Research Article

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A modified predictor–corrector scheme with graded mesh for numerical solutions of nonlinear Ψ-caputo fractional-order systems

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Abstract: The aim of this article is to develop a modified predictor–corrector scheme for solving the system of nonlinear Ψ-Caputo fractional differential equations with order $0 < \alpha < 1$. By using the graded mesh and considering the predictor–corrector scheme for Ψ-Caputo fractional derivative, the numerical solutions of nonlinear Ψ-Caputo fractional-order systems are derived. Moreover, the error estimations of predictor–corrector scheme with graded mesh are also investigated. Particularly, the accuracy of numerical solutions depended on the function Ψ and the partition size on graded mesh. Numerical examples for linear and nonlinear fractional differential systems with various kernels and meshes are considered to explain the value and effectiveness of the proposed schemes.

Keywords: Ψ-Caputo nonlinear fractional-order systems, predictor–corrector scheme, graded mesh, error estimation

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1 Introduction

Over the last few decades, the concepts of fractional calculus have increased interest in related fields of science and engineering. Some important results in applications of fractional calculus are reported in [1]. Additionally, fractional calculus can be described as complex procedures in real world applications such as signal and image processing [2], biology [3], environmental science [4], economics [5], multidisciplinary engineering fields [6], etc. In mathematical models, fractional derivatives are suitable tools for explaining memory and hereditary properties of several materials and processes. Moreover, there are many definitions of fractional derivatives for applying to fractional-order models, e.g., Caputo-Hadamard, Hadamard, Caputo-Erdélyi-Kober, Erdélyi-Kober, Caputo, and Riemann-Liouville. We recommend reading previous studies [7–10] for further information.

There is a specific type of kernel dependency represented in each of those definitions. To investigate fractional differential equations in a comprehensive way, Almeida [11] proposed the definition of fractional derivatives with arbitrary kernel and called Ψ-Caputo derivative. In particular, Caputo-Erdélyi-Kober, Caputo-Hadamard, and Caputo are the specific cases of Ψ-Caputo derivatives.

The study of Ψ-Caputo fractional differential equation is currently increasing. Many researchers have worked on the solution of Ψ-Caputo fractional differential equations, and have not yet succeeded. The work of

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Almeida et al. [12] suggested that the Ψ-Caputo fractional derivatives in mathematical models are more practical and capable of extracting hidden parts of real-world conditions. The applications of Ψ-Caputo fractional differential equation are also shown in the research of Almeida et al. [13]. Moreover, the approximation on the extremal solutions of Ψ-Caputo fractional differential equations is shown and obtained by using the monotone iteration of upper and lower solutions [14,15]. Furthermore, the linear Ψ-Caputo fractional differential systems are represented by the work of Almeida et al. [16] and solved in the form of Mittag-Leffler function. They also proved theorems on the existence and uniqueness of the solution of linear Ψ-Caputo fractional differential systems. However, the work of Almeida et al. [16] considered only the case of linear Ψ-Caputo fractional differential equations. Nevertheless, numerous practical research problems are still represented by linear Ψ-Caputo fractional differential equations. For example, the solution of a linear non-homogeneous fractional differential system involving Ψ-Caputo fractional derivatives is derived in the form of matrix Mittag-Leffler functions in [17].

To extend the idea of Almeida et al. [16], solving the system of nonlinear Ψ-Caputo fractional differential equations is investigated in this study. However, the system of nonlinear Ψ-Caputo fractional differential equations is difficult to solve analytically. Consequently, numerical schemes are necessary to estimate the solution of the system of nonlinear Ψ-Caputo fractional differential equations. To solve nonlinear fractional differential equations, several studies constructed numerical schemes under the assumption of uniform meshes, as referenced in [18–21]. In order to acquire the optimal convergence order for numerical schemes, Liu et al. [22] applied predictor–corrector scheme with graded mesh to solve nonlinear fractional differential equations. Furthermore, the error estimation for the predictor–corrector scheme with graded mesh shows that the optimal convergence order of this scheme is adjusted uniformly. In this study, we consider the predictor–corrector scheme with graded mesh for solving nonlinear Ψ-Caputo fractional-order differential systems on $\mathbb{R}^M$,

$$\begin{cases} {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}\mathbf{Y}(t) = \mathbf{F}(t, \mathbf{Y}(t)), \\ \mathbf{Y}(t_0) = \mathbf{Y}_0, \end{cases}$$

where ${}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}$ denotes the Ψ-Caputo fractional derivative with $\alpha > 0$, $\Psi \in C^1([t_0, T])$ is an increasing function such that $\Psi'(t) \neq 0$ for all $t \in [t_0, T]$, $\mathbf{Y}_0 \in \mathbb{R}^M$ denotes the initial condition and $\mathbf{F} : [t_0, T] \times \mathbb{R}^M \to \mathbb{R}^M$ is a nonlinearity term. Additionally, several illustrations are shown, and solutions are obtained by the related predictor–corrector scheme with uniform and graded meshes. Consequently, the presented scheme with graded mesh can save numerical accuracy and reduce the computation fee.

This work is divided to six sections as follows. The first section is introduction. It includes the review of the related research and work. In Section 2, important definitions for the system of fractional differential equations with Ψ-Caputo derivative are presented. A modified predictor–corrector scheme with graded mesh is described in Section 3. Next the error estimation of this scheme is demonstrated. In Section 5, various examples are offered to demonstrate the performance of our numerical schemes. The final section gives the conclusion.

2 Ψ-Caputo nonlinear fractional-order systems

Some definitions and theorems in this section will be used to declare and verify our essential results. Let $\mathbb{J} = [t_0, T]$ be finite interval. $f \in \mathbb{C}(\mathbb{J}, \mathbb{R})$ is the continuous function from interval $\mathbb{J}$ into $\mathbb{R}$. Moreover, $\mathbb{C}^n(\mathbb{J}, \mathbb{R})$ is the continuous function and n times differentiable from interval $\mathbb{J}$ into $\mathbb{R}$. We suppose that $\alpha > 0$ and Ψ is increasing for all $t \in \mathbb{J}$.

