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Research Article

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Local minimizers for the NLS equation with localized nonlinearity on noncompact metric graphs

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Abstract: We investigate the existence of local minimizers for the nonlinear Schrödinger (NLS) equation with localized nonlinearity on noncompact metric graphs. In the absence of ground states, we prove that normalized local minimizers of the NLS equation do exist under suitable topological and metric assumptions of the graphs. In particular, we provide a criterion for the existence of local minimizers for the NLS equation in this article. Our results rely on the variational method and an application of Gagliardo-Nirenberg inequalities.

Keywords: normalized local minimizer, NLS equation, localized nonlinearity, noncompact metric graph

MSC 2020: 35R02, 35Q55, 49J40

1 Introduction and main results

In this article, we deal with the so-called L^2 mass-critical nonlinear Schrödinger (NLS) energy functional with localized nonlinearity

$$E(u, G, \mathcal{K}) = \frac{1}{2}\int_G |u'|^2 \mathrm{d}x - \frac{1}{6}\int_{\mathcal{K}} |u|^6 \mathrm{d}x, \tag{1.1}$$

under the mass constraint

$$u \in H^1_\mu(G) := \left\{u \in H^1(G), \int_G |u|^2 \mathrm{d}x = \mu\right\}, \tag{1.2}$$

where $\mu > 0$, G is a noncompact metric graph and $\mathcal{K}$ is the compact core of the graph G.

We briefly recall that the noncompact metric graph G has finite number of edges $\{e_j\}_{j=1}^n$ (bounded or unbounded) and vertices (that not at infinity), with a metric structure on any edge. A bounded edge e is identified with an interval $I_e = [0, l_e]$, where l_e denotes the length of e. Any unbounded edge e is referred to as the positive half-line $I_e = \mathbb{R}^+ = [0, +\infty)$. G is noncompact, which means that it has at least one unbounded edge. The compact core $\mathcal{K}$ is defined as the subgraph by removing all the half-lines of the metric graph G (see [1–4] for more details). It is clear that $\mathcal{K}$ is connected. We refer the interested readers to [5–9] for different kinds of graphs that have infinite many edges and vertices.

A function $u = u(x)$ on the graph G can be defined as a vector

$$u = (u_{e_1}, u_{e_2}, \ldots, u_{e_n}), \quad \text{with } u_{e_j} = u_{e_j}(x), x \in e_j, \text{for every } j = 1, 2, \ldots, n.$$

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The continuity of u in the interior of any edge e is the normal continuity, while the continuity of u at a intersection x_0 of any two different edges e_1 and e_2 means that $u_{e_1}(x_0) = u_{e_2}(x_0)$. The Lebesgue spaces $L^p(G)$ $(1 < p < +\infty)$ are defined as

$$L^p(G) := \left\{ u : G \to \mathbb{R}, \int_G |u(x)|^p \mathrm{d}x = \sum_{j=1}^{n} \int_{I_{e_j}} |u_{e_j}(x)|^p \mathrm{d}x < +\infty \right\},$$

with the corresponding norm

$$\|u\|_{L^p(G)} := \left(\int_G |u(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

In a similar way, we can define the Sobolev spaces

$$H^1(G) := \left\{ u \text{ is continuous on } G \text{ and } \sum_{j=1}^{n} \int_{I_{e_j}} (|u_{e_j}(x)|^2 + |u'_{e_j}(x)|^2)\mathrm{d}x < +\infty \right\},$$

with the corresponding norm

$$\|u\|_{H^1(G)} := (\|u'\|^2_{L^2(G)} + \|u\|^2_{L^2(G)})^{\frac{1}{2}} = \left(\sum_{j=1}^{n} \int_{I_{e_j}} (|u_{e_j}|^2 + |u'_{e_j}|^2)\mathrm{d}x \right)^{\frac{1}{2}}.$$

In this framework, Dovetta and Tentarelli have studied the existence of ground states (global minimizers) of NLS energy functional (1.1) under the mass constraint (1.2) in [10] and [11], namely, the minimization problem

$$\mathcal{E}(\mu, G, \mathcal{K}) := \inf_{u \in H^1_\mu(G)} E(u, G, \mathcal{K}), \tag{1.3}$$

where $E(u, G, \mathcal{K})$ is the energy functional in (1.1).

Precisely, Dovetta and Tentarelli first proved the existence and nonexistence of ground states for $\mathcal{E}(\mu, G, \mathcal{K})$ on a special metric graph – the tadpole graph in [10]. Ground states existed when the given mass μ belonged to a suitable interval, whereas for large masses and small masses, ground states did not exist.

Subsequently, Dovetta and Tentarelli [11] had a full discussion of the existence and nonexistence of ground states influenced by the topological and metric properties of the graphs (we will provide these conclusions in the following text of this section). It is worth mentioning that the discussion in [11] involves the classification of the noncompact metric graphs. We point out that the classification method in [11] is borrowed directly from [12], where all of the metric graphs are divided into four mutually exclusive types.

The main purpose of this article is to investigate, under what topology and metric conditions, the existence of local minimizers for the energy functional $E(u, G, \mathcal{K})$ in the space $H^1_\mu(G)$ when the ground states do not exist. The inspiration for this article comes from Pierotti et al. [13], where they showed that, under suitable topology and metric assumptions, the energy functional (2.4) did have local minimizer under the mass constraint (1.2) in the absence of ground states.

As is known, critical points (including ground states and local minimizers) of energy functional $E(u, G, \mathcal{K})$ in the space $H^1_\mu(G)$ satisfy the following NLS equation:

$$u'' + \chi_{\mathcal{K}} |u|^4 u = \omega u, \quad x \in G, \tag{1.4}$$

where $\chi_{\mathcal{K}}$ is the characteristic function of $\mathcal{K}$ and the parameter ω plays the role of a Lagrange multiplier (which is the same on every edge, but may be different for different solutions). Through the usual ansatz

$\Phi(x,t) = e^{i\omega t}u(x)$, the solutions of (1.4) correspond to solitary wave solutions of the time-dependent NLS equation:

$$i\partial_t\Phi + \partial_{xx}\Phi + \chi_{\mathcal{K}}\,|\Phi|^4\Phi = 0, \quad x \in G,\ t > 0. \tag{1.5}$$

The study of the differential models on metric graphs has recently become a quite popular topic and has been witnessing a significant growth. For the linear case, we refer the readers to [14–17] and references therein for some of the most recent developments. For the nonlinear case, an obvious preference has been devoted to the Schrödinger equations, and we refer the readers to [6,13,18–20] and references therein. We also mention two other nonlinear models, [21] for the Korteweg-de Vries equation and [22,23] for the nonlinear Dirac equation. In particular, previous works [24–26] are devoted to the NLS equations with L^2 mass-subcritical localized nonlinearities (the exponent of the nonlinearity belongs to the interval (2,6)), while Borthwick et al. [27] dealt with the L^2 mass-supercritical localized nonlinearities (the exponent of the nonlinearity is greater than 6).

Due to the characteristics of the energy functional $E(u, G, \mathcal{K})$ in this article, we limit ourselves to the noncompact metric graphs G satisfying

(**A**): G is connected, noncompact, with a finite number of edges and with nonempty compact core $\mathcal{K}$.

In this case, some special metric graphs (such as the half-line $\mathbb{R}^+$, the real line $\mathbb{R}$, and the general star graphs) are not within the scope of our discussion. A general star graph is made up of $N \geq 3$ half-lines that meet at a unique vertex (see an example in Figure 1). In order to better state our main results in the following, we assume without loss of generality that any vertex of G has a degree different from 2 unless G is (isometric to) $\mathbb{R}$, which can be decomposed into two half-lines only allowing a vertex of degree 2. See Remark 5.9 in [13] for more about the discussion of the "fake vertex" (with degree 2) of the metric graph G.

