Research Article

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On the dimension of the algebraic sum of subspaces

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Abstract: We provide a recursive formula for the dimension of the algebraic sum of finitely many subspaces in a finite-dimensional vector space over an arbitrary field.

Keywords: linear subspaces, lattices, algebraic sum, dimension

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1 Introduction

The main purpose of this article is to prove the following fundamental result.

Theorem 1.1. *Let U be a finite-dimensional vector space over an arbitrary field $\mathbb{F}$. Let $U_1, \ldots, U_n$ be linear subspaces of U. Then,*

$$(n-1)\dim(U_1 + U_2 + \ldots + U_n) = \dim(U_1^\star) + \dim(U_2^\star) + \ldots + \dim(U_n^\star) - \dim(U_1^\star \cap U_2^\star \cap \ldots \cap U_n^\star), \tag{1}$$

where

$$U_i^\star \coloneqq U_1 + \ldots + U_{i-1} + U_{i+1} + \ldots + U_n,$$

for $i = 1, \ldots, n$.

For $n = 2$, Theorem 1.1 reduces to a simple exercise in a standard linear algebra course to the effect that

$$\begin{aligned} \dim(U_1 + U_2) &= \dim(U_1^*) + \dim(U_2^*) - \dim(U_1^* \cap U_2^*) \\ &= \dim(U_2) + \dim(U_1) - \dim(U_2 \cap U_1). \end{aligned} \tag{2}$$

It is one of the most common misconceptions in mathematics that formula (2) can be generalized to more than two summands using the inclusion-exclusion principle [1]. Following this reasoning, one might expect the formula

$$\begin{aligned} \dim(U_1 + U_2 + U_3) = {} & \dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ & + \dim(U_1 \cap U_2 \cap U_3), \end{aligned} \tag{3}$$

to hold for three summands. However, if we take three distinct lines in a two-dimensional vector space U over an arbitrary field $\mathbb{F}$ as the subspaces U_1, U_2, and U_3, we immediately encounter a contradiction.

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A deeper reason why (3) fails in general is that the lattice of vector spaces, with the operations of taking algebraic sums and intersections, is not *distributive*. For more details on lattices, see [2].

In some textbooks on linear algebra authors warn that generalizations of (2) as stipulated in (3) fail (see for example [3, Chapter 2, Exercises 19 and 20] or [4, Aufgabe 2.7.6]). However, we were not able to find the formula (1) in any of the textbooks we looked through.

We found formula (1) in the research article [5, Theorem 2] by Yongge Tian. However, his proof is quite intricate and primarily applicable to complex numbers, as it frequently relies on pseudo-inverse matrices. In contrast, our proof is much more elementary and holds true over any field.

2 Proof

In this section, we will prove our main result. The strategy is to prove first the case when the intersection on the right of formula (1) is just the zero vector and then to reduce the general situation to that case.

We begin with a simple but useful lemma.

Lemma 2.1. *Let U be a finite-dimensional vector space over an arbitrary field $\mathbb{F}$ and let $U_1, \dots, U_n$ be the subspaces of U. The following conditions are equivalent*:
(*a*) *the algebraic sum $U_1 + \dots + U_n$ is the direct sum $U_1 \oplus \dots \oplus U_n$;*
(*b*) *for all $1 \le i \le n$, there is $U_i \cap U_i^* = 0$.*

Proof. The implication from (a) to (b) is straightforward. For the opposite direction, let us assume (b) and show that any vector $u \in U_1 + \dots + U_n$ has a unique presentation as

$$u = u_1 + \dots + u_n, \quad \text{with } u_i \in U_i.$$

Suppose that we have two such presentations

$$u = u_1 + \dots + u_n = u_1' + \dots + u_n'.$$

Then, for any $1 \le i \le n$, we have

$$U_i \ni u_i - u_i' = (u_1 - u_1') + \dots + (u_{i-1} - u_{i-1}') + (u_{i+1} - u_{i+1}') + \dots + (u_n - u_n') \in U_i^*.$$

By (b), this implies $u_i = u_i'$ for all $i = 1, \dots, n$, and we are done. □

We can now pass directly to the proof of Theorem 1.1.