Definition 2.1. [8] The Riemann-Liouville fractional integral of the function $f \in \mathbb{C}(\mathbb{J}, \mathbb{R})$ with the order α is defined as

$$\mathbb{I}_{t_0}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t \in \mathbb{J}, \tag{1}$$

where $\Gamma(\alpha)$ is the Gamma function such that

$$\Gamma(\alpha) := \int_{0}^{\infty} e^{-t} t^{\alpha-1} \mathrm{d}t.$$

Definition 2.2. [11] The Ψ-Riemann-Liouville fractional integral of the function $f \in \mathbb{C}(\mathbb{J}, \mathbb{R})$ with the order α is given by

$$\mathbb{I}_{t_0}^{\alpha,\Psi} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} f(s) \mathrm{d}s. \tag{2}$$

Definition 2.3. [11] Let $n = [\alpha] + 1$, the Ψ-Riemann-Liouville derivative of the function $f \in \mathbb{C}(\mathbb{J}, \mathbb{R})$ with the order α is defined as

$$\mathbb{D}_{t_0}^{\alpha,\psi} f(t) := \left(\frac{1}{\Psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \right)^{n} \mathbb{I}_{t_0}^{n-\alpha,\Psi} f(t). \tag{3}$$

Definition 2.4. [12] Given $\Psi \in \mathbb{C}^{n}(\mathbb{J})$ and $f \in \mathbb{C}^{n-1}(\mathbb{J})$, the Ψ-Caputo fractional derivative with order α is defined as

$${}^{c}\mathbb{D}_{t_0}^{\alpha,\Psi} f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_{t_0}^{t} \Psi'(s)(\Psi(t) - \Psi(s))^{n-\alpha-1} f_{\Psi}^{[n]}(s) \mathrm{d}s, & \text{if } \alpha \notin \mathbb{N}, \\ f_{\Psi}^{[n]}(t), & \text{if } \alpha = n \in \mathbb{N}, \end{cases} \tag{4}$$

where

$$f_{\Psi}^{[n]}(t) := \left(\frac{1}{\Psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \right)^{n} f(t). \tag{5}$$

Remark 1. From Definition 2.4, the examples of specific kernels ψ are presented as $\Psi(t) = t$, $\Psi(t) = \ln(t)$, and $\Psi(t) = t^{\varrho}$ reduce to Caputo, Caputo-Hadamard, and Caputo-Erdélyi-Kober fractional derivatives, respectively.

Then, the important properties of fractional ψ-integrals and Ψ-derivatives in [11] are introduced below.

Theorem 2.1. [12] *If* $f \in \mathbb{C}^{n-1}(\mathbb{J}, \mathbb{R})$, *then*

$$\mathbb{I}_{t_0}^{\alpha,\Psi} {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi} f(t) = f(t) - \sum_{i=0}^{n-1} \frac{f_{\psi}^{[i]}(t_0)}{i!} (\psi(t) - \psi(t_0))^{i}$$

and

$${}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi} \mathbb{I}_{t_0}^{\alpha,\Psi} f(t) = f(t).$$

In this study, the Ψ-Caputo nonlinear fractional-order systems are defined as

$$\begin{cases} {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi} \mathbf{Y}(t) = \mathbf{F}(t, \mathbf{Y}(t)), & t \in \mathbb{J}, \\ \mathbf{Y}(t_0) = \mathbf{Y}_0, \end{cases} \tag{6}$$

where
- $\mathbf{Y}(t) = [y_1(t), y_2(t), \ldots, y_M(t)]^T$,
- ${}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}\mathbf{Y}(t) = [{}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}y_1(t), {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}y_2(t), \ldots, {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}y_M(t)]^T$ with $\alpha \in (0,1)$,
- The function $\Psi \in C^1([t_0, T])$ is increasing such that $\Psi'(t) \neq 0$ for all $t \in [t_0, T]$,
- $\mathbf{Y}_0$ is fixed vectors in $\mathbb{R}^M$,
- $\mathbf{F} = [f_1, f_2, \ldots, f_M]^T$ with $\mathbf{F} \in \mathbb{C}(\mathbb{J} \times \mathbb{R}^M, \mathbb{R}^M)$.

In particular, we found that system (6) can be reformulated to the system of Volterra integral equations (SVIEs):

$$\mathbf{Y}(t) = \mathbf{Y}(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1}\mathbf{F}(s, y(s))\mathrm{d}s \tag{7}$$

or

$$\mathbf{Y}(t) = \mathbf{Y}(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1}\,{}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}\mathbf{Y}(s)\mathrm{d}s. \tag{8}$$

Based on the idea of Songsanga and Sa Ngiamsunthorn [23], we extend the concept of predictor–corrector scheme for solving Ψ-Caputo fractional differential systems in (6) by adding graded mesh. In order to ensure the existence of a unique solution for (6), we found that the detail of existence and uniqueness for the system of Ψ-Caputo fractional differential equations can be proved in [16]. In [22,24,25], the concept of smoothness properties is suggested for improving the optimal convergence of predictor–corrector scheme. To develop predictor–corrector scheme for the approximation solution of (7), the smoothness properties of [22,24,25] are applied in this study.

2.1 Smoothness properties

Theorem 2.2. [22]
- *Assume that $\alpha \in (0,1)$, U is an appropriate set, and $\mathbf{F} \in \mathbb{C}^2(U)$. Then, there exists a function $\Lambda \in \mathbb{C}^1(\mathbb{J})$, such that the solution $\mathbf{Y}(t)$ of equation* (6) *can be smoother than being continuous as*

$$\mathbf{Y}(t) = \Lambda(t) + \sum_{\omega=1}^{\lceil 1/\alpha\rceil-1} c_\omega(\Psi(t) - \Psi_0)^{\alpha\omega},$$

  where c_ω is the real number and $\lceil 1/\alpha\rceil$ is the ceiling value.
- *Assume that $\alpha \in (0,1)$, U is the appropriate set and $\mathbf{F} \in \mathbb{C}^3(U)$. Then, there exists a function $\widetilde{\Lambda} \in \mathbb{C}^2(\mathbb{J})$, such that the solution $\mathbf{Y}(t)$ of equation* (6) *can be smoother than being continuous as*

$$\mathbf{Y}(t) = \widetilde{\Lambda}(t) + \sum_{\omega=1}^{\lceil 2/\alpha\rceil-1} c_\omega(\Psi(t) - \Psi_0)^{\alpha\omega} + \sum_{\mu=1}^{\lceil 1/\alpha\rceil-1} d_\mu(\Psi(t) - \Psi_0)^{1+\alpha\mu},$$

  where c_ω and d_μ are real numbers.

Additionally, we applied Theorem 2.2 and modified the smoothness assumptions of [22] to the solution of Ψ-Caputo nonlinear fractional-order systems in (6).

Assumption 1. Given $\varrho \in (0,1)$ and $\mathbf{Y}$ can be the solution of (6). Denote that $\rho(\Psi(t)) \coloneqq {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}\mathbf{Y}(t) \in \mathbb{C}^2(\mathbb{J})$ with $\alpha \in (0,1)$. There is a positive constant ζ such that

$$|\rho'(\Psi(t))| \leq \zeta(\Psi(t) - \Psi_0)^{\varrho-1}, \quad |\rho''(\Psi(t))| \leq \zeta(\Psi(t) - \Psi_0)^{\varrho-2}. \tag{9}$$

where $\rho'(\cdot)$ and $\rho''(\cdot)$ are the first- and second-order derivatives of ρ, respectively.

