities for (m, ψ)-lower convex, (M, ψ)-upper convex, and (m, M, ψ)-convex functions, we start this section with consideration on the Sherman-type inequalities.

Throughout the section, I denotes a real interval.

Theorem 1. *Let* $\varphi, \psi : I \to \mathbb{R}$ *be differentiable functions such that* ψ *is convex. Suppose* $(\mathbf{x}, \mathbf{a}) \in I^n \times [0, \infty)^n$, $(\mathbf{y}, \mathbf{b}) \in I^k \times [0, \infty)^k$, *and* $\mathbf{S} \in C_{nk}([0, \infty))$ *are such that* $(\mathbf{y}, \mathbf{b}) \prec (\mathbf{x}, \mathbf{a})$, *and let* $m, M \in \mathbb{R}$.

(*a*) *If φ is an (m, ψ)-lower convex function, then*

$$m\left(\sum_{i=1}^{n} a_i\psi(x_i) - \sum_{j=1}^{k} b_j\psi(y_j)\right) \le \sum_{i=1}^{n} a_i\varphi(x_i) - \sum_{j=1}^{k} b_j\varphi(y_j). \tag{3.1}$$

(*b*) *If φ is an (M, ψ)-upper convex function, then*

$$\sum_{i=1}^{n} a_i\varphi(x_i) - \sum_{j=1}^{k} b_j\varphi(y_j) \le M\left(\sum_{i=1}^{n} a_i\psi(x_i) - \sum_{j=1}^{k} b_j\psi(y_j)\right). \tag{3.2}$$

(*c*) *If φ is an (m, M, ψ)-convex function, where $m \le M$, then* (3.1) *and* (3.2) *both hold.*

Proof.

(a) Making use of (1.7) while applying the characterization (2.2) of (m, ψ)-lower convex functions, we have

$$\begin{aligned}\sum_{j=1}^{k} b_j\varphi(y_j) &= \sum_{j=1}^{k} b_j\varphi\left(\sum_{i=1}^{n} x_i s_{ij}\right)\\ &\le \sum_{j=1}^{k} b_j\left[\sum_{i=1}^{n} s_{ij}\varphi(x_i) - m\left(\sum_{i=1}^{n} s_{ij}\psi(x_i) - \psi\left(\sum_{i=1}^{n} x_i s_{ij}\right)\right)\right]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{k} b_j s_{ij}\varphi(x_i) - m\left(\sum_{i=1}^{n}\sum_{j=1}^{k} b_j s_{ij}\psi(x_i) - \sum_{j=1}^{k} b_j\psi\left(\sum_{i=1}^{n} x_i s_{ij}\right)\right)\\ &= \sum_{i=1}^{n} a_i\varphi(x_i) - m\left(\sum_{i=1}^{n} a_i\psi(x_i) - \sum_{j=1}^{k} b_j\psi(y_j)\right),\end{aligned}$$

what we need to prove.

(b) Making use of (1.7) while applying the characterization (2.9) of (M, ψ)-upper convex functions, we have

$$\begin{aligned}\sum_{j=1}^{k} b_j\varphi(y_j) &= \sum_{j=1}^{k} b_j\varphi\left(\sum_{i=1}^{n} x_i s_{ij}\right)\\ &\ge \sum_{j=1}^{k} b_j\left[\sum_{i=1}^{n} s_{ij}\varphi(x_i) - M\left(\sum_{i=1}^{n} s_{ij}\psi(x_i) - \psi\left(\sum_{i=1}^{n} x_i s_{ij}\right)\right)\right]\\ &= \sum_{i=1}^{n} a_i\varphi(x_i) - M\left(\sum_{i=1}^{n} a_i\psi(x_i) - \sum_{j=1}^{k} b_j\psi(y_j)\right),\end{aligned}$$

i.e., we proved (3.2).

(c) This case is proved by a) and b) combined. □

In the context of majorization, next we present the Lah-Ribarich-type inequalities for (m, ψ)-lower convex, (M, ψ)-upper convex, and (m, M, ψ)-convex functions under conditions (1.7).