With respect to this article, it is worth recalling some results in [11] that deal with the existence (or nonexistence) of ground states on noncompact metric graphs. If G satisfies condition (**A**), then there exists a so-called *reduced critical mass* $\mu_{\mathcal{K}} \in [\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$ such that

$$\mathcal{E}(\mu, G, \mathcal{K}) \begin{cases} = 0, & \text{if } \mu \leq \mu_{\mathcal{K}}, \\ < 0, & \text{if } \mu \in (\mu_{\mathcal{K}}, \mu_{\mathbb{R}}], \\ = -\infty, & \text{if } \mu > \mu_{\mathbb{R}}, \end{cases} \tag{1.6}$$

where

$$\mu_{\mathcal{K}} := \sqrt{\frac{3}{C_{\mathcal{K}}}}, \tag{1.7}$$

and $C_{\mathcal{K}}$ denotes the best constant of the following Gagliardo-Nirenberg-type inequality

$$\|u\|^6_{L^6(\mathcal{K})} \leq C_{\mathcal{K}}\, \|u\|^4_{L^2(G)}\|u'\|^2_{L^2(G)}, \quad \forall u \in H^1(G), \tag{1.8}$$

namely,

$$C_{\mathcal{K}} := \sup_{u \in H^1(G)\setminus\{0\}} \frac{\|u\|^6_{L^6(\mathcal{K})}}{\|u\|^4_{L^2(G)}\|u'\|^2_{L^2(G)}}. \tag{1.9}$$

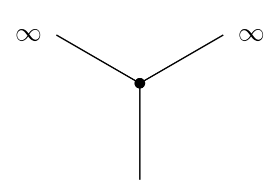

Figure 1: N-star graph ($N = 3$).

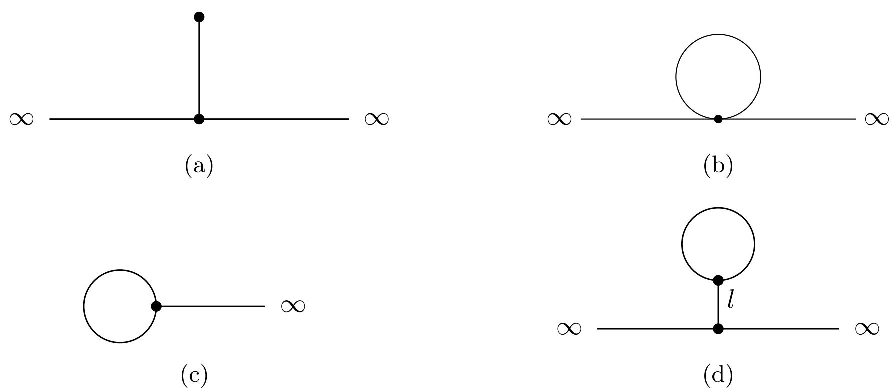

Figure 2: Examples of the four mutually exclusive kinds of metric graphs: (a) a graph with a terminal point, (b) a graph with a cycle-covering, (c) a tadpole graph, and (d) a sign-post graph.

Moreover,

(i) if G has at least one terminal edge (Figure 2 (a)), then

$$\mu_{\mathcal{K}} = \mu_{\mathbb{R}^+}, \mathcal{E}(\mu, G, \mathcal{K}) = -\infty, \quad \text{for all } \mu > \mu_{\mathcal{K}},$$

and ground states never exist;

(ii) if G admits a cycle covering (Figure 2 (b)), then

$$\mu_{\mathcal{K}} = \mu_{\mathbb{R}},$$

and ground states never exist;

(iii) if G has exactly one half-line and no terminal edges (Figure 2 (c)), then

$$\mu_{\mathbb{R}^+} < \mu_{\mathcal{K}} < \sqrt{3},$$

and ground states of mass μ exist if and only if $\mu \in [\mu_{\mathcal{K}}, \mu_{\mathbb{R}}]$;

(iv) if G does not belong to any of the previous cases (Figure 2 (d)), then

$$\mu_{\mathbb{R}^+} < \mu_{\mathcal{K}} \leq \mu_{\mathbb{R}},$$

and ground states of mass μ exist if and only if $\mu \in [\mu_{\mathcal{K}}, \mu_{\mathbb{R}}]$, provided that $\mu_{\mathcal{K}} \neq \mu_{\mathbb{R}}$ (by Proposition 4.3 of [11], we know that $\mu_{\mathcal{K}} \neq \mu_{\mathbb{R}}$ when G is the sign-post graph in Figure 2 (d) accompanied by the “cut edge” l long enough).

In this article, we do not intend to have a comprehensive discussion on all types of the metric graphs mentioned earlier. Throughout this article, we focus on the first two types:

(1) G has at least one terminal edge;
(2) G admits a cycle covering.

A terminal edge means an edge that ends with a vertex of degree 1. A metric graph G admits a cycle covering if and only if every edge of G belongs to a cycle, which is either a loop (a closed path of consecutive bounded edges) or an unbounded path joining the finite endpoints of two distinct half-lines.

Our main results are the following.

Theorem 1.1. *Let G satisfy* (**A**) *and have at least a terminal edge. If G has at least two half-lines, then there exists $\mu_1 \in (0, \mu_{\mathcal{K}})$ such that for every $\mu \in (\mu_1, \mu_{\mathcal{K}})$, the energy functional* (1.1) *has a critical point in the space $H^1_\mu(G)$, which is a local minimizer.*

Theorem 1.2. *Let G satisfy* (**A**) *and admit a cycle-covering. If $G \backslash l$ admits a cycle-covering for any half-line l, then there exists $\mu_2 \in (0, \mu_{\mathcal{K}})$ such that for every $\mu \in (\mu_2, \mu_{\mathcal{K}})$, the energy functional* (1.1) *has a critical point in the space $H^1_\mu(G)$, which is a local minimizer.*

Remark 1.3. Note that the simplest metric graph described in Theorem 1.1 is in Figure 2 (a), which is made up of one bounded edge and two half-lines. The symbol $G\backslash l$ in Theorem 1.2 denotes the remaining graph by removing any half-line l from G. By the definition of metric graph G admitting a cycle covering, we know that G has at least two half-lines. As a consequence, the graphs described in Theorem 1.2 have at least three half-lines (Figure 3).

Remark 1.4. Based on the aforementioned results in cases (i) and (ii) of [11], it is clear that there are no ground states for $\mathcal{E}(\mu, G, \mathcal{K})$ when the given mass $\mu \in (0, \mu_{\mathcal{K}})$. Actually, the ground states do not exist for any $\mu > 0$. As a consequence, it is meaningful to investigate the existence of local minimizers in Theorems 1.1 and 1.2 because the local minimizers are natural candidates to correspond to orbitally stable solitary waves [28]. Finally, we point out that there are some interesting open problems, such as the search for local minimizers of $E(u, G, \mathcal{K})$ in the space $H^1_\mu(G)$ when $\mu \in (0, \mu_{\mathcal{K}})$ and the metric graph G belongs to case (iii) or (iv).

This article is organized as follows. In Section 2, we introduce some preliminary results that will be helpful for the subsequent proof and briefly state the idea of searching for the local minimizers. In Section 3, we give a criterion for the existence of local minimizers by the variational principle. Finally, Section 4 is devoted to the proofs of Theorems 1.1 and 1.2 using the criterion established in Section 3.

2 Preliminaries

In this section, we first recall a classic Gagliardo-Nirenberg inequality on noncompact metric graph and the corresponding critical mass.

As is known, for any $q > 2$ and any non-compact metric graph G, there exists an optimal constant $C_q(G) > 0$ depending on the exponent q and the metric graph G such that (see [29])

$$\|u\|^q_{L^q(G)} \le C_q(G)\|u\|^{\frac{q}{2}+1}_{L^2(G)}\|u'\|^{\frac{q}{2}-1}_{L^2(G)}, \quad \forall u \in H^1(G), \tag{2.1}$$

where $C_q(G)$ is characterized as

$$C_q(G) := \sup_{u\in H^1(G)\backslash\{0\}} \frac{\|u\|^q_{L^q(G)}}{\|u\|^{\frac{q+2}{2}}_{L^2(G)}\|u'\|^{\frac{q-2}{2}}_{L^2(G)}}.$$

When $q = 6$, it reads

$$\|u\|^6_{L^6(G)} \le C_6(G)\|u\|^4_{L^2(G)}\|u'\|^2_{L^2(G)}, \quad \forall u \in H^1(G), \tag{2.2}$$

where the sharpest constant $C_6(G)$ depends only on G. In [12], the definition of μ_G is

$$\mu_G := \sqrt{\frac{3}{C_6(G)}},$$

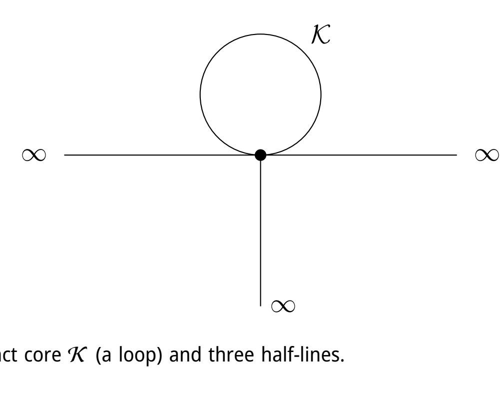

Figure 3: Metric graph with a compact core $\mathcal{K}$ (a loop) and three half-lines.