Let $\mathbf{Y}_\Psi(t) \coloneqq \rho(\Psi(t))$. By (5) in Definition 2.4, we have

$$\begin{aligned}\mathbf{Y}_\Psi^{[1]}(t) &\coloneqq \left(\frac{1}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{Y}(t) = \rho'(\Psi(t)),\\ \mathbf{Y}_\Psi^{[2]}(t) &\coloneqq \left(\frac{1}{\Psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^2\mathbf{Y}(t) = \rho''(\Psi(t)).\end{aligned} \tag{10}$$

Therefore, condition (9) can be rewritten as follows:

$$|\mathbf{Y}_\Psi^{[1]}(t)| \le \zeta(\Psi(t) - \Psi_0)^{\varrho-1}, \quad |\mathbf{Y}_\Psi^{[2]}(t)| \le \zeta(\Psi(t) - \Psi_0)^{\varrho-2}. \tag{11}$$

Remark 2. Assumption 1 provides the behavior of $\mathbf{Y}_\Psi(t)$ near $t = t_0$. We imply that $\mathbf{Y}_\Psi(t)$ has the singularity near $t = t_0$. It is obvious that $\mathbf{Y}_\Psi(t) \notin \mathbb{C}^2(\mathbb{J})$. For instance, we choose $\mathbf{Y}_\Psi(t) = (\Psi(t) - \Psi_0)^\varrho$ with $\varrho \in (0, 1)$.

3 Predictor-corrector scheme with graded mesh

In this section, we investigate the predictor–corrector scheme for solving the system of nonlinear fractional differential equations involving Ψ-Caputo fractional derivative with graded mesh.

To divide the partition on the interval $\mathbb{J}$, we assumed that $t_0 < t_1 < \ldots < t_M = T$, where M is a positive integer. Motivated by [22,25], the graded mesh on $[\Psi(t_0), \Psi(T)]$ can be represented below

$$\frac{\Psi(t_j) - \Psi(t_0)}{\Psi(t_M) - \Psi(t_0)} = \left(\frac{j}{M}\right)^r, \tag{12}$$

where $\Psi(t_0) < \Psi(t_1) < \ldots < \Psi(t_M) = \Psi(T)$ and $r \ge 1$. In the case $r = 1$, equation (12) is called uniform mesh. Moreover, we denote $\Psi_k \approx \Psi(t_k)$, $\mathbf{F}_k \approx \mathbf{F}(t_k, \mathbf{Y}_k)$, and $\mathbf{Y}_k \approx \mathbf{Y}(t_k)$, $k = 0, 1, 2, \ldots, M - 1$. Therefore, the solution of the SVIEs (7) at t_{k+1} is rewritten in a piece-wise way

$$\mathbf{Y}_{k+1} = \mathbf{Y}_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\Psi'(s)(\Psi_{k+1} - \Psi(s))^{\alpha-1}\mathbf{F}(s, y(s))\mathrm{d}s. \tag{13}$$

The predictor–corrector scheme is proposed in the work of Liu et al. [22] for solving the numerical solution of nonlinear fractional differential equations. It is suggested to be one of the most reliable, consistent, and effective approaches. The graded mesh is applied to recover the optimal convergence order for Volterra integral equations. The SVIEs (13) can be solved through the modification of predictor–corrector scheme with graded mesh [22]. To approximate $\mathbf{F}(s, y(s))$ in the SVIEs (13), we apply the rectangular interpolation to obtain $\mathbf{F}(s, y(s)) \approx P_0(s) = \mathbf{F}_j$ on $[t_j, t_{j+1}]$, $j = 0, 1, 2, \ldots, k$. The mentioned step is known as the predictor step and can be defined as follows:

$$\mathbf{Y}_{k+1} = \mathbf{Y}_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\Psi'(s)(\Psi_{k+1} - \Psi(s))^{\alpha-1}P_0(s)\mathrm{d}s = \mathbf{Y}_0 + \sum_{j=0}^{k} b_{k+1,j}^{r,\alpha,\Psi}\mathbf{F}_j, \tag{14}$$

where $b_{k+1,j}^{r,\alpha,\Psi}$ with $k = 0, 1, 2, \ldots, M - 1$ is defined as

$$b_{k+1,j}^{r,\alpha,\Psi} = \frac{(\Psi_{k+1} - \Psi_j)^\alpha - (\Psi_{k+1} - \Psi_{j+1})^\alpha}{\Gamma(\alpha + 1)}. \tag{15}$$

Next the function $\mathbf{F}(s, y(s))$ on the right-hand side of (13) is approximated by using trapezoidal interpolation, which is $\mathbf{F}(s, y(s)) \approx P_1(s)$ on $[t_j, t_{j+1}]$, $j = 0, 1, 2, \ldots, k + 1$. $P_1(s)$ denotes the trapezoidal interpolation function

$$P_1(s) = \frac{\Psi(s) - \Psi_{j+1}}{\Psi_j - \Psi_{j+1}}\mathbf{F}_j + \frac{\Psi(s) - \Psi_j}{\Psi_{j+1} - \Psi_j}\mathbf{F}_{j+1}. \tag{16}$$

Then, this step is called corrector step and is defined as

$$\mathbf{Y}_{k+1} = \mathbf{Y}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \Psi'(s)(\Psi_{k+1} - \Psi(s))^{\alpha-1} P_1(s) \mathrm{d}s = \mathbf{Y}_0 + \sum_{j=0}^{k+1} a_{k+1,j}^{r,\alpha,\Psi} \mathbf{F}_j, \tag{17}$$

where $a_{k+1,j}^{r,\alpha,\Psi}$, $k = 0, 1, 2, \ldots, M-1$ is defined as

$$a_{k+1,j}^{r,\alpha,\Psi} = \frac{1}{\Gamma(\alpha+2)} \times \begin{cases} \dfrac{A_{k+1,0}}{\Psi_0 - \Psi_1}, & j = 0 \\ \dfrac{A_{k+1,j}}{\Psi_j - \Psi_{j+1}} + \dfrac{B_{k+1,j}}{\Psi_j - \Psi_{j-1}}, & 1 \le j \le k \\ (\Psi_{k+1} - \Psi_k)^{\alpha}, & j = k+1, \end{cases} \tag{18}$$

$$\begin{aligned} A_{k+1,j} &= \alpha(\Psi_{k+1} - \Psi_j)^{\alpha+1} + (\Psi_{k+1} - \Psi_{j+1})^{\alpha+1} \\ &\quad - (\alpha+1)(\Psi_{k+1} - \Psi_{j+1})(\Psi_{k+1} - \Psi_j)^{\alpha}, \qquad j = 0, 1, \ldots, k, \\ B_{k+1,j} &= \alpha(\Psi_{k+1} - \Psi_j)^{\alpha+1} + (\Psi_{k+1} - \Psi_{j-1})^{\alpha+1} \\ &\quad - (\alpha+1)(\Psi_{k+1} - \Psi_{j-1})(\Psi_{k+1} - \Psi_j)^{\alpha}, \qquad j = 1, 2, \ldots, k. \end{aligned}$$

Therefore, the predictor–corrector scheme is defined as

$$\begin{aligned} \mathbf{Y}_{k+1}^{P} &= \mathbf{Y}_0 + \sum_{j=0}^{k} b_{k+1,j}^{r,\alpha,\Psi} \mathbf{F}_j \\ \mathbf{Y}_{k+1} &= \mathbf{Y}_0 + \sum_{j=0}^{k} a_{k+1,j}^{r,\alpha,\Psi} \mathbf{F}_j + a_{k+1,k+1}^{r,\alpha,\Psi} \mathbf{F}(t_{k+1}, \mathbf{Y}_{k+1}^{P}), \end{aligned} \tag{19}$$

where $b_{k+1,j}^{r,\alpha,\Psi}$ and $a_{k+1,j}^{r,\alpha,\Psi}$ are defined in (15) and (18), respectively.