Theorem 2. *Let $\varphi, \psi : I \to \mathbb{R}$ be differentiable functions such that ψ is convex. Suppose $\alpha, \beta \in I$, $\alpha < \beta$, and m, $M \in \mathbb{R}$. Let $(\mathbf{x}, \mathbf{a}) \in [\alpha, \beta]^n \times [0, \infty)^n$, $(\mathbf{y}, \mathbf{b}) \in [\alpha, \beta]^k \times [0, \infty)^k$, and $\mathbf{S} \in C_{nk}([0, \infty))$ be such that $(\mathbf{y}, \mathbf{b}) \prec (\mathbf{x}, \mathbf{a})$. Let us denote $\bar{\beta} = \beta\sum_{j=1}^{k} b_j$, $\bar{\alpha} = \alpha\sum_{i=1}^{n} a_i$, and $\bar{y} = \sum_{j=1}^{k} b_j y_j$.*

(*a*) *If φ is an (m, ψ)-lower convex function, then*

$$m\left(\frac{\bar{\beta} - \bar{y}}{\beta - \alpha}\psi(\alpha) + \frac{\bar{y} - \bar{\alpha}}{\beta - \alpha}\psi(\beta) - \sum_{i=1}^{n} a_i\psi(x_i)\right) \le \frac{\bar{\beta} - \bar{y}}{\beta - \alpha}\varphi(\alpha) + \frac{\bar{y} - \bar{\alpha}}{\beta - \alpha}\varphi(\beta) - \sum_{i=1}^{n} a_i\varphi(x_i). \tag{3.3}$$

(*b*) *If* φ *is an* (M, ψ)*-upper convex function, then*

$$\frac{\bar{\beta} - \bar{y}}{\beta - \alpha}\varphi(\alpha) + \frac{\bar{y} - \bar{\alpha}}{\beta - \alpha}\varphi(\beta) - \sum_{i=1}^{n} a_i\varphi(x_i) \leq M\left(\frac{\bar{\beta} - \bar{y}}{\beta - \alpha}\psi(\alpha) + \frac{\bar{y} - \bar{\alpha}}{\beta - \alpha}\psi(\beta) - \sum_{i=1}^{n} a_i\psi(x_i)\right). \tag{3.4}$$

(c) *If* φ *is an* (m, M, ψ)*-convex function, where* $m \leq M$*, then* (3.3) *and* (3.4) *both hold.*

Proof. a) By substituting $x = x_i$, $y_2 = \beta$, and $y_1 = \alpha$ into characterization (2.4) for (m, ψ)-lower convex functions, we obtain

$$\varphi(x_i) - \frac{\beta - x_i}{\beta - \alpha}\varphi(\alpha) - \frac{x_i - \alpha}{\beta - \alpha}\varphi(\beta) \leq m\left(\psi(x_i) - \frac{\beta - x_i}{\beta - \alpha}\psi(\alpha) - \frac{x_i - \alpha}{\beta - \alpha}\psi(\beta)\right).$$

Multiplying it with s_{ij} and summing over i, $i = 1, \ldots, n$, we have

$$\begin{aligned}&\sum_{i=1}^{n} s_{ij}\varphi(x_i) - \frac{\beta - \sum_{i=1}^{n} s_{ij}x_i}{\beta - \alpha}\varphi(\alpha) - \frac{\sum_{i=1}^{n} s_{ij}x_i - \alpha}{\beta - \alpha}\varphi(\beta)\\ &\leq m\left(\sum_{i=1}^{n} s_{ij}\psi(x_i) - \frac{\beta - \sum_{i=1}^{n} s_{ij}x_i}{\beta - \alpha}\psi(\alpha) - \frac{\sum_{i=1}^{n} s_{ij}x_i - \alpha}{\beta - \alpha}\psi(\beta)\right).\end{aligned}$$