which is the L^2-critical mass. Comparing μ_G and $\mu_{\mathcal{K}}$ defined in (1.7), then for a general noncompact metric graph G satisfying (**A**), one can check that

$$\mu_{\mathcal{K}} \geq \mu_G.$$

In addition, let us also mention some useful results on the L^2-critical problem with the nonlinearity extended on the whole metric graph G, namely (see [12]),

$$\mathcal{E}(\mu, G) := \inf_{u \in H^1_\mu(G)} E(u, G), \tag{2.3}$$

where

$$E(u, G) := \frac{1}{2}\int_G |u'|^2 \mathrm{d}x - \frac{1}{6}\int_G |u|^6 \mathrm{d}x. \tag{2.4}$$

When $G = \mathbb{R}$, it holds

$$\mathcal{E}(\mu, \mathbb{R}) = \begin{cases} 0, & \text{if } \mu \leq \mu_{\mathbb{R}}, \\ -\infty, & \text{if } \mu > \mu_{\mathbb{R}}, \end{cases} \tag{2.5}$$

where $\mu_{\mathbb{R}} = \frac{\sqrt{3}\pi}{2}$ and $\mathcal{E}(\mu, \mathbb{R})$ is achieved if and only if $\mu = \mu_{\mathbb{R}}$. In this case, the ground states, called solitons, can be written as

$$\phi_\lambda(x) = \sqrt{\lambda}\,\phi(\lambda x), \quad \lambda > 0, \tag{2.6}$$

where

$$\phi(x) = \operatorname{sech}^{\frac{1}{2}}\left(\frac{2x}{\sqrt{3}}\right).$$

Analogously, when $G = \mathbb{R}^+$, it holds

$$\mathcal{E}(\mu, \mathbb{R}^+) = \begin{cases} 0, & \text{if } \mu \leq \mu_{\mathbb{R}^+}, \\ -\infty, & \text{if } \mu > \mu_{\mathbb{R}^+}, \end{cases} \tag{2.7}$$

where $\mu_{\mathbb{R}^+} = \frac{\mu_{\mathbb{R}}}{2}$ and $\mathcal{E}(\mu, \mathbb{R}^+)$ is achieved if and only if $\mu = \mu_{\mathbb{R}^+}$, with the ground states called half-solitons whose expressions are also (2.6). It is obvious that

$$E(\phi_\lambda, \mathbb{R}) = E(\phi_\lambda, \mathbb{R}^+) = 0, \quad \forall \lambda > 0. \tag{2.8}$$

For a general metric graph G, the critical mass μ_G satisfies (see Proposition 2.3 in [12])

$$\mu_{\mathbb{R}^+} \leq \mu_G \leq \mu_{\mathbb{R}}.$$

Moreover, observe that we focus on the mass $\mu \in (0, \mu_{\mathcal{K}})$ in this article. In view of this, fixing $\mu \in (0, \mu_{\mathcal{K}})$, then by the Gagliardo-Nirenberg-type inequality (1.8) and the definition of the so-called critical mass $\mu_{\mathcal{K}}$, we have

$$\begin{aligned} E(u, G, \mathcal{K}) &= \frac{1}{2}\,\|u'\|^2_{L^2(G)} - \frac{1}{6}\,\|u\|^6_{L^6(\mathcal{K})} \\ &\geq \frac{1}{2}\left(1 - \left(\frac{\mu}{\mu_{\mathcal{K}}}\right)^2\right)\|u'\|^2_{L^2(G)}, \quad \forall u \in H^1_\mu(G). \end{aligned} \tag{2.9}$$

This follows that the energy functional $E(u, G, \mathcal{K})$ is always bounded from below in the space $H^1_\mu(G)$. Without loss of generality, throughout this article we can restrict ourselves to the non-negative real functions.

Although the lower boundedness is satisfied, there are still two difficulties in obtaining the critical point of the energy functional $E(u, G, \mathcal{K})$ in the space $H^1_\mu(G)$. On the one hand, G is noncompact and this will lead to a failure of convergence of the minimizing or Palais-Smale sequences. On the other hand, G has a nonempty compact core $\mathcal{K}$ and this feature will prevent us from using the scaling method to obtain solutions of given mass.

In order to search the local minimizers (critical points) for this article, in Section 3, we construct a suitable subspace $\mathcal{A}$ of $H^1_\mu(G)$ using the compact core $\mathcal{K}$ of graph G. Then, by the variational principle, we find a special minimizing sequence in $\mathcal{A}$ and prove its strong convergence in $\mathcal{A}$. We know that ground states never exist for every $\mu \in (0, \mu_{\mathcal{K}})$, and thus, the strong limit of the special minimizing sequence is a local minimizer in the set $\mathcal{A}$.

3 Criterion for the existence of local minimizers

In this section, we deal with a general compactness discussion for suitable locally minimizing sequences and give a criterion for the existence of local minimizers. Then, the criterion will be applied directly in the proof of our main results in Section 4.

Let G satisfy (**A**). Throughout this section, we denote the finite number of vertices of G as $\{v_1, v_2, \ldots, v_m\}$, the bounded edges as $\{e_1, e_2, \ldots, e_p\}$, and the half-lines as $\{l_1, l_2, \ldots, l_s\}$. Any function $u \in H^1(G)$ is identified with a vector

$$u \coloneqq (u_1, u_2, \ldots, u_s, v),$$

where $u_i \in H^1(l_i)$ is the restriction of u on the half-line l_i and $v \in H^1(\mathcal{K})$ is the restriction of u on the compact core $\mathcal{K}$. Analogously, we can denote $v_k \in H^1(e_k)$ as the restriction of v on every bounded edge e_k. Then, function u can be denoted as

$$u \coloneqq (u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_p).$$

If any edge l_i or e_k is incident to the vertex v_j, we record it as $l_i \succ v_j$ or $e_k \succ v_j$.

Let us consider $0 < \zeta < \mu < \mu_{\mathcal{K}}$, $\delta > 0$ such that

$$\mu - \zeta < \mu_{\mathbb{R}^+} \quad \text{and} \quad (1+\delta)\zeta < \mu. \tag{3.1}$$

We introduce two subsets of $H^1_\mu(G)$ as

$$\mathcal{A} \coloneqq \left\{ u \in H^1_\mu(G) : \zeta \le \int_{\mathcal{K}} |u|^2 \mathrm{d}x \right\}$$

and

$$\mathcal{B} \coloneqq \left\{ u \in H^1_\mu(G) : \zeta \le \int_{\mathcal{K}} |u|^2 \mathrm{d}x \le (1+\delta)\zeta \right\},$$

where $\mathcal{K}$ is the compact core of G. It is clear that

$$\mathcal{B} \subseteq \mathcal{A} \subseteq H^1_\mu(G).$$

Then, we can obtain a variational principle as follows.

Proposition 3.1. *Let* (3.1) *hold and*

$$-\infty < \inf_{u \in \mathcal{A}} E(u, G, \mathcal{K}) < \inf_{u \in \mathcal{B}} E(u, G, \mathcal{K}). \tag{3.2}$$

Then, the energy functional $E(\cdot, G, \mathcal{K})$ constrained in the space $H^1_\mu(G)$ has a critical point, which is a local minimizer in set $\mathcal{A}$.

Remark 3.2. According to the argument in Section 2, the energy functional $E(\cdot, G, \mathcal{K})$ is bounded from below for every $\mu \in (0, \mu_{\mathcal{K}})$. Naturally, it is also bounded from below in $\mathcal{A}$.

Let $\{\bar{u}_n\}$ be a minimizing sequence for the energy functional $E(\cdot, G, \mathcal{K})$ in set $\mathcal{A}$. Then, by the Ekeland variational principle, Theorem 1.1 in [30], there exists a (possibly different) minimizing sequence $\{u_n\}$ in $\mathcal{A}$ such that

$$\|\bar{u}_n - u_n\|_{H^1_\mu(G)} \to 0, \quad \text{as } n \to \infty.$$

Let us define the tangent space of $H^1_\mu(G)$ at u_n as

$$T_{u_n}(H^1_\mu(G)) := \left\{ v \in H^1(G) : \int_G u_n v \mathrm{d}x = 0 \right\}.$$

In light of Proposition 5.1 in [30], we have

$$\|\mathrm{d}E(u_n, G, \mathcal{K})\|_* \to 0, \quad \text{as } n \to \infty,$$

where $\|\cdot\|_*$ denotes the norm in the dual space of $T_{u_n}(H^1_\mu(G))$ (see [24] for more details).