Proof of Theorem 1.1. Case 1. Assume that

$$U_1^* \cap \dots \cap U_n^* = 0.$$

We claim that under this assumption, all algebraic sums appearing in formula (1) are direct sums. Of course, it suffices to show that it is the case for the sum of all involved subspaces, so we claim that

$$U_1 + \dots + U_n = U_1 \oplus \dots \oplus U_n.$$

We want to use Lemma 2.1, so let

$$u_i \in U_i \cap U_i^*,$$

for some $1 \le i \le n$. Let j be an index different from i. Then, $u_i \in U_j^*$ because $U_i \subset U_j^*$ for all $j \ne i$. Since also u_i was taken as an element of U_i^*, we conclude that

$$u_i \in U_1^* \cap \dots \cap U_n^*, \quad \text{hence } u_i = 0 \text{ by assumption}.$$

It follows that the right-hand side of (1) is

$$\begin{aligned}&\dim(U_1^*) + \dim(U_2^*) + \ldots + \dim(U_n^*) = \dim(U_2) + \dim(U_3) + \ldots + \dim(U_n) + \dim(U_1) + \dim(U_3) + \ldots + \dim(U_n) + \\ &\vdots \\ &\dim(U_1) + \dim(U_2) + \ldots + \dim(U_{n-1}) = (n-1)(\dim(U_1) + \dim(U_2) + \ldots + \dim(U_n)),\end{aligned}$$

and we are done in this case.

Case 2. Let $W \coloneqq U_1^* \cap \ldots \cap U_n^*$ be now arbitrary. The idea is to reduce the situation to Case 1, taking quotients of all involved spaces by W. In order to control the dimensions, we show that

$$W \subset U_1 + \ldots + U_n \quad \text{and} \quad W \subset U_i^*, \quad \text{for all } i = 1, \ldots, n.$$

The containment $W \subset U_i^*$ follows by the definition of W, and of course, every U_i^* is contained in $U_1 + \ldots + U_n$.

Let $\pi : U \to U/W$ be the quotient map. Then,

(a) $\dim(\pi(U_1 + \ldots + U_n)) = \dim(U_1 + \ldots + U_n) - \dim(W)$ and;

(b) $\dim(\pi(U_i^*)) = \dim(U_i^*) - \dim(W)$.

Since taking the image of a linear map commutes with taking algebraic sums, we have

$$\pi(U_i^*) = \pi(U_i)^* = \pi(U_1) + \ldots + \pi(U_{i-1}) + \pi(U_{i+1}) + \ldots + \pi(U_n).$$

Moreover, formula (1) holds for subspaces $\pi(U_1), \ldots, \pi(U_n)$ in U/W because by construction,

$$\pi(U_1)^* \cap \ldots \cap \pi(U_n)^* = 0.$$

Indeed, let $y \in \pi(U_1^*) \cap \ldots \cap \pi(U_n^*)$. So there are elements $x_i \in U_i^*$ with $\pi(x_i) = y$ for $i = 1, \ldots, n$. Hence, $x_i - x_j \in W$ for all $i, j \in \{1, \ldots, n\}$. But then $x_i = (x_i - x_j) + x_j \in U_j^*$ for all $1 \le i, j \le n$, or equivalently, $x_i \in U_1^* \cap \ldots \cap U_n^* = W$. Thus, $y = \pi(x_i) = 0$ in U/W.

Using (a) and (b) as mentioned earlier, it is now elementary to check that formula (1) holds for the spaces before taking the quotient. □

The following corollary, proposed by one of the referees, results in a more direct way the dimension of the sum of the spaces U_i and the sum of their dimensions.

Corollary 2.2. *Let U be a finite-dimensional vector space over an arbitrary field* $\mathbb{F}$. *Let* $U_1, \ldots, U_n$ *be linear subspaces of U. Then,*

$$\dim\left(\sum_{i=1}^{n} U_i\right) + \sum_{i=1}^{n} \dim(U_i \cap U_i^*) = \sum_{i=1}^{n} \dim(U_i) + \dim\left(\bigcap_{i=1}^{n} U_i^*\right). \tag{4}$$

Proof. The idea is to use formula (2) for spaces U_i and U_i^*. We have

$$\dim(U_i + U_i^*) + \dim(U_i \cap U_i^*) = \dim(U_i) + \dim(U_i^*). \tag{5.i}$$

Since $U_i + U_i^* = \sum_{i=1}^{n} U_i$, adding equation (5.i) for $i = 1, \ldots, n$, we obtain

$$n \cdot \dim\left(\sum_{i=1}^{n} U_i\right) + \sum_{i=1}^{n} \dim(U_i \cap U_i^*) = \sum_{i=1}^{n} \dim(U_i) + \sum_{i=1}^{n} \dim(U_i^*). \tag{5}$$

Now, subtracting (1) on both sides does the job. □

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