Furthermore, multiplying with b_j and summing over j, $j = 1, \ldots, m$, we obtain

$$\begin{aligned}&\sum_{j=1}^{k} b_j \sum_{i=1}^{n} s_{ij}\varphi(x_i) - \frac{\beta\sum_{j=1}^{k} b_j - \sum_{j=1}^{k} b_j \sum_{i=1}^{n} s_{ij}x_i}{\beta - \alpha}\varphi(\alpha) - \frac{\sum_{j=1}^{k} b_j \sum_{i=1}^{n} s_{ij}x_i - \alpha\sum_{j=1}^{k} b_j}{\beta - \alpha}\varphi(\beta)\\ &\leq m\sum_{j=1}^{k} b_j\left(\sum_{i=1}^{n} s_{ij}\psi(x_i) - \frac{\beta - \sum_{i=1}^{n} s_{ij}x_i}{\beta - \alpha}\psi(\alpha) - \frac{\sum_{i=1}^{n} s_{ij}x_i - \alpha}{\beta - \alpha}\psi(\beta)\right).\end{aligned} \tag{3.5}$$

Since by (1.7), we have

$$\begin{aligned}&\sum_{j=1}^{k} b_j\left(\sum_{i=1}^{n} s_{ij}\varphi(x_i)\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{k} b_j s_{ij}\right)\varphi(x_i) = \sum_{i=1}^{n} a_i\varphi(x_i),\\ &\sum_{j=1}^{k} b_j\left(\sum_{i=1}^{n} s_{ij}\psi(x_i)\right) = \sum_{i=1}^{n} a_i\psi(x_i),\end{aligned}$$

and

$$\sum_{j=1}^{k} b_j \sum_{i=1}^{n} s_{ij}x_i = \sum_{j=1}^{k} b_j\left(\sum_{i=1}^{n} s_{ij}x_i\right) = \sum_{j=1}^{k} b_j y_j,$$

then from (3.5), we obtain (3.3).

(b) We proceed analogously, except that this time we start from characterization (2.10) for (M, ψ)-upper convex functions. Hence, we omit the details.

(c) This case is proved by (a) and (b) combined. □

Remark 2. Note that

$$\bar{y} = \sum_{j=1}^{k} b_j y_j = \sum_{j=1}^{k} b_j\left(\sum_{i=1}^{n} s_{ij}x_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{k} b_j s_{ij}\right)x_i = \sum_{i=1}^{n} a_i x_i = \bar{x},$$

so we can replace each occurrence of the expression $\bar{y}$ in (3.3) and (3.4) with $\bar{x}$.

In the trivial case of ψ being a zero function, (3.3) reduces to

$$\sum_{i=1}^{n} a_i\varphi(x_i) \leq \frac{\bar{\beta} - \bar{x}}{\beta - \alpha}\varphi(\alpha) + \frac{\bar{y} - \bar{x}}{\beta - \alpha}\varphi(\beta). \tag{3.6}$$

This inequality is known as the Lah-Ribarich inequality for convex functions [17].

A generalization of inequality (3.6) for positive linear functionals is known as the Lupaş-Beesack-Pečarić inequality (see [18, p. 98]). Inequalities of the Lupaş-Beesack-Pečarić type for (m, ψ)-lower convex, (M, ψ)-upper convex, and (m, M, ψ)-convex functions were proved by Dragomir [9].

If in the previous results, we put $n = m$ and take $\mathbf{a} = \mathbf{b} = \mathbf{e} = (1, \ldots, 1) \in [0, \infty)^n$, then we consider a classical majorization with notation $\mathbf{y} \prec \mathbf{x}$. Thus, as easy consequences of Theorems 1 and 2, we obtain the following corollaries that include inequalities of the Hardy-Littlewood-Pólya type.

Corollary 1. *Let $\varphi, \psi : I \to \mathbb{R}$ be differentiable functions such that ψ is convex, and let $m, M \in \mathbb{R}$. Suppose $\mathbf{x}, \mathbf{y} \in I^n$ are such that $\mathbf{y} \prec x$.*

(*a*) *If φ is an (m, ψ)-lower convex function, then*

$$m\left(\sum_{i=1}^{n}\psi(x_i) - \sum_{i=1}^{n}\psi(y_i)\right) \leq \sum_{i=1}^{n}\varphi(x_i) - \sum_{i=1}^{n}\varphi(y_i). \tag{3.7}$$