Based on the aforementioned discussion, we call $\{u_n\}$ the minimizing Palais-Smale sequence of the energy functional $E(\cdot, G, \mathcal{K})$ in $\mathcal{A}$ and denote

$$u_n := (u_{1,n}, u_{2,n}, \ldots, u_{s,n}, v_{1,n}, v_{2,n}, \ldots, v_{p,n}).$$

By the way, we point out that the assumption of the second strict inequality in (3.2) is to ensure that the minimizing sequence in $\mathcal{A}$ maintains a certain distance away from the boundary of $\mathcal{A}$.

The proof process of Proposition 3.1 is somewhat complex, and we will divide it into the following lemmas.

Lemma 3.3. *Let $\{\bar{u}_n\}$ be a minimizing sequence for the energy functional $E(\cdot, G, \mathcal{K})$ in $\mathcal{A}$, then there exists a minimizing Palais-Smale sequence $\{u_n\}$ for the energy functional $E(\cdot, G, \mathcal{K})$ in $\mathcal{A}$. Moreover, $\{u_n\}$ is bounded in $H^1(G)$, and thus, up to a subsequence, there exists a function $u \in H^1(G)$ such that*

$$\begin{cases} \bar{u}_n, u_n \rightharpoonup u, & \text{in } H^1(G), \text{ as } n \to \infty, \\ \bar{u}_n, u_n \to u, & \text{in } L^\infty_{\mathrm{loc}}(G), \text{ as } n \to \infty. \end{cases} \tag{3.3}$$

Proof. By the argument in Remark 3.2, we are left to show that $\{u_n\}$ is bounded in $H^1(G)$. The results in (3.3) are a direct consequence of the boundedness of $\{u_n\}$ in $H^1(G)$. Indeed, by (3.2) and (2.9), one can check that

$$\frac{1}{2}\left(1 - \left(\frac{\mu}{\mu_{\mathcal{K}}}\right)^2\right) \|u_n'\|^2_{L^2(G)} < \inf_{u \in \mathcal{B}} E(u_n, G, \mathcal{K}), \quad \text{as } n \to \infty,$$

which implies that $\{u_n\}$ is bounded in $H^1(G)$ since $\mu \in (0, \mu_{\mathcal{K}})$. □

Lemma 3.4. *Let $\{u_n\}$ be the minimizing Palais-Smale sequence for the energy functional $E(\cdot, G, \mathcal{K})$ and u be the weak limit in* Lemma 3.3, *then there exists $\omega \in \mathbb{R}$ such that*

$$\begin{cases} -u_i'' = \omega u_i, & u_i > 0, \text{ on } l_i, \text{ for every } i = 1, 2, \ldots, s, \\ -v_k'' = \omega v_k + v_i^5, & v_k > 0, \text{ on } e_k, \text{ for every } k = 1, 2, \ldots, p, \\ \sum\limits_{l_i \succ v_j} u_i'(v_j) + \sum\limits_{e_k \succ v_j} v_k'(v_j) = 0, & \text{for every } j = 1, 2, \ldots, m, \end{cases} \tag{3.4}$$

and

$$u(v_j) > 0, \quad \text{for every vertex } v_j. \tag{3.5}$$

Proof. Let us preliminarily do some explanations for Lemma 3.4. The symbol $v_k'(v_j)$ in (3.4) is a shorthand notation for $v_k'(0^+)$ or $-v_k'(l_{e_k}^-)$, according to whether the coordinate is equal to 0 or l_{e_k} at v_j. Similarly,

$u_i'(v_j)$ represents $u_i'(0^+)$ excluding the vertex at infinity. We say $u(v_j) > 0$ means that

$$u_i(v_j) = v_k(v_j) > 0,$$

for every $i = 1, 2, \ldots, s$ or $k = 1, 2, \ldots, p$ as long as the edge l_i or e_k is incident at v_j.

First, we show that the three equations in (3.4) hold, namely,

$$\begin{cases} -u_i'' = \omega u_i, & \text{on } l_i, \text{ for every } i = 1, 2, \ldots, s, \\ -v_k'' = \omega v_k + v_i^5, & \text{on } e_k \text{ for every } k = 1, 2, \ldots, p, \\ \sum\limits_{l_i \succ v_j} u_i'(v_j) + \sum\limits_{e_k \succ v_j} v_k'(v_j) = 0, & \text{for every } j = 1, 2, \ldots, m. \end{cases} \tag{3.6}$$

In fact, let $\psi \in H^1(G)$. Obviously, the energy functional $E(\cdot, G, \mathcal{K})$ is C^1. According to Lemma 3 in [31] and the argument in [24], since $\{u_n\}$ is bounded in $H^1(G)$, we have for some approximate Lagrange multiplier $\omega_n \in \mathbb{R}$ that

$$\mathrm{d}E(u_n, G, \mathcal{K})[\psi] - \omega_n \int_G u_n \psi \mathrm{d}x = o(1), \quad \text{as } n \to \infty, \tag{3.7}$$

where ω_n is given by

$$\omega_n = \frac{1}{\mu}\left(\int_G |u_n'|^2 \mathrm{d}x - \int_{\mathcal{K}} |u_n|^6 \mathrm{d}x\right).$$

By the boundedness of $\{u_n\}$ in $H^1(G)$ and the Gagliardo-Nirenberg-type inequality (1.8), we immediately obtain that ω_n is bounded and

$$\omega_n \to \omega \in \mathbb{R}.$$

By weak convergence in (3.7), then as $n \to \infty$, one can check that

$$\int_G u'\psi' \mathrm{d}x - \int_{\mathcal{K}} |u|^4 u\psi \mathrm{d}x - \omega \int_G u\psi \mathrm{d}x = 0, \quad \forall \psi \in H^1(G), \tag{3.8}$$

where ω plays the role of a Lagrange multiplier.

For every edge I_e (including the half-line l_i and bounded edge e_k) of G, choosing $\psi \in C_0^\infty(I_e)$ and integrating by parts into (3.8), we obtain that the first two equations in (3.6) hold. For any given vertex v_j (not at infinity), such as v_1, let $\psi \in H^1(G)$ such that

$$\psi(v_j) \begin{cases} = 0, & \text{for every } j = 2, \ldots, m, \\ \neq 0, & \text{for } j = 1. \end{cases} \tag{3.9}$$

Integrating by parts into (3.8) and noting the fact that the first two equations in (3.6) hold, we have

$$\left(\sum_{l_i \succ v_1} u_i'(v_1) + \sum_{e_k \succ v_1} v_k'(v_1)\right)\psi(v_1) = 0.$$

Due to the arbitrariness of $\psi(v_1)$, the third equation in (3.6) holds for $j = 1$. Other cases can be proved by repeating the aforementioned process.

Second, we will prove the two inequalities in (3.4) and the inequality in (3.5) hold, namely,

$$\begin{cases} u_i > 0, & \text{on } l_i \text{ for every } i = 1, 2, \ldots, s, \\ v_k > 0, & \text{on } e_k \text{ for every } k = 1, 2, \ldots, p, \\ u(v_j) > 0, & \text{for every } j = 1, 2, \ldots, m. \end{cases} \tag{3.10}$$

In fact, noting that $u_n \geq 0$, we also have $u \geq 0$.

To prove $u(v_j) > 0$ for any vertex v_j, without loss of generality, we suppose $u(v_1) = 0$ by contradiction. Since $u \geq 0$ on G, we have

$$u_i'(v_1), v_k'(v_1) \geq 0, \quad \text{whenever } l_i, e_k \succ v_1. \tag{3.11}$$

Combining (3.11) with the Kirchhoff condition (the third equality) in (3.4), we immediately obtain that all the derivatives at v_1 are equal to 0. Thus, by uniqueness of solutions for the Cauchy problems associated with the first two equations in (3.4) on every edge that is incident at v_1, one sees that

$$u_i \equiv 0, \quad \text{for } l_i \succ v_1,$$

and

$$v_k \equiv 0, \quad \text{for } e_k \succ v_1,$$

which imply that $u(v_j) = 0$ for every vertex v_j directly connected to v_1. In fact, since G satisfies (**A**), we deduce that $u \equiv 0$ on G by iterating the aforementioned argument a finite number of times. However, this is a contradiction with

$$\int_{\mathcal{K}} |u|^2 \mathrm{d}x = \lim_{n\to\infty} \int_{\mathcal{K}} |u_n|^2 \mathrm{d}x \geq \zeta > 0,$$

where we utilize the fact that $u_n \to u$ locally uniformly on G.