4 Error estimation of the approximation

Next we introduce some properties of the coefficients in (15) and (18), respectively, and several useful lemmas.

Lemma 4.1. *If $\alpha \in (0,1)$ and $Q(t)$ satisfies Assumption* 1, *then*

$$\left| \int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} (\mathbf{Y}_\Psi(s) - P_1(s)) \mathrm{d}\Psi(s) \right| \le \begin{cases} \zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha) < 2 \\ \zeta M^{-2} \log M, & \text{if } r(\varrho+\alpha) = 2 \\ \zeta M^{-2}, & \text{if } r(\varrho+\alpha) > 2. \end{cases}$$

Proof.

$$\begin{aligned} &\int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} (\mathbf{Y}_\Psi(s) - P_1(s)) \mathrm{d}\Psi(s) \\ &= \left(\int_{t_0}^{t_1} + \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} + \int_{t_k}^{t_{k+1}} \right) (\Psi_{k+1} - \Psi(s))^{\alpha-1} (\mathbf{Y}_\Psi(s) - P_1(s)) \mathrm{d}\Psi(s) \\ &= \mathbb{S}_1 + \mathbb{S}_2 + \mathbb{S}_3, \end{aligned}$$

where

$$\mathbb{S}_1 := \int_{t_0}^{t_1} (\Psi_{k+1} - \Psi(s))^{\alpha-1} [\mathbf{Y}_\Psi(s) - P_1(s)] \mathrm{d}\Psi(s), \tag{20}$$

$$\mathbb{S}_2 \coloneqq \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}[\mathbf{Y}_\Psi(s) - P_1(s)]\mathrm{d}\Psi(s), \tag{21}$$

and

$$\mathbb{S}_3 \coloneqq \int_{t_k}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}[\mathbf{Y}_\Psi(s) - P_1(s)]\mathrm{d}\Psi(s). \tag{22}$$

First, we consider $\mathbf{Y}_\Psi(s) - P_1(s)$ with $s \in [t_0, t_1]$,

$$\begin{aligned} \mathbf{Y}_\Psi(s) - P_1(s) &= \mathbf{Y}_\Psi(s) - \left[\mathbf{Y}_\Psi(t_1) + \frac{\Psi(s) - \Psi_1}{\Psi_1 - \Psi_0}(\mathbf{Y}_\Psi(t_1) - \mathbf{Y}_\Psi(t_0))\right] \\ &= \left[\frac{\Psi(s) - \Psi_1}{\Psi_0 - \Psi_1}(\mathbf{Y}_\Psi(s) - \mathbf{Y}_\Psi(t_0)) + \frac{\Psi(s) - \Psi_0}{\Psi_1 - \Psi_0}(\mathbf{Y}_\Psi(s) - \mathbf{Y}_\Psi(t_1))\right] \\ &= \frac{\Psi(s) - \Psi_1}{\Psi_0 - \Psi_1}\int_{t_0}^{s}\rho'(\Psi(\eta))\mathrm{d}\Psi(\eta) + \frac{\Psi(s) - \Psi_0}{\Psi_1 - \Psi_0}\int_{t_1}^{s}\rho'(\Psi(\eta))\mathrm{d}\Psi(\eta). \end{aligned} \tag{23}$$

By Assumption 1 and equation (23), we obtain

$$\begin{aligned} |\mathbf{Y}_\Psi(s) - P_1(s)| &\le \int_{t_0}^{s}|\rho'(\Psi(\eta))|\mathrm{d}\Psi(\eta) + \int_{s}^{t_1}|\rho'(\Psi(\eta))|\mathrm{d}\Psi(\eta) \\ &\le \zeta\int_{t_0}^{s}(\Psi(\eta) - \Psi_0)^{\varrho-1}d(\Psi(\eta) - \Psi_0) + \zeta\int_{s}^{t_1}(\Psi(\eta) - \Psi_0)^{\varrho-1}d(\Psi(\eta) - \Psi_0) \\ &\le \zeta(\Psi(s) - \Psi_0)^{\varrho} + \zeta(\Psi_1 - \Psi_0)^{\varrho}. \end{aligned} \tag{24}$$

For $k = 1, 2, \ldots, M - 1$, there exists a constant $\zeta > 0$ such that

$$(\Psi_{k+1} - \Psi_0) \ge (\Psi_{k+1} - \Psi_1) \ge \zeta(\Psi_{k+1} - \Psi_0),$$

which implies that

$$1 \le \frac{(\Psi_{k+1} - \Psi_0)}{(\Psi_{k+1} - \Psi_1)} = \frac{(k+1)^r}{(k+1)^r - 1^r} = \frac{1}{(k+1)^r - 1} + 1 \le \frac{1}{2^r - 1} + 1 \le \zeta.$$

For $\mathbb{S}_1$ in (20), we have

$$\begin{aligned} |\mathbb{S}_1| &\le \zeta\int_{t_0}^{t_1}(\Psi_{k+1} - \Psi(s))^{\alpha-1}(\Psi(s) - \Psi_0)^{\varrho}\mathrm{d}\Psi(s) + \zeta\int_{t_0}^{t_1}(\Psi_{k+1} - \Psi(s))^{\alpha-1}(\Psi_1 - \Psi_0)^{\varrho}\mathrm{d}\Psi(s) \\ &\le \zeta(\Psi_{k+1} - \Psi_1)^{\alpha-1}\int_{t_0}^{t_1}(\Psi(s) - \Psi_0)^{\varrho}\mathrm{d}\Psi(s) + \zeta(\Psi_{k+1} - \Psi_1)^{\alpha-1}(\Psi_1 - \Psi_0)^{\varrho+1} \\ &\le \zeta(\Psi_{k+1} - \Psi_1)^{\alpha-1}(\Psi_1 - \Psi_0)^{\varrho+1} \\ &\le \zeta(\Psi_{k+1} - \Psi_0)^{\alpha-1}(\Psi_1 - \Psi_0)^{\varrho+1} \\ &\le \zeta(\Psi_k - \Psi_0)^{\alpha-1}(\Psi_1 - \Psi_0)^{\varrho+1} \\ &= \zeta(\Psi_M - \Psi_0)^{\alpha-1}\left(\frac{k}{M}\right)^{r(\alpha-1)}(\Psi_M - \Psi_0)^{\varrho+1}\left(\frac{1}{M}\right)^{r(\varrho+1)} \\ &= \zeta(k^{r(\alpha-1)}M^{-r(\alpha+\varrho)}). \end{aligned}$$