(*b*) *If φ is an (M, ψ)-upper convex function, then*

$$\sum_{i=1}^{n}\varphi(x_i) - \sum_{i=1}^{n}\varphi(y_i) \leq M\left(\sum_{i=1}^{n}\psi(x_i) - \sum_{i=1}^{n}\psi(y_i)\right). \tag{3.8}$$

(*c*) *If φ is an (m, M, ψ)-convex function, where $m \leq M$, then* (3.7) *and* (3.8) *both hold.*

Corollary 2. *Let $\varphi, \psi : I \to \mathbb{R}$ be differentiable functions such that ψ is convex. Let $\alpha, \beta \in I$, $\alpha < \beta$, and $m, M \in \mathbb{R}$. Suppose $\mathbf{x}, \mathbf{y} \in [\alpha, \beta]^n$ are such that $\mathbf{y} \prec \mathbf{x}$. Let us denote $\bar{\beta} = \beta n$, $\bar{\alpha} = \alpha n$, and $\bar{y} = \sum_{j=1}^{n} y_j$.*

(*a*) *If φ is an (m, ψ)-lower convex function, then*

$$m\left(\frac{\bar{\beta} - \bar{y}}{\beta - \alpha}\psi(\alpha) + \frac{\bar{y} - \bar{\alpha}}{\beta - \alpha}\psi(\beta) - \sum_{i=1}^{n}a_i\psi(x_i)\right) \leq \frac{\bar{\beta} - \bar{y}}{\beta - \alpha}\varphi(\alpha) + \frac{\bar{y} - \bar{\alpha}}{\beta - \alpha}\varphi(\beta) - \sum_{i=1}^{n}a_i\varphi(x_i). \tag{3.9}$$

(*b*) *If φ is an (M, ψ)-upper convex function, then*

$$\frac{\bar{\beta} - \bar{y}}{\beta - \alpha}\varphi(\alpha) + \frac{\bar{y} - \bar{\alpha}}{\beta - \alpha}\varphi(\beta) - \sum_{i=1}^{n}a_i\varphi(x_i) \leq M\left(\frac{\bar{\beta} - \bar{y}}{\beta - \alpha}\psi(\alpha) + \frac{\bar{y} - \bar{\alpha}}{\beta - \alpha}\psi(\beta) - \sum_{i=1}^{n}a_i\psi(x_i)\right). \tag{3.10}$$

(*c*) *If φ is an (m, M, ψ)-convex function, where $m \leq M$, then* (3.9) *and* (3.10) *both hold.*

4 Applications

In this section, we derive Sherman-type inequalities for twice-differentiable (m, ψ)-lower convex, (M, ψ)-upper convex, and (m, M, ψ)-convex functions with some specified choices for function ψ.

Proposition 3. *Let $I \subseteq (0, \infty)$ be an open interval. Suppose $(\mathbf{x}, \mathbf{a}) \in I^n \times [0, \infty)^n$, $(\mathbf{y}, \mathbf{b}) \in I^k \times [0, \infty)^k$, and $\mathbf{S} \in C_{nk}([0, \infty))$ are such that $(\mathbf{y}, \mathbf{b}) \prec (\mathbf{x}, \mathbf{a})$. Let $\varphi : I \to \mathbb{R}$ be a twice-differentiable function and $g_p : I \to \mathbb{R}$ be defined by $g_p(t) = \varphi''(t)t^{2-p}$, where $p \in (-\infty, 0) \cup (1, \infty)$.*

(*a*) *If $\inf_{t \in I} g_p(t) = \gamma > -\infty$, then*

$$\frac{\gamma}{p(p-1)}\left(\sum_{i=1}^{n}a_i x_i^p - \sum_{j=1}^{k}b_j y_j^p\right) \leq \sum_{i=1}^{n}a_i\varphi(x_i) - \sum_{j=1}^{k}b_j\varphi(y_j). \tag{4.1}$$

(*b*) *If* $\sup_{t\in I} g_p(t) = \delta < \infty$, *then*

$$\sum_{i=1}^{n} a_i\varphi(x_i) - \sum_{j=1}^{k} b_j\varphi(y_j) \le \frac{\delta}{p(p-1)}\left(\sum_{i=1}^{n} a_i x_i^p - \sum_{j=1}^{k} b_j y_j^p\right). \tag{4.2}$$