To prove the first two inequalities in (3.10), without loss of generality, we suppose $u_1(x) = 0$ (or $v_1(x) = 0$) for any interior point of the half-line l_1 (or bounded edge e_1) by contradiction. Since $u \geq 0$ on G, by the definition of the derivative at point x, we have

$$u_1'(x) = 0 \quad (\text{or } v_1'(x) = 0).$$

By uniqueness of solutions for the Cauchy problems associated with the first two equations in (3.4) on every edge, we obtain that

$$u_1 \equiv 0 \text{ on } l_1 \quad (\text{or } v_1 \equiv 0 \text{ on } e_1),$$

which indicates that $u(v) = 0$ at the vertex $v \prec l_1$ (or $v \prec e_1$). Arguing as mentioned earlier, we have that the first two inequalities in (3.10) hold. □

According to the discussion of Proposition 3.3 in [32], it can be concluded that the weak limit u obtained in Lemma 3.3 has satisfied some important properties of the solution to (1.4). Is this function u actually a critical point? The following lemma provides the answer.

Lemma 3.5. *Let $\{u_n\}$ be the minimizing Palais-Smale sequence for the energy functional $E(\cdot, G, \mathcal{K})$ and u be the weak limit in* Lemma 3.3, *then $u_n \to u$ strongly in $H^1(G)$ and u is a local minimizer for the energy functional $E(\cdot, G, \mathcal{K})$ in set $\mathcal{A}$.*

Proof. First, by (3.6), we deduce that the Lagrange multiplier ω is negative. Indeed, by the property of u_i on the half-line l_i, it is not difficult to deduce that $\omega \neq 0$. On the other hand, if $\omega > 0$, then the function u_i is C^2 on $l_i = (0, +\infty)$, concave, strictly positive, and tending to 0 as $x \to \infty$. This is impossible.

Second, since $\{u_n\}$ is a bounded minimizing Palais-Smale sequence and $\omega_n \to \omega \in \mathbb{R}$, for every $\psi \in H^1(G)$, there exist

$$\mathrm{d}E(u_n, G, \mathcal{K})[\psi] - \omega \int_G u_n \psi \mathrm{d}x = o(1), \quad \text{as } n \to \infty. \tag{3.12}$$

Moreover, by weak convergence in (3.12), we have

$$\mathrm{d}E(u, G, \mathcal{K})[\psi] - \omega \int_G u \psi \mathrm{d}x = 0. \tag{3.13}$$

Let $\psi = u_n - u$, and subtracting (3.13) from (3.12), we have

$$(\mathrm{d}E(u_n, G, \mathcal{K}) - \mathrm{d}E(u, G, \mathcal{K}))[u_n - u] - \omega \int_G |u_n - u|^2 \mathrm{d}x = o(1),$$

which yields

$$\int_G |(u_n - u)'|^2 \mathrm{d}x - \int_{\mathcal{K}} |u_n - u|^6 \mathrm{d}x - \omega \int_G |u_n - u|^2 \mathrm{d}x = o(1). \tag{3.14}$$

Noting the facts that $\mathcal{K}$ is compact and $u_n \to u$ locally uniformly on G, then from (3.14), we have

$$\int_G |(u_n - u)'|^2 \mathrm{d}x - \omega \int_G |u_n - u|^2 \mathrm{d}x = o(1). \tag{3.15}$$

Since $\omega < 0$, the left-hand side of (3.15) is the square of a norm that is equivalent to the standard one in the space $H^1(G)$. As a consequence, we have

$$u_n \to u \quad \text{strongly in } H^1(G),$$

and $u \in H^1_\mu(G)$. Moreover, since $u_n \to u$ in $L^\infty_{\mathrm{loc}}(G)$, we have

$$\int_{\mathcal{K}} |u|^2 \mathrm{d}x = \lim_{n\to\infty} \int_{\mathcal{K}} |u_n|^2 \mathrm{d}x \geq \zeta,$$

and $u \in \mathcal{A}$. So we conclude that $u \in H^1_\mu(G)$ is a local minimizer for the energy functional $E(\cdot, G, \mathcal{K})$ in set $\mathcal{A}$. The proof is complete. □

4 Proof of Theorems 1.1 and 1.2

Throughout this section, we present the proofs of Theorems 1.1 and 1.2. Precisely, based on the premise of the criterion in Section 3, we show that there exist local minimizers for the energy functional $E(u, G, \mathcal{K})$ when G satisfies suitable topological and metric assumptions.

4.1 Graphs with a terminal edge

We first focus on the graphs with a terminal edge and having at least two half-lines (this indicates that $s \geq 2$ based on the argument in Section 3), for which it is true that

$$\mu_{\mathcal{K}} = \mu_{\mathbb{R}^+}, \tag{4.1}$$

and ground states do not exist for $\mu \in (0, \mu_{\mathcal{K}})$. To prove Theorem 1.1, we preliminarily consider $\zeta, \mu, \delta > 0$ satisfying

$$(1 + 2\delta)\zeta < \mu_{\mathcal{K}} \quad \text{and} \quad \mu \in [(1 + 2\delta)\zeta, \mu_{\mathcal{K}}]. \tag{4.2}$$

Thus, we have

$$\mu - \zeta < \mu_{\mathcal{K}} = \mu_{\mathbb{R}^+}$$

and

$$\mu \geq (1 + 2\delta)\zeta > (1 + \delta)\zeta.$$

These imply that (3.1) is established. Here, we introduce two corresponding subsets, which are denoted as

$$\mathcal{A}^\zeta_\mu \coloneqq \left\{ u \in H^1_\mu(G) : \zeta \leq \int_{\mathcal{K}} |u|^2 \mathrm{d}x \right\}$$

and

$$\mathcal{B}^\zeta_\mu \coloneqq \left\{ u \in H^1_\mu(G) : \zeta \leq \int_{\mathcal{K}} |u|^2 \mathrm{d}x \leq (1 + \delta)\zeta \right\},$$

where $\mathcal{K}$ is the compact core of G.

Proof of Theorem 1.1. By the criterion for the existence of local minimizers established in Proposition 3.1, we need to show that there exists $\mu_1 \in (0, \mu_{\mathcal{K}})$ such that for every $\mu \in (\mu_1, \mu_{\mathcal{K}})$, (3.2) holds, namely,

$$-\infty < \inf_{u \in \mathcal{A}_\mu^\zeta} E(u, G, \mathcal{K}) < \inf_{u \in \mathcal{B}_\mu^\zeta} E(u, G, \mathcal{K}).$$

To proceed, we divide the proof into the following four steps.

Step 1: The energy functional $E(u, G, \mathcal{K})$ is bounded from below in set $\mathcal{A}_\mu^\zeta$, namely,

$$\inf_{u \in \mathcal{A}_\mu^\zeta} E(u, G, \mathcal{K}) > -\infty.$$

Indeed, by (2.9) and the definition of set $\mathcal{A}_\mu^\zeta$, this is valid since $\mu \le \mu_{\mathcal{K}}$.

Step 2: The energy functional $E(u, G, \mathcal{K})$ is bounded from below in set $\mathcal{B}_\mu^\zeta$, i.e., there exists $C_1 = C_1(\delta, \zeta, s) > 0$ (independent of μ) such that

$$E(u, G, \mathcal{K}) \ge C_1 \|u'\|^2_{L^2(G)}, \tag{4.3}$$

for every $u \in \mathcal{B}_\mu^\zeta$ and every $\mu \in [(1 + 2\delta)\eta, \mu_{\mathcal{K}}]$.