Therefore,

$$|\mathbb{S}_1| \le \zeta M^{-r(\alpha+\varrho)}.$$

It follows from the mean value theorem that for $s \in (t_j, t_{j+1})$ there exists $t_j \leq \tau_j \leq t_{j+1}$ with $j = 1, 2, \ldots, k-1$ and $k = 2,3,\ldots, M-1$, such that

$$\mathbf{Y}_\Psi(s) - \left[\frac{\Psi(s) - \Psi_{j+1}}{\Psi(t_j) - \Psi_{j+1}}\mathbf{Y}_\Psi(t_j) + \frac{\Psi(s) - \Psi_j}{\Psi(t_{j+1}) - \Psi_j}\mathbf{Y}_\Psi(t_{j+1})\right] = (\Psi(s) - \Psi_j)(\Psi(s) - \Psi_{j+1})\mathbf{Y}_\Psi^{[2]}(\tau_j).$$

In the case of $\mathbb{S}_2$ in (21), the condition above is applied as

$$|\mathbb{S}_2| = \left|\sum_{j=1}^{k-1}\int_{t_j}^{t_{j+1}}(\Psi_{k+1} - \Psi(s))^{\alpha-1}(\Psi(s) - \Psi_j)(\Psi(s) - \Psi_{j+1})\mathbf{Y}_\Psi^{[2]}(\tau_j)\mathrm{d}\Psi(s)\right|.$$

Applying the result of Stynes et al. [26] and using the Assumption 1, we obtain

$$\begin{aligned}
|\mathbb{S}_2| &\leq \zeta\left|\sum_{j=1}^{k-1}(\Psi_{j+1} - \Psi_j)^2(\Psi_j - \Psi_0)^{\varrho-2}\int_{t_j}^{t_{j+1}}(\Psi_{k+1} - \Psi_j)^{\alpha-1}\mathrm{d}\Psi(s)\right| \\
&\leq \zeta\left|\sum_{j=1}^{\lceil\frac{k-1}{2}\rceil-1}(\Psi_{j+1} - \Psi_j)^2(\Psi_j - \Psi_0)^{\varrho-2}\int_{t_j}^{t_{j+1}}(\Psi_{k+1} - \Psi_j)^{\alpha-1}\mathrm{d}\Psi(s)\right| \\
&\quad + \zeta\left|\sum_{j=\lceil\frac{k-1}{2}\rceil}(\Psi_{j+1} - \Psi_j)^2(\Psi_j - \Psi_0)^{\varrho-2}\int_{t_j}^{t_{j+1}}(\Psi_{k+1} - \Psi_j)^{\alpha-1}\mathrm{d}\Psi(s)\right| \\
&= \mathbb{A}_1 + \mathbb{A}_2,
\end{aligned}$$

where

$$\mathbb{A}_1 \coloneqq \zeta\left|\sum_{j=1}^{\lceil\frac{k-1}{2}\rceil-1}(\Psi_{j+1} - \Psi_j)^2(\Psi_j - \Psi_0)^{\varrho-2}\int_{t_j}^{t_{j+1}}(\Psi_{k+1} - \Psi_j)^{\alpha-1}\mathrm{d}\Psi(s)\right|, \tag{25}$$

$$\mathbb{A}_2 \coloneqq \zeta\left|\sum_{j=\lceil\frac{k-1}{2}\rceil}(\Psi_{j+1} - \Psi_j)^2(\Psi_j - \Psi_0)^{\varrho-2}\int_{t_j}^{t_{j+1}}(\Psi_{k+1} - \Psi_j)^{\alpha-1}\mathrm{d}\Psi(s)\right|, \tag{26}$$

and $\lceil\frac{k-1}{2}\rceil$ can be the least integer greater than or equal to $\frac{k-1}{2}$.

Assuming that $j \leq \tau_j \leq j+1$ with $j = 1, 2, \ldots, \lceil\frac{k-1}{2}\rceil - 1$, we have the conditions, such that

$$\begin{aligned}
(\Psi_{j+1} - \Psi_j) &= (\Psi_M - \Psi_0)((j+1)^r - j^r)M^{-r} \\
&= \zeta r\tau_j^{r-1}M^{-r} \\
&\leq \zeta r(j+1)^{r-1}M^{-r} \\
&\leq \zeta j^{r-1}M^{-r}
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
(\Psi_{k+1} - \Psi_{j+1})^{\alpha-1} &= \left(\frac{M^r}{(k+1)^r - (j+1)^r}\right)^{1-\alpha}(\Psi_M - \Psi_0)^{\alpha-1} \\
&\leq \left(\frac{M^r}{(k+1)^r - \lceil\frac{k+1}{2}\rceil^r}\right)^{1-\alpha}(\Psi_M - \Psi_0)^{\alpha-1} \\
&\leq \zeta((k+1)^{-r}M^r)^{1-\alpha} \leq \zeta(M/k)^{r(1-\alpha)}.
\end{aligned} \tag{28}$$

For $k \geq 4$,

$$\begin{aligned}\mathbb{A}_1 &\leq \zeta \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} (\Psi_{j+1}-\Psi_j)^2(\Psi_j-\Psi_0)^{\varrho-2}(\Psi_{k+1}-\Psi_{j+1})^{\alpha-1}(\Psi_{j+1}-\Psi_j)\\
&\leq \zeta \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} (\Psi_{j+1}-\Psi_j)^3(\Psi_j-\Psi_0)^{\varrho-2}(\Psi_{k+1}-\Psi_{j+1})^{\alpha-1}\\
&\leq \zeta \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} (j^{r-1}M^{-r})^3\left(\frac{j}{M}\right)^{r(\varrho-2)}\left(\frac{M}{k}\right)^{r(1-\alpha)}\\
&\leq \zeta \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} (j^{r-1}M^{-r})^3\left(\frac{j}{M}\right)^{r(\varrho-2)}\left(\frac{M}{k}\right)^{r(1-\alpha)}\\
&= \zeta \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} j^{r(\alpha+\varrho)-3}M^{-r(\varrho+\alpha)}\left(\frac{j}{k}\right)^{r(1-\alpha)}\\
&= \zeta M^{-r(\varrho+\alpha)} \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} j^{r(\alpha+\varrho)-3}.\end{aligned} \tag{29}$$

Inequality (29) is considered in three cases

- If $r(\varrho+\alpha) < 2$,

$$\mathbb{A}_1 \leq \zeta M^{-r(\varrho+\alpha)} \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} j^{r(\varrho+\alpha)-3} \leq \zeta M^{-r(\varrho+\alpha)}.$$