(*c*) *If* $-\infty < \gamma \le g_p(t) \le \delta < \infty$, $t \in I$, *then* (4.1) *and* (4.2) *hold.*

Proof. (a) Let us consider the function $h_p(t) = \varphi(t) - \frac{\gamma}{p(p-1)}t^p$. Then,

$$h_p''(t) = \varphi''(t) - \gamma t^{p-2} = t^{p-2}(t^{2-p}\varphi''(t) - \gamma) = t^{p-2}(g_p(t) - \gamma) \ge 0,$$

i.e., h_p is convex, and then, the function φ is $\left(\frac{\gamma}{p(p-1)}, (\cdot)^p\right)$-lower convex.

Applying (3.1) to the $\left(\frac{\gamma}{p(p-1)}, (\cdot)^p\right)$-lower convex function φ, we obtain (4.1).

(b) Let us consider the function $i_p(t) = \frac{\delta t^p}{p(p-1)} - \varphi(t)$. Then,

$$i_p''(t) = \delta t^{p-2} - \varphi''(t) = t^{p-2}(\delta - t^{2-p}\varphi''(t)) = t^{p-2}(\delta - g_p(t)) \ge 0,$$

i.e., i_p is convex, and then, the function φ is $\left(\frac{\delta}{p(p-1)}, (\cdot)^p\right)$-upper convex.

Applying (3.2) to the $\left(\frac{\delta}{p(p-1)}, (\cdot)^p\right)$-upper convex function φ, we obtain (4.2).

(c) This case is a combination of (a) and (b). □

Remark 3. The special case of $p = 2$ observed in the previous proposition leads to inequalities

$$\frac{\gamma}{2}\left(\sum_{i=1}^{n} a_i x_i^2 - \sum_{j=1}^{k} b_j y_j^2\right) \le \sum_{i=1}^{n} a_i\varphi(x_i) - \sum_{j=1}^{k} b_j\varphi(y_j) \tag{4.3}$$

and

$$\sum_{i=1}^{n} a_i\varphi(x_i) - \sum_{j=1}^{k} b_j\varphi(y_j) \le \frac{\delta}{2}\left(\sum_{i=1}^{n} a_i x_i^2 - \sum_{j=1}^{k} b_j y_j^2\right), \tag{4.4}$$

which provide the tool for mean inequality estimations analyzed in the sequel.

Proposition 4. *Let* $I \subseteq (0, \infty)$ *be an open interval. Suppose* $(\mathbf{x}, \mathbf{a}) \in I^n \times [0, \infty)^n$, $(\mathbf{y}, \mathbf{b}) \in I^k \times [0, \infty)^k$ *and* $\mathbf{S} \in C_{nk}([0, \infty))$ *are such that* $(\mathbf{y}, \mathbf{b}) \prec (\mathbf{x}, \mathbf{a})$. *Let* $\varphi : I \to \mathbb{R}$ *be a twice-differentiable function and* $g : I \to \mathbb{R}$ *be defined by* $g(t) = t^2\varphi''(t)$.
(*a*) *If* $\inf_{t\in I} g(t) = \gamma > -\infty$, *then*

$$\ln\left(\frac{\prod_{j=1}^{k} y_j^{b_j}}{\prod_{i=1}^{n} x_i^{a_i}}\right)^{\gamma} \le \sum_{i=1}^{n} a_i\varphi(x_i) - \sum_{j=1}^{k} b_j\varphi(y_j). \tag{4.5}$$

(*b*) *If* $\sup_{t\in I} g(t) = \delta < \infty$, *then*

$$\sum_{i=1}^{n} a_i\varphi(x_i) - \sum_{j=1}^{k} b_j\varphi(y_j) \le \ln\left(\frac{\prod_{j=1}^{k} y_j^{b_j}}{\prod_{i=1}^{n} x_i^{a_i}}\right)^{\delta}. \tag{4.6}$$

(*c*) *If* $-\infty < \gamma \le g(t) \le \delta < \infty$, $t \in I$, *then* (4.5) *and* (4.6) *hold.*

Proof. (a) Let us consider the function $h(t) = \varphi(t) + \gamma \ln t$. Then,

$$h''(t) = \varphi''(t) - \gamma\frac{1}{t^2} = \frac{1}{t^2}(t^2\varphi''(t) - \gamma) = t^{p-2}(g(t) - \gamma) \ge 0,$$

i.e., h is convex, and then, the function φ is $(\gamma, -\ln(\cdot))$-lower convex.