Indeed, let $\mu \in [(1 + 2\delta)\eta, \mu_{\mathcal{K}}]$, if $u \in \mathcal{B}_\mu^\zeta$, then

$$\mu = \int_G |u|^2 \mathrm{d}x = \sum_{i=1}^s \int_{l_i} |u|^2 \mathrm{d}x + \int_{\mathcal{K}} |u|^2 \mathrm{d}x,$$

which yields

$$\sum_{i=1}^s \int_{l_i} |u|^2 \mathrm{d}x = \mu - \int_{\mathcal{K}} |u|^2 \mathrm{d}x \in [\mu - (1 + \delta)\zeta, \mu - \zeta].$$

Thus, there must be an index $i \in \{1, 2, \ldots, s\}$, such as $i = 1$, satisfying

$$\int_{l_1} |u|^2 \mathrm{d}x \ge \frac{1}{s}[\mu - (1 + \delta)\zeta] \ge \frac{\delta\zeta}{s}. \tag{4.4}$$

For every $u \in \mathcal{B}_\mu^\zeta$, it holds

$$E(u, G, \mathcal{K}) = E(u, l_1, \mathcal{K}) + E(u, G \backslash l_1, \mathcal{K}). \tag{4.5}$$

Note that the graph $G \backslash l_1$ (the graph by removing the half-line l_1 from G) has a terminal point, and thus, (4.1) holds. On the one hand, for the graph $G \backslash l_1$, by (2.9), we have

$$E(u, G \backslash l_1, \mathcal{K}) \ge \frac{1}{2}\left(1 - \left(\frac{\int_{G \backslash l_1} |u|^2 \mathrm{d}x}{\mu_{\mathbb{R}^+}}\right)^2\right) \|u'\|^2_{L^2(G \backslash l_1)},$$

where

$$\int_{G \backslash l_1} |u|^2 \mathrm{d}x = \int_G |u|^2 \mathrm{d}x - \int_{l_1} |u|^2 \mathrm{d}x \le \mu - \frac{\delta\zeta}{s} \le \mu_{\mathbb{R}^+} - \frac{\delta\zeta}{s},$$

which is strictly smaller than $\mu_{\mathbb{R}^+}$. Let us define

$$C_1 = C_1(\delta, \zeta, s) =: \frac{1}{2}\left(1 - \left(\frac{\int_{G \backslash l_1} |u|^2 \mathrm{d}x}{\mu_{\mathbb{R}^+}}\right)^2\right).$$

Then, from the aforementioned argument, we obtain that

$$0 < C_1 < \frac{1}{2}.$$

Thus, there exists

$$E(u, G\backslash l_1, \mathcal{K}) \geq C_1 \|u'\|^2_{L^2(G\backslash l_1)}. \tag{4.6}$$

On the other hand, for the half-line l_1, we have

$$E(u, l_1, \mathcal{K}) = \frac{1}{2} \|u'\|^2_{L^2(l_1)}. \tag{4.7}$$

Noting the fact that $C_1 < \frac{1}{2}$ and combining (4.5) with (4.6) and (4.7), we have

$$E(u, G, \mathcal{K}) \geq C_1\|u'\|^2_{L^2(G)},$$

which is the desired result.

Step 3: The energy functional $E(u, G, \mathcal{K})$ is uniformly bounded from below in $\mathcal{B}^\zeta_\mu$, i.e., there exists $C_2 = C_2(\delta, \zeta, s) > 0$ such that

$$E(u, G, \mathcal{K}) \geq C_2, \tag{4.8}$$

for every $u \in \mathcal{B}^\zeta_\mu$ and every $\mu \in [(1 + 2\delta)\eta, \mu_{\mathcal{K}}]$.

Indeed, if, by contradiction, there exists a sequence $\{u_n\}$ in $H^1(G)$, with $u_n \in \mathcal{B}^\zeta_{\mu_n}$ and $\{\mu_n\} \subset [(1 + 2\delta)\zeta, \mu_{\mathcal{K}}]$, such that

$$E(u_n, G, \mathcal{K}) \to 0, \quad \text{as } n \to \infty.$$

By (4.3), we deduce that $\{u_n\}$ is bounded in $H^1(G)$, and furthermore, it holds

$$\|u_n'\|^2_{L^2(G)} \to 0, \quad \text{as } n \to \infty.$$

Thus, up to a subsequence, one sees that

$$u_n \rightharpoonup u \quad \text{in } H^1(G), \text{as } n \to \infty,$$

and

$$u_n \to u \quad \text{in } L^\infty_{\text{loc}}(G), \text{as } n \to \infty.$$

Moreover, by weak lower semi-continuity, it holds

$$\|u'\|^2_{L^2(G)} \leq \liminf_{n\to\infty} \|u_n'\|^2_{L^2(G)} = 0, \tag{4.9}$$

which implies that u is constant on G. Since G is noncompact and $u \in H^1(G)$, it is necessary that $u \equiv 0$ on G. This is a contradiction with

$$\int_{\mathcal{K}} |u|^2 \mathrm{d}x = \lim_{n\to\infty} \int_{\mathcal{K}} |u_n|^2 \mathrm{d}x \geq \zeta > 0,$$

where we utilize the fact that $u_n \to u$ locally uniformly on G.

Step 4: There exists $\mu_1 \in ((1 + 2\delta)\zeta, \mu_{\mathcal{K}})$ such that

$$\inf_{u\in\mathcal{A}^\zeta_\mu} E(u, G, \mathcal{K}) < C_2, \quad \text{for every } \mu \in (\mu_1, \mu_{\mathcal{K}}), \tag{4.10}$$

where C_2 is the constant obtained from Step 3.

Indeed, there are two difficulties in the proof of this step. One is the existence of mass μ_1, and the other is the strict inequality in (4.10). To overcome these difficulties, our strategy is to show that, for any $\varepsilon > 0$, there exists a mass $\mu_\varepsilon \in ((1 + 2\delta)\zeta, \mu_{\mathcal{K}})$, and if $\mu \in (\mu_\varepsilon, \mu_{\mathcal{K}})$, there exists a function $\varphi_\mu \in \mathcal{A}^\zeta_\mu$ such that

$$E(\varphi_\mu, G, \mathcal{K}) < \varepsilon.$$

Thus, the proof will be completed when we choose $\varepsilon = C_2$ and $\mu_\varepsilon = \mu_1$.

According to the aforementioned analysis, the key to the problem lies in the construction of function φ_μ. For this purpose, we consider using the half-soliton $\phi(x)$ defined in (2.6) to construct function φ_μ. Since G has at least a terminal edge, without loss of generality, let $e_p = [0, 1]$ be a terminal edge, with the coordinate 0 at the terminal point. Define

$$\varphi_\lambda(x) := (\phi_\lambda(x) - \phi_\lambda(1))^+, \quad \forall \lambda > 0.$$

Naturally, one can see that $\varphi_\lambda \in H^1(0, 1)$, accompanied by

$$\varphi_\lambda > 0 \text{ on } [0, 1), \quad \text{and} \quad \varphi_\lambda(1) = 0.$$

The mass of φ_λ on $e_p = [0, 1]$ is

$$m_\lambda := \int_0^1 |\varphi_\lambda|^2 \mathrm{d}x = \int_0^\infty |\phi(z) - \phi(\lambda)|^2_{\chi_{[0,\lambda]}}(z)\mathrm{d}z. \tag{4.11}$$

Taking the derivative of (4.11) with respect to λ, we obtain that

$$\frac{\mathrm{d}m_\lambda}{\mathrm{d}\lambda} = -\int_0^\infty (\phi(y) - \phi(\lambda))\phi'(\lambda)\chi_{[0,\lambda]}(y)\mathrm{d}y = -\int_0^\lambda (\phi(y) - \phi(\lambda))\phi'(\lambda)\mathrm{d}y. \tag{4.12}$$

Through simple calculations, we know that $\phi'(\lambda) < 0$ for every $\lambda > 0$. Thereby, by monotone and dominated convergence principle, one can see that $\frac{\mathrm{d}m_\lambda}{\mathrm{d}\lambda} > 0$ for every $\lambda > 0$ and m_λ is continuous and strictly monotone increasing with respect to λ. Moreover, we have

$$\lim_{\lambda \to 0^+} m_\lambda = 0, \quad \text{and} \quad \lim_{\lambda \to +\infty} m_\lambda = \int_0^{+\infty} |\phi|^2 \mathrm{d}x = \mu_{\mathbb{R}^+} = \mu_{\mathcal{K}}.$$

As a consequence, for every $\mu \in (0, \mu_{\mathcal{K}})$, there exists a unique $\lambda = \lambda(\mu) > 0$ such that $m_{\lambda(\mu)} = \mu$. Then, we can construct the function φ_μ on G as

$$\varphi_\mu := \begin{cases} \varphi_{\lambda(\mu)}, & \text{on } e_p = [0, 1], \\ 0, & \text{elsewhere.} \end{cases} \tag{4.13}$$

Here, we need to do some remarks. First of all, the function φ_μ satisfies the continuity condition at the other vertex of edge e_p (corresponding to the coordinate 1). Second, the mass of φ_μ on both the compact core $\mathcal{K}$ and the whole G is $m_{\lambda(\mu)} = \mu$ so that $\varphi_\mu \in \mathcal{A}^\zeta_\mu$.