- If $r(\varrho+\alpha) = 2$,

$$\mathbb{A}_1 \leq \zeta M^{-2} \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} j^{-1} \leq \zeta M^{-2}\left(1+\frac{1}{2}+\ldots+\frac{1}{M}\right) \leq \zeta M^{-2}\log M.$$

- If $r(\varrho+\alpha) > 2$,

$$\begin{aligned}\mathbb{A}_1 &\leq \zeta M^{-r(\varrho+\alpha)} \sum_{j=1}^{\left\lceil\frac{k-1}{2}\right\rceil-1} j^{r(\varrho+\alpha)-3}\\
&\leq \zeta M^{-r(\varrho+\alpha)}k^{r(\varrho+\alpha)-2}\\
&= \zeta (k/M)^{r(\varrho+\alpha)-2M^{-2}}\\
&\leq \zeta M^{-2}.\end{aligned}$$

Hence,

$$\mathbb{A}_1 \leq \begin{cases}\zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha) < 2\\ \zeta M^{-2}\log M, & \text{if } r(\varrho+\alpha) = 2\\ \zeta M^{-2}, & \text{if } r(\varrho+\alpha) > 2.\end{cases}$$

For $k \geq 2$ and $\left\lceil\frac{k-1}{2}\right\rceil \leq j \leq k-1$, we obtain that

$$\begin{aligned}(\Psi_j-\Psi_0)^{\varrho-2} &= (\Psi_M-\Psi_0)^{\varrho-2}(j/M)^{r(\varrho-2)}\\
&= (\Psi_M-\Psi_0)(M/j)^{r(2-\varrho)}\\
&\leq \zeta\left(\frac{M}{k}\right)^{r(2-\varrho)}\end{aligned} \tag{30}$$

and

$$\begin{aligned}\int_{t_{\lceil\frac{k-1}{2}\rceil}}^{t_k}(\Psi_{k+1}-\Psi(s))^{\alpha-1}\mathrm{d}\Psi(s)&=\frac{1}{\alpha}\left[\left(\Psi_{k+1}-\Psi_{\lceil\frac{k-1}{2}\rceil}\right)^{\alpha}-(\Psi_{k+1}-\Psi_k)^{\alpha}\right]\\&\leq\frac{1}{\alpha}\left(\Psi_{k+1}-\Psi_{\lceil\frac{k-1}{2}\rceil}\right)^{\alpha}\\&\leq\frac{1}{\alpha}(\Psi_{k+1}-\Psi_0)^{\alpha}\\&=\frac{1}{\alpha}(\Psi_M-\Psi_0)^{\alpha}\left(\frac{k+1}{M}\right)^{r\alpha}\leq\zeta\left(\frac{k}{M}\right)^{r\alpha}.\end{aligned}\tag{31}$$

From (30) and (31), we conclude that

$$\begin{aligned}\mathbb{A}_2&\leq\zeta\left|\sum_{j=\lceil\frac{k-1}{2}\rceil}^{k-1}(k^{r-1}M^{-r})^2\left(\frac{M}{k}\right)^{r(2-\varrho)}\int_{t_j}^{t_{j+1}}(\Psi_{k+1}-\Psi(s))^{\alpha-1}\mathrm{d}\Psi(s)\right|\\&\leq\zeta k^{r\varrho-2}M^{-r\varrho}\int_{t_{\lceil\frac{k-1}{2}\rceil}}^{t_k}(\Psi_{k+1}-\Psi(s))^{\alpha-1}\mathrm{d}\Psi(s)\\&\leq\zeta M^{-r(\varrho+\alpha)}k^{r(\varrho+\alpha)-2}\\&\leq\begin{cases}\zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha)<2\\ \zeta M^{-2}, & \text{if } r(\varrho+\alpha)\geq 2.\end{cases}\end{aligned}$$

Let $\tau_k\in(t_k,t_{k+1})$, $k=1,2,\ldots,M-1$. By Assumption 1, we obtain

$$\begin{aligned}|\mathbb{S}_3|&=\left|\int_{t_k}^{t_{k+1}}(\Psi_{k+1}-\Psi(s))^{\alpha-1}[\mathbf{Y}_\Psi(s)-P_1(s)]\mathrm{d}\Psi(s)\right|\\&=\left|\int_{t_k}^{t_{k+1}}(\Psi_{k+1}-\Psi(s))^{\alpha-1}(\Psi(s)-\Psi_k)(\Psi(s)-\Psi_{k+1})\mathbf{Y}_\Psi^{[2]}(\tau_k)\mathrm{d}\Psi(s)\right|\\&\leq\zeta(\Psi_{k+1}-\Psi_k)^2(\Psi_k-\Psi_0)^{\varrho-2}\int_{t_k}^{t_{k+1}}(\Psi_{k+1}-\Psi(s))^{\alpha-1}\mathrm{d}\Psi(s)\\&=\zeta(\Psi_{k+1}-\Psi_k)^2(\Psi_k-\Psi_0)^{\varrho-2}\frac{1}{\alpha}(\Psi_{k+1}-\Psi_k)^{\alpha}\\&=\zeta(\Psi_{k+1}-\Psi_k)^{2+\alpha}(\Psi_k-\Psi_0)^{\varrho-2}\\&\leq\zeta(\Psi_M-\Psi_0)^{2+\alpha}(k^{r-1}M^{r})^{2+\alpha}(\Psi_M-\Psi_0)^{\varrho-2}\left(\frac{k}{M}\right)^{r(\varrho-2)}\\&=\zeta k^{r(\alpha+\varrho)-2-\alpha}M^{-r(\varrho+\alpha)}\\&\leq\begin{cases}\zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha)<2+\alpha\\ \zeta M^{-(2+\alpha)}, & \text{if } r(\varrho+\alpha)\geq 2+\alpha.\end{cases}\end{aligned}\tag{32}$$

It is obvious that the bound of $\mathbb{S}_3$ is stronger than the bound of $\mathbb{A}_1$. The proof of this lemma is completed. □

Lemma 4.2. *Assume that M is a positive integer, $r\geq 1$, $\Psi\geq 0$, $j=0,1,2,\ldots,k+1$ with $k=0,1,2,\ldots,M-1$, and $\alpha\in(0,1)$. We have two conditions*

- $b_{k+1,j}^{r,\alpha,\Psi}$ *in* (15) *and* $a_{k+1,j}^{r,\alpha,\Psi}$ *in* (18) *are positive constants.*
- $a_{k+1,k+1}^{r,\alpha,\Psi}\leq\zeta M^{-r\alpha}k^{(r-1)\alpha}$

Proof. It is obvious that $b_{k+1,j}^{r,\alpha,\Psi}$ and $a_{k+1,j}^{r,\alpha,\Psi}$ are positive constants. Hence, we skip the proof of $b_{k+1,j}^{r,\alpha,\Psi}$ and $a_{k+1,j}^{r,\alpha,\Psi}$.