Now, applying (3.1) to the $(\gamma, -\ln(\cdot))$-lower convex function φ, we obtain (4.5).

(b) Let us consider the function $i(t) = -\delta \ln t - \varphi(t)$. Then,

$$i''(t) = \delta \frac{1}{t^2} - \varphi''(t) = \frac{1}{t^2}(\delta - t^2 \varphi''(t)) = \frac{1}{t^2}(\delta - g(t)) \geq 0,$$

i.e., i is convex, and then, the function φ is $(\delta, -\ln(\cdot))$-upper convex.

Now, applying (3.2) to the $(\delta, -\ln(\cdot))$-upper convex function φ, we obtain (4.6).

(c) This case is a combination of (a) and (b). □

Proposition 5. *Let $I \subseteq (0, \infty)$ be an open interval. Suppose $(\mathbf{x}, \mathbf{a}) \in I^n \times [0, \infty)^n$, $(\mathbf{y}, \mathbf{b}) \in I^k \times [0, \infty)^k$, and $\mathbf{S} \in C_{nk}([0, \infty))$ are such that $(\mathbf{y}, \mathbf{b}) \prec (\mathbf{x}, \mathbf{a})$. Let $\varphi : I \to \mathbb{R}$ be a twice-differentiable function and $g : I \to \mathbb{R}$ be defined by $g(t) = t\varphi''(t)$.*

(*a*) *If* $\inf_{t \in I} g(t) = \gamma > -\infty$, *then*

$$\ln\left(\frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}}\right)^{\gamma} \leq \sum_{i=1}^{n} a_i \varphi(x_i) - \sum_{j=1}^{k} b_j \varphi(y_j). \tag{4.7}$$

(*b*) *If* $\sup_{t \in I} g(t) = \delta < \infty$, *then*

$$\sum_{i=1}^{n} a_i \varphi(x_i) - \sum_{j=1}^{k} b_j \varphi(y_j) \leq \ln\left(\frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}}\right)^{\delta}. \tag{4.8}$$

(*c*) *If* $-\infty < \gamma \leq g(t) \leq \delta < \infty$, $t \in I$, *then* (4.7) *and* (4.8) *hold.*

Proof. (a) Let us consider the function $h(t) = \varphi(t) - \gamma t \ln t$. Then,

$$h''(t) = \varphi''(t) - \gamma \frac{1}{t} = \frac{1}{t}(t\varphi''(t) - \gamma) = \frac{1}{t}(g(t) - \gamma) \geq 0,$$

i.e., h is convex, and then, the function φ is $(\gamma, (\cdot)\ln(\cdot))$-lower convex.

Now, applying (3.1) to the $(\gamma, (\cdot)\ln(\cdot))$-lower convex function φ, we obtain (4.7).

(b) We prove analogously that φ is $(\delta, (\cdot)\ln(\cdot))$-upper convex by observing the function $i(t) = \delta t \ln t - \varphi(t)$, and applying (3.2) to such function, we obtain (4.8).

(c) This case is a combination of (a) and (b). □

In the sequel, we apply results from the previous three propositions in order to derive new lower and upper bound for the well-known mean inequalities.