Next, what we need to do is to show that the energy functional $E(\varphi_\mu, G, \mathcal{K}) = E(\varphi_{\lambda(\mu)}, (0, 1), \mathcal{K})$ can be made arbitrarily small as $\mu \to \mu_{\mathcal{K}}$ (that is equivalent to $\lambda \to +\infty$). Indeed, we have

$$\begin{aligned} E(\varphi_\lambda, (0, 1), \mathcal{K}) &= \frac{1}{2}\int_0^1 |\varphi'_\mu|^2 \,\mathrm{d}x - \frac{1}{6}\int_0^1 |\varphi_\mu|^6 \mathrm{d}x \\ &= \lambda^2 \int_0^\lambda \left[\frac{1}{2}|\phi'(z)|^2 - \frac{1}{6}(\phi(z) - \phi(\lambda))^6\right]\mathrm{d}z \\ &\le \lambda^2 \int_0^\lambda \left[\frac{1}{2}|\phi'(z)|^2 - \frac{1}{6}|\phi(z)|^6\right]\mathrm{d}z + \lambda^2\phi(\lambda)\int_0^\lambda \phi^5(z)\mathrm{d}z, \end{aligned} \tag{4.14}$$

where we utilize the mean value theorem in the last inequality. Precisely, for every $z \in (0, \lambda)$, there exists $\theta \in (0, 1)$ such that

$$|\phi^6(z) - (\phi(z) - \phi(\lambda))^6| = 6\,|\phi(z) - \theta\phi(\lambda)|^5\phi(\lambda) \le 6\phi^5(z)\phi(\lambda).$$

By the property of function $\phi(\cdot)$, one can see that

$$\lambda^2\phi(\lambda) \to 0, \quad \text{as } \lambda \to +\infty,$$

and

$$\int_0^\lambda \phi^5(z)\mathrm{d}z \le \int_0^\infty \phi^5(z)\mathrm{d}z < +\infty.$$

Thus, the estimate in (4.14) can be denoted as

$$E(\varphi_\lambda, (0,1), \mathcal{K}) \le \lambda^2 E(\phi, (0,\lambda)) + o(1), \quad \text{as } \lambda \to +\infty,$$

where the energy functional $E(\phi, (0, \lambda))$ is defined in (2.4). In fact, by (2.8), there exists

$$E(\phi, \mathbb{R}^+) = E(\phi, (0,\lambda)) + E(\phi, (\lambda, +\infty)) = 0.$$

Thus,

$$E(\phi, (0,\lambda)) = -E(\phi, (\lambda, +\infty)),$$

which yields

$$E(\varphi_\lambda, (0,1), \mathcal{K}) \le -\lambda^2 E(\phi, (\lambda, +\infty)) + o(1), \quad \text{as } \lambda \to +\infty. \tag{4.15}$$

To proceed further, since the mass of function ϕ on $(\lambda, +\infty)$ satisfies

$$0 < \tilde{\mu} = \int_\lambda^\infty \phi^2(x)\mathrm{d}x = \int_0^\infty \phi^2(z+\lambda)\mathrm{d}z < \mu_{\mathbb{R}^+} \quad \text{for every } \lambda > 0,$$

one can see that $\phi(\cdot+\lambda) \in H^1(\mathbb{R}^+)$, and moreover, by applying (2.2), we observe that

$$E(\phi, (\lambda, +\infty)) = E(\phi(\cdot+\lambda), (0, +\infty)) \ge \frac{1}{2}\left(1 - \left(\frac{\tilde{\mu}}{\mu_{\mathbb{R}^+}}\right)^2\right) \|\phi'(\cdot+\lambda)\|^2_{L^2(\mathbb{R}^+)} \ge 0. \tag{4.16}$$

By (4.15) and (4.16), we immediately have

$$E(\varphi_\lambda, (0,1), \mathcal{K}) \le o(1) < \varepsilon, \tag{4.17}$$

for $\lambda \to +\infty$ (that is, equivalent to $\mu \to \mu_{\mathcal{K}}$). This implies the existence of μ_1 in (4.10).

To conclude the proof, let $\mathcal{A} = \mathcal{A}_\mu^\zeta$, $\mathcal{B} = \mathcal{B}_\mu^\zeta$ and μ_1 be the mass obtained in Step 4, then Theorem 1.1 is a direct result of Proposition 3.1, i.e., for every $\mu \in (\mu_1, \mu_{\mathcal{K}})$, the energy functional (1.1) has a critical point in the space $H_\mu^1(G)$, which is a local minimizer. □

4.2 Graphs admitting a cycle-covering

In this section, we deal with the graphs that admit a cycle-covering and have at least three half-lines (this means that $s \ge 3$), for which it holds

$$\mu_{\mathcal{K}} = \mu_{\mathbb{R}}, \tag{4.18}$$

and ground states do not exist for $\mu \in (0, \mu_{\mathcal{K}})$. It is worth mentioning that the graph $G \backslash l_i$ (by removing any half-line l_i from G) also admits a cycle-covering according to the argument in Section 1. The proof of Theorem 1.2 is analogous to the one of Theorem 1.1. Precisely, let $\zeta, \mu, \delta > 0$ be such that

$$\zeta > \mu_{\mathbb{R}^+}, \quad (1+2\delta)\zeta < \mu_{\mathcal{K}} \quad \text{and} \quad \mu \in [(1+2\delta)\zeta, \mu_{\mathcal{K}}]. \tag{4.19}$$

Then, we have

$$\mu - \zeta < \mu_{\mathcal{K}} - \mu_{\mathbb{R}^+} = \mu_{\mathbb{R}^+}$$

and

$$\mu \ge (1+2\delta)\zeta > (1+\delta)\zeta.$$

Thus, we obtain that (3.1) holds. We still introduce two subsets as

$$\mathcal{A}_\mu^\zeta := \left\{u \in H_\mu^1(G) : \zeta \le \int_{\mathcal{K}} |u|^2 \mathrm{d}x\right\}$$

and

$$\mathcal{B}_\mu^\zeta := \left\{u \in H_\mu^1(G) : \zeta \le \int_{\mathcal{K}} |u|^2 \mathrm{d}x \le (1+\delta)\zeta\right\},$$

where $\mathcal{K}$ is the compact core of G.

Proof of Theorem 1.2. By Proposition 3.1, we need to show that there exists $\mu_2 \in (0, \mu_{\mathcal{K}})$ such that for every $\mu \in (\mu_2, \mu_{\mathcal{K}})$, (3.2) holds, namely,

$$-\infty < \inf_{u \in \mathcal{A}_\mu^\zeta} E(u, G, \mathcal{K}) < \inf_{u \in \mathcal{B}_\mu^\zeta} E(u, G, \mathcal{K}).$$

For the sake of completeness, we still divide the proof into the following four steps.

Step 1: The energy functional $E(u, G, \mathcal{K})$ is bounded from below in set $\mathcal{A}_\mu^\zeta$, namely,

$$\inf_{u \in \mathcal{A}_\mu^\zeta} E(u, G, \mathcal{K}) > -\infty.$$

Indeed, by (2.9) and the definition of set $\mathcal{A}_\mu^\zeta$, the result is clearly valid since $\mu \le \mu_{\mathcal{K}}$.

Step 2: The energy functional $E(u, G, \mathcal{K})$ is bounded from below in set $\mathcal{B}_\mu^\zeta$, i.e., there exists $C_3 = C_3(\delta, \zeta, s) > 0$ (independent of μ) such that

$$E(u, G, \mathcal{K}) \ge C_3 \|u'\|_{L^2(G)}^2, \tag{4.20}$$

for every $u \in \mathcal{B}_\mu^\zeta$ and every $\mu \in [(1+2\delta)\eta, \mu_{\mathcal{K}}]$.