According to (15) and by mean value theorem, there is $\tau_k \in (k, k+1)$, such that

$$\begin{aligned} a_{k+1,k+1}^{r,\alpha,\Psi} &\le \frac{1}{\Gamma(\alpha+2)}(\Psi_{k+1} - \Psi_k)^{\alpha} \\ &\le \zeta(\Psi_M - \Psi_0)^{\alpha} M^{-r\alpha}((k+1)^r - k^r)^{\alpha} \\ &= \zeta M^{-r\alpha}(r\tau_k^{r-1})^{\alpha} = \zeta M^{-r\alpha}(r(k+1)^{(r-1)})^{\alpha} = \zeta M^{-r\alpha} k^{(r-1)\alpha}. \end{aligned}$$ □

Lemma 4.3. *If $\alpha \in (0,1)$ and $\mathbf{Y}_{\Psi}(t)$ satisfies Assumption 1, then*

$$\begin{aligned} &\left| a_{k+1,k+1}^{r,\alpha,\Psi} \int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}(\mathbf{Y}_{\Psi}(s) - P_0(s))\mathrm{d}\Psi(s) \right| \\ &\le \begin{cases} \zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha) < 1+\alpha \\ \zeta M^{-r(\varrho+\alpha)} \log M, & \text{if } r(\varrho+\alpha) = 1+\alpha \\ \zeta M^{-1-\alpha}, & \text{if } r(\varrho+\alpha) > 1+\alpha. \end{cases} \end{aligned} \tag{33}$$

Proof. Similar to the proof of Lemma 4.1, we denote

$$\hat{S}_1 \coloneqq a_{k+1,k+1}^{r,\alpha,\Psi} \int_0^{t_1} (\Psi_{k+1} - \Psi(s))^{\alpha-1}(\mathbf{Y}_{\Psi}(s) - P_0(s))\mathrm{d}\Psi(s),$$

$$\hat{S}_2 \coloneqq a_{k+1,k+1}^{r,\alpha,\Psi} \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}(\mathbf{Y}_{\Psi}(s) - P_0(s))\mathrm{d}\Psi(s),$$

and

$$\hat{S}_3 \coloneqq a_{k+1,k+1}^{r,\alpha,\Psi} \int_{t_k}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}(\mathbf{Y}_{\Psi}(s) - P_0(s))\mathrm{d}\Psi(s).$$

Then, we consider

$$a_{k+1,k+1}^{r,\alpha,\Psi} \int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}(\mathbf{Y}_{\Psi}(s) - P_0(s))\mathrm{d}\Psi(s) = \hat{S}_1 + \hat{S}_2 + \hat{S}_3.$$

For $|\mathbf{Y}_{\Psi}(s)| = |\rho(\Psi(s))| \le \zeta(\Psi(t) - \Psi_0)^{\varrho}$ and $|P_0(s)| = |\mathbf{Y}_{\Psi}(t_0)| = 0$ in Assumption 1,

$$\begin{aligned} |\hat{S}_1| &\le a_{k+1,k+1}^{r,\alpha,\Psi} \left(\int_{t_0}^{t_1} (\Psi_{k+1} - \Psi(s))^{\alpha-1} |\mathbf{Y}_{\Psi}(s)| \mathrm{d}\Psi(s) + \int_{t_0}^{t_1} (\Psi_{k+1} - \Psi(s))^{\alpha-1} |P_0(s)| \mathrm{d}\Psi(s) \right) \\ &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha}) \int_{t_0}^{t_1} (\Psi_{k+1} - \Psi(s))^{\alpha-1}(\Psi(s) - \Psi_0)^{\varrho} \mathrm{d}\Psi(s) \\ &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha})(\Psi_{k+1} - \Psi_1)^{\alpha-1}(\Psi_1 - \Psi_0)^{\varrho+1} \\ &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha})(\Psi_{k+1} - \Psi_0)^{\alpha-1}(\Psi_1 - \Psi_0)^{\varrho+1} \\ &= (\zeta M^{-r\alpha} k^{(r-1)\alpha})(\Psi_M - \Psi_0)^{\alpha-1} \left(\frac{k+1}{M} \right)^{r(\alpha-1)} (\Psi_M - \Psi_0)^{\varrho+1} \left(\frac{1}{M} \right)^{r(\varrho+1)} \\ &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha})(\zeta M^{-r(\alpha+\varrho)}) \\ &= \zeta \left(\frac{k}{M} \right)^{r\alpha} k^{-\alpha}(\zeta M^{-r(\alpha+\varrho)}). \end{aligned}$$

Thus,

$$|\hat{S}_1| \leq \zeta M^{-r(\alpha+\varrho)}.$$

We have Assumption 1 and the mean value theorem, which is $\tau_j \in (t_j, t_{j+1})$, $j = 1, 2, \ldots, k-1$, such that

$$\begin{aligned}|\hat{S}_2| &= \left| a^{r,\alpha,\Psi}_{k+1,k+1} \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} (\mathbf{Y}_\Psi(s) - P_0(s)) \mathrm{d}\Psi(s) \right| \\ &\leq a^{r,\alpha,\Psi}_{k+1,k+1} \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} (\Psi(s) - \Psi_j) |\mathbf{Y}^{[1]}_\Psi(\tau_j)| \mathrm{d}\Psi(s) \\ &= \mathbb{B}_1 + \mathbb{B}_2,\end{aligned}$$

where

$$\mathbb{B}_1 = \zeta a^{r,\alpha,\Psi}_{k+1,k+1} \left(\sum_{j=1}^{\left\lceil \frac{k-1}{2} \right\rceil - 1} \right) (\Psi_{j+1} - \Psi_j)(\Psi_j - \Psi_0)^{\varrho-1} \int_{t_j}^{t_{j+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} \mathrm{d}\Psi(s)$$

and

$$\mathbb{B}_2 = \zeta a^{r,\alpha,\Psi}_{k+1,k+1} \left(\sum_{\left\lceil \frac{k-1}{2} \right\rceil}^{k-1} \right) (\Psi_{j+1} - \Psi_j)(\Psi_j - \Psi_0)^{\varrho-1} \int_{t_j}^{t_{j+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} \mathrm{d}\Psi(s).$$

For $k \geq 4$, we consider

$$\begin{aligned}\mathbb{B}_1 &\leq (\zeta M^{-r\alpha} k^{(r-1)\alpha}) \sum_{j=1}^{\left\lceil \frac{k-1}{2} \right\rceil - 1} (\Psi_{j+1} - \Psi_j)^2 (\Psi_j - \Psi_0)^{\varrho-1} (\Psi_{k+1} - \Psi_{j+1})^{\alpha-1} \\ &= \zeta M^{-r\alpha} k^{(r-1)\alpha} \sum_{j=1}^{\left\lceil \frac{k-1}{2} \right\rceil - 1} (j^{r-1} M^{-r})^2 \left(\frac{j}{M} \right)^{r(\varrho-1)} \left(\frac{M}{k} \right)^{r(1-\alpha)} \\ &\leq \zeta M^{-r(\alpha+\varrho)} \sum_{j=1}^{\left\lceil \frac{k-1}{2} \right\rceil - 1} j^{r(\alpha+\varrho)-2-\alpha}.\end{aligned}$$