For this purpose, let us recall that for $0 < l < L$, $\mathbf{x} = (x_1, \ldots, x_n) \in (l, L)^n$, and a nonnegative n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ such that $\sum_{i=1}^{n} a_i = 1$, the weighted power mean of order $s \in \mathbb{R}$ is defined by

$$M_s(\boldsymbol{x}, \boldsymbol{a}) := \begin{cases} \left(\sum_{i=1}^{n} a_i x_i^s\right)^{\frac{1}{s}}, & s \neq 0, \\ \prod_{i=1}^{n} x_i^{a_i}, & s = 0, \\ \min\{x_1, \ldots, x_n\}, & s \to -\infty, \\ \max\{x_1, \ldots, x_n\}, & s \to \infty. \end{cases} \tag{4.9}$$

As a special case of the aforementioned definition, we obtain the classical weighted means as follows:

- arithmetic mean $A(\boldsymbol{x}, \boldsymbol{a}) = M_1(\boldsymbol{x}, \boldsymbol{a}) = \sum_{i=1}^{n} a_i x_i$,
- geometric mean $G(\boldsymbol{x}, \boldsymbol{a}) = M_0(\boldsymbol{x}, \boldsymbol{a}) = \prod_{i=1}^{n} x_i^{a_i}$,
- harmonic mean $H(\boldsymbol{x}, \boldsymbol{a}) = M_{-1}(\boldsymbol{x}, \boldsymbol{a}) = \frac{1}{\sum_{i=1}^{n} \frac{w_i}{x_i}}$.

In an analogous way, for $\mathbf{y} = (y_1, \ldots, y_k) \in (l, L)^k$ and a nonnegative k-tuple $\mathbf{b} = (b_1, \ldots, b_k)$ with $\sum_{j=1}^{k} b_j = 1$, we define

$$M_s(\boldsymbol{y}, \boldsymbol{b}) := \begin{cases} \left(\sum_{j=1}^{k} b_j y_j^s\right)^{\frac{1}{s}}, & s \neq 0, \\ \prod_{j=1}^{n} y_j^{b_j}, & s = 0, \\ \min\{y_1, \ldots, y_k\}, & s \to -\infty, \\ \max\{y_1, \ldots, y_k\} & s \to \infty, \end{cases} \tag{4.10}$$

and then, accordingly, the classical weighted means $A(\boldsymbol{y}, \boldsymbol{b})$, $G(\boldsymbol{y}, \boldsymbol{b})$, and $H(\boldsymbol{y}, \boldsymbol{b})$.

Example 1. Let $0 < l < L$ and the functions $\varphi, g_p : (l, L) \to \mathbb{R}$ be such that $g_p(t) = \varphi''(t)t^{2-p}$, where $p \in (-\infty, 0) \cup (1, \infty)$. Applying Proposition 3 to particular choices for the function φ, we obtain the following results.

(a) If $\varphi(t) = -\ln t$, then $g_p(t) = t^{-p}$ and $\inf_{t\in(l,L)} g_p(t) = L^{-p}$ and $\sup_{t\in(l,L)} g_p(t) = l^{-p}$. Applying (4.1) and (4.2), we obtain

$$\frac{L^{-p}}{p(p-1)}(M_p^p(\boldsymbol{x}, \boldsymbol{a}) - M_p^p(\boldsymbol{y}, \boldsymbol{b})) \leq \ln \frac{G(\boldsymbol{y}, \boldsymbol{b})}{G(\boldsymbol{x}, \boldsymbol{a})} \leq \frac{l^{-p}}{p(p-1)}(M_p^p(\boldsymbol{x}, \boldsymbol{a}) - M_p^p(\boldsymbol{y}, \boldsymbol{b})).$$

In a special case $p = 2$, we have

$$\frac{1}{2L^2}(M_2^2(\boldsymbol{x}, \boldsymbol{a}) - M_2^2(\boldsymbol{y}, \boldsymbol{b})) \leq \ln \frac{G(\boldsymbol{y}, \boldsymbol{b})}{G(\boldsymbol{x}, \boldsymbol{a})} \leq \frac{1}{2l^2}(M_2^2(\boldsymbol{x}, \boldsymbol{a}) - M_2^2(\boldsymbol{y}, \boldsymbol{b})).$$