Indeed, let $\mu \in [(1+2\delta)\eta, \mu_{\mathcal{K}}]$, if $u \in \mathcal{B}_\mu^\zeta$, then

$$\mu = \int_G |u|^2 \mathrm{d}x = \sum_{i=1}^{s} \int_{l_i} |u|^2 \mathrm{d}x + \int_{\mathcal{K}} |u|^2 \mathrm{d}x,$$

which yields

$$\sum_{i=1}^{s} \int_{l_i} |u|^2 \mathrm{d}x = \mu - \int_{\mathcal{K}} |u|^2 \mathrm{d}x \in [\mu - (1+\delta)\zeta, \mu - \zeta].$$

Thus, there must be an index $i \in \{1, 2, \dots, s\}$, such as $i = 1$, satisfying

$$\int_{l_1} |u|^2 \mathrm{d}x \ge \frac{1}{s}[\mu - (1+\delta)\zeta] \ge \frac{\delta\zeta}{s}. \tag{4.21}$$

For every $u \in \mathcal{B}_\mu^\zeta$, we have

$$E(u, G, \mathcal{K}) = E(u, l_1, \mathcal{K}) + E(u, G \backslash l_1, \mathcal{K}). \tag{4.22}$$

Note that both the graph $G \backslash l_1$ (the graph by removing the half-line l_1 from G) and the graph G admit a cycle covering, and thus, (4.18) holds. On the one hand, for the graph $G \backslash l_1$, by (2.9), we have

$$E(u, G \backslash l_1, \mathcal{K}) \ge \frac{1}{2}\left(1 - \left(\frac{\int_{G \backslash l_1} |u|^2 \mathrm{d}x}{\mu_{\mathbb{R}}}\right)^2\right) \|u'\|_{L^2(G \backslash l_1)}^2,$$

where

$$\int_{G\setminus l_1} |u|^2 \mathrm{d}x = \int_G |u|^2 \mathrm{d}x - \int_{l_1} |u|^2 \mathrm{d}x \le \mu - \frac{\delta\zeta}{s} \le \mu_{\mathbb{R}} - \frac{\delta\zeta}{s},$$

which is strictly smaller than $\mu_{\mathbb{R}}$. Let us define

$$C_3 = C_3(\delta, \zeta, s) =: \frac{1}{2}\left(1 - \left(\frac{\int_{G\setminus l_1} |u|^2 \mathrm{d}x}{\mu_{\mathbb{R}}}\right)^2\right).$$

Then, from the aforementioned argument, we obtain that

$$0 < C_3 < \frac{1}{2}.$$

Thus, there exists

$$E(u, G\setminus l_1, \mathcal{K}) \ge C_3 \, \|u'\|^2_{L^2(G\setminus l_1)}. \tag{4.23}$$

On the other hand, for the half-line l_1, we have

$$E(u, l_1, \mathcal{K}) = \frac{1}{2} \, \|u'\|^2_{L^2(l_1)}. \tag{4.24}$$

Since $C_3 < \frac{1}{2}$, combining (4.22) with (4.23) and (4.24), we have

$$E(u, G, \mathcal{K}) \ge C_3 \|u'\|^2_{L^2(G)},$$

which is the desired result.

Step 3: The energy functional $E(u, G, \mathcal{K})$ is uniformly bounded from below in $\mathcal{B}^\zeta_\mu$, i.e., there exists $C_4 = C_4(\delta, \zeta, s) > 0$ such that

$$E(u, G, \mathcal{K}) \ge C_4, \tag{4.25}$$

for every $u \in \mathcal{B}^\zeta_\mu$ and every $\mu \in [(1 + 2\delta)\eta, \mu_{\mathcal{K}}]$.

Indeed, the proof of the existence of constant C_4 is completely analogous to the derivation of C_2 in the proof of Theorem 1.1 and thus is omitted.

Step 4: There exists $\mu_2 \in ((1 + 2\delta)\zeta, \mu_{\mathcal{K}})$ such that

$$\inf_{u \in \mathcal{A}^\zeta_\mu} E(u, G, \mathcal{K}) < C_4, \quad \text{for every } \mu \in (\mu_2, \mu_{\mathcal{K}}), \tag{4.26}$$

where C_4 is the constant obtained from Step 3.

Indeed, as in the proof of Theorem 1.1, there are still two difficulties in this step. One is the existence of mass μ_2, and the other is the strict inequality in (4.26). To overcome them, our strategy is to show that, for any $\varepsilon > 0$, there exists a mass $\mu_\varepsilon \in ((1 + 2\delta)\zeta, \mu_{\mathcal{K}})$, and if $\mu \in (\mu_\varepsilon, \mu_{\mathcal{K}})$, there exists a function $\varphi_\mu \in \mathcal{A}^\zeta_\mu$ such that

$$E(\varphi_\mu, G, \mathcal{K}) < \varepsilon.$$

The proof will be completed when we choose $\varepsilon = C_4$ and $\mu_\varepsilon = \mu_2$.

We point out that the most important thing is still to construct the function φ_μ. Since the compact core $\mathcal{K}$ is nonempty, without loss of generality, we can assume $e_p = [-1,1]$ be a bounded edge with the coordinates ±1 corresponding to the two vertices of e_p. In order to construct φ_μ, we consider the soliton $\phi(x)$ defined in (2.6), and let

$$\varphi_\lambda(x) := (\phi_\lambda(x) - \phi_\lambda(1))^+, \quad \forall \lambda > 0.$$

One can see that $\varphi_\lambda \in H^1(-1, 1)$, with

$$\varphi_\lambda > 0 \text{ on } (-1, 1), \quad \text{and} \quad \varphi_\lambda(-1) = \varphi_\lambda(1) = 0.$$

In particular, by the property of the soliton $\phi(x)$, we have that φ_λ is symmetric with respect to the middle point (corresponding to the coordinate 0) of the bounded edge e_p for every $\lambda > 0$.

In addition, we can see that

$$\|\varphi_\lambda\|^2_{L^2(-1,1)} = \int_{-1}^{1} |\varphi_\lambda|^2 \mathrm{d}x = 2\int_0^1 |\varphi_\lambda|^2 \mathrm{d}x = 2\,\|\varphi_\lambda\|^2_{L^2(0,\,1)}$$

and

$$\|\varphi'_\lambda\|^2_{L^2(-1,1)} = \int_{-1}^{1} |\varphi'_\lambda|^2 \mathrm{d}x = 2\int_0^1 |\varphi'_\lambda|^2 \mathrm{d}x = 2\,\|\varphi'_\lambda\|^2_{L^2(0,\,1)}.$$

If we denote the mass of φ_λ on $e_p = [-1, 1]$ as m_λ, through the same computations of Step 4 in the proof of Theorem 1.1, we know that m_λ is continuous and strictly monotone increasing with respect to λ. Moreover, there exist

$$\lim_{\lambda\to 0^+} m_\lambda = 0, \quad \text{and} \quad \lim_{\lambda\to+\infty} m_\lambda = 2\int_0^{+\infty} |\phi|^2 \mathrm{d}x = \mu_{\mathbb{R}} = \mu_{\mathcal{K}}.$$

Thus, for every $\mu \in (0, \mu_{\mathcal{K}})$, there exists a unique $\lambda(\mu) > 0$ such that $m_{\lambda(\mu)} = \mu$. Define

$$\varphi_\mu \coloneqq \begin{cases} \varphi_{\lambda(\mu)}, & \text{on } e_p = [-1, 1], \\ 0, & \text{elsewhere.} \end{cases} \tag{4.27}$$

Obviously, φ_μ satisfies the continuity condition at the two vertices of edge e_p and $\varphi_\mu \in \mathcal{A}^\zeta_\mu$.

Finally, by the symmetry properties of the two functions φ_λ and φ'_λ on the interval $[-1, 1]$, then through a similar analysis in the proof of Theorem 1.1, we can obtain

$$E(\varphi_\mu, G, \mathcal{K}) = E(\varphi_\lambda, (-1, 1), \mathcal{K}) < \varepsilon, \tag{4.28}$$

for $\lambda \to +\infty$ (that is, equivalent to $\mu \to \mu_{\mathcal{K}}$). This implies the existence of μ_2 in (4.26).

To conclude the proof, let $\mathcal{A} = \mathcal{A}^\zeta_\mu$, $\mathcal{B} = \mathcal{B}^\zeta_\mu$, and μ_2 be the mass obtained in Step 4, then Theorem 1.2 is a direct result of Proposition 3.1, i.e., for every $\mu \in (\mu_2, \mu_{\mathcal{K}})$, the energy functional (1.1) has a critical point in the space $H^1_\mu(G)$, which is a local minimizer. □

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