(b) If $\varphi(t) = t \ln t$, then $g_p(t) = t^{1-p}$. If $p \in (-\infty, 0)$, then $\inf_{t\in(l,L)} g_p(t) = l^{1-p}$ and $\sup_{t\in(l,L)} g_p(t) = L^{1-p}$ and we have

$$\frac{M_p^p(\boldsymbol{x}, \boldsymbol{a}) - M_p^p(\boldsymbol{y}, \boldsymbol{b})}{p(p-1)l^{p-1}} \leq \ln \frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}} \leq \frac{M_p^p(\boldsymbol{x}, \boldsymbol{a}) - M_p^p(\boldsymbol{y}, \boldsymbol{b})}{p(p-1)L^{p-1}}. \tag{4.11}$$

If $p \in (1, \infty)$, then $\inf_{t\in(l,L)} g_p(t) = L^{1-p}$ and $\sup_{t\in(l,L)} g_p(t) = l^{1-p}$, and the inequalities in (4.11) are reversed. In a special case $p = 2$, we have

$$\frac{1}{2L}(M_2^2(\boldsymbol{x}, \boldsymbol{a}) - M_2^2(\boldsymbol{y}, \boldsymbol{b})) \leq \ln \frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}} \leq \frac{1}{2l}(M_2^2(\boldsymbol{x}, \boldsymbol{a}) - M_2^2(\boldsymbol{y}, \boldsymbol{b})).$$

Example 2. Let $0 < l < L$ and the functions $\varphi, g : (l, L) \to \mathbb{R}$ be such that $\varphi(t) = t \ln t$ and $g(t) = t^2\varphi''(t) = t$. Then, $\inf_{t\in(l,L)} g(t) = l$ and $\sup_{t\in(l,L)} g(t) = L$.

Applying (4.5) and (4.6), we obtain

$$\left(\frac{G(\boldsymbol{y}, \boldsymbol{b})}{G(\boldsymbol{x}, \boldsymbol{a})}\right)^l \leq \frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}} \leq \left(\frac{G(\boldsymbol{y}, \boldsymbol{b})}{G(\boldsymbol{x}, \boldsymbol{a})}\right)^L.$$

Example 3. Let $0 < l < L$ and the functions $\varphi, g_p : (l, L) \to \mathbb{R}$ be such that $g_p(t) = t\varphi''(t)$, where $p \in (-\infty, 0) \cup (1, \infty)$. Applying Proposition 5 to particular choices for the function φ, we obtain the following results.

(a) If $\varphi(t) = -\ln t$, then $g_p(t) = \frac{1}{t}$ and $\inf_{t\in(l,L)} g_p(t) = \frac{1}{L}$ and $\sup_{t\in(l,L)} g_p(t) = \frac{1}{l}$. Now, applying (4.7) and (4.8), we have

$$\left(\frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}}\right)^{\frac{1}{L}} \le \frac{G(\boldsymbol{y},\boldsymbol{b})}{G(\boldsymbol{x},\boldsymbol{a})} \le \left(\frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}}\right)^{\frac{1}{l}}.$$

(b) If $\varphi(t) = e^t$, then $g_p(t) = te^t$ and $\inf_{t\in(l,L)} g_p(t) = le^l$ and $\sup_{t\in(l,L)} g_p(t) = Le^L$.

Applying (4.7) and (4.8), we obtain

$$\ln\left(\frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}}\right)^{le^l} \le \sum_{i=1}^{n} a_i e^{x_i} - \sum_{j=1}^{k} b_j e^{y_j} \le \ln\left(\frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\prod_{j=1}^{k} y_j^{b_j y_j}}\right)^{Le^L}.$$

5 Conclusion

In this article, we deal with the generalized class of convex functions known as (m, M, ψ)-convex functions and its subclasses (m, ψ)-lower convex and (M, ψ)-upper convex functions. Since these classes encompass numerous other types of convexity obtained results we can use and interpret in the context of different classes of convex functions. Moreover, our results generalize and refine the corresponding results valid for other classes of convex functions. To specify a little, in this study, we prove some useful characterizations of these generalized convex functions that enable us to connect them with an important mathematical concept vector majorization. We derive the appropriate majorization inequalities, which we then use to obtain lower and upper bounds between well-known means. The presented ideas explained in propositions from Section 4 can be extended to different specified functions, which opens up the possibility of different applications and proving new results for different classes of convex functions.

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