Thus,

$$\mathbb{B}_1 \leq \begin{cases} \zeta M^{-r(\alpha+\varrho)}, & \text{if } r(\alpha+\varrho) < 1+\alpha \\ \zeta M^{-r(\alpha+\varrho)} \log M, & \text{if } r(\alpha+\varrho) = 1+\alpha \\ \zeta M^{-1-\alpha}, & \text{if } r(\alpha+\varrho) > 1+\alpha. \end{cases}$$

According to $\left\lceil \frac{k-1}{2} \right\rceil \leq j \leq k-1$ with $k \geq 2$,

$$(\Psi_j - \Psi_0)^{\varrho-1} = (\Psi_M - \Psi_0)^{\varrho-1} \left(\frac{j}{M} \right)^{r(\varrho-1)} = (\Psi_M - \Psi_0)^{\varrho-1} \left(\frac{M}{j} \right)^{r(1-\varrho)} \leq \zeta \left(\frac{M}{k} \right)^{r(1-\varrho)}.$$

From the above inequality, we obtain

$$\begin{aligned}\mathbb{B}_2 &\le \zeta M^{-r\alpha} k^{(r-1)\alpha} \sum_{\lceil \frac{k-1}{2} \rceil}^{k-1} \left((\Psi_{j+1} - \Psi_j)(\Psi_j - \Psi_0)^{\varrho-1} \int_{t_j}^{t^{j+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} \mathrm{d}\Psi(s) \right) \\ &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha}) \sum_{\lceil \frac{k-1}{2} \rceil}^{k-1} \left((\zeta k^{r-1} M^{-r})(M/k)^{r(1-\varrho)} \int_{t_j}^{t_{j+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} \mathrm{d}\Psi(s) \right) \\ &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha}) k^{r-1-r+\varrho} M^{-r+r-r\varrho} \left(\frac{k}{M} \right)^{r\alpha} \\ &\le \zeta k^{r(\varrho+\alpha)-1-\alpha} M^{-r(\varrho+\alpha)}.\end{aligned}$$

Hence, we have

$$\mathbb{B}_2 \le \begin{cases} \zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha) < 1+\alpha \\ \zeta M^{-1-\alpha}, & \text{if } r(\varrho+\alpha) \ge 1+\alpha. \end{cases}$$

In the case of $\hat{S}_3$, we find that

$$\begin{aligned}|\hat{S}_3| &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha})(\Psi_{k+1} - \Psi_k)(\Psi_k - \Psi_0)^{\varrho-1}(\Psi_{k+1} - \Psi_k)^{\alpha} \\ &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha})(\Psi_{k+1} - \Psi_k)^{\alpha+1}(\Psi_k - \Psi_0)^{\varrho-1} \\ &\le (\zeta M^{-r\alpha} k^{(r-1)\alpha})(k^{r-1} M^{-r})^{1+\alpha} \left(\frac{k}{M} \right)^{r(\varrho-1)} \\ &= \zeta \left(\frac{k}{M} \right)^{r\alpha} k^{r(\alpha+\varrho)-1} M^{-r(\alpha+\varrho)} \\ &\le \zeta k^{r(\alpha+\varrho)-\alpha-1} M^{-r(\alpha+\varrho)}\end{aligned}$$

Therefore,

$$|\hat{S}_3| \le \begin{cases} \zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha) < 1+\alpha \\ \zeta M^{-1-\alpha,} & \text{if } r(\varrho+\alpha) \ge 1+\alpha. \end{cases}$$

This completes the proof. □

Lemma 4.4. *Let* $\mathbf{Y}_\Psi(s) = 1$, *there exists* $\zeta > 0$ *such that*

$$\sum_{j=0}^{k} a_{k+1,j}^{r,\alpha,\Psi} \le \zeta (\Psi_M - \Psi_0)^{\alpha} \tag{34}$$

and

$$\sum_{j=0}^{k} b_{k+1,j}^{r,\alpha,\Psi} \le \zeta (\Psi_M - \Psi_0)^{\alpha}. \tag{35}$$

Proof. We have

$$\int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} \mathbf{Y}_\Psi(s) \mathrm{d}\Psi(s) = \sum_{j=0}^{k+1} a_{j,k+1} g(t_j) + \text{ remainder term},$$

which implies that

$$\sum_{j=0}^{k+1} a_{k+1,j}^{r,\alpha,\Psi} = \int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1} \cdot 1 \mathrm{d}\Psi(s) = \frac{1}{\alpha} (\Psi_{k+1} - \Psi_0)^{\alpha} \le \zeta (\Psi_M - \Psi_0)^{\alpha}.$$

Because the proof of (35) is similar to the proof of (34), we only prove the case of (34). Therefore, the proof is complete. □

To prove Theorem 4.5, we recall that $\mathbf{Y}(t_j)$ and $\mathbf{Y}_j$ are the solutions of (7) and (19), respectively. Moreover, we denote $E_j = \mathbf{Y}(t_j) - \mathbf{Y}_j$ and apply all the above lemmas.

Theorem 4.5. *If* $\alpha \in (0,1)$ *and* $\mathbf{Y}_\Psi(t) := {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}\mathbf{Y}(t)$ *satisfy Assumption* 1,

$$\max_{0\le j\le M}|E_j| \le \begin{cases} \zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha) < 1+\alpha \\ \zeta M^{-r(\varrho+\alpha)}\log M, & \text{if } r(\varrho+\alpha) = 1+\alpha \\ \zeta M^{-(1+\alpha)}, & \text{if } r(\varrho+\alpha) > 1+\alpha. \end{cases}$$

Proof. We suppose that

$$\begin{aligned} \acute{S}_1 &:= \frac{1}{\Gamma(\alpha)}\left\{\int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}(\mathbf{F}(s, y(s)) - P_1(s))\mathrm{d}\Psi(s)\right. \\ \acute{S}_2 &:= \sum_{j=0}^{k} a_{k+1,j}^{r,\alpha,\Psi}(\mathbf{F}(t_j, \mathbf{Y}(t_j)) - \mathbf{Y}_\Psi(t_0)) \\ \acute{S}_3 &:= a_{k+1,k+1}^{r,\alpha,\Psi}(\mathbf{F}(t_{k+1}, \mathbf{Y}(t_{k+1})) - \mathbf{F}(t_{k+1}, \mathbf{Y}_{k+1}^{P})). \end{aligned}$$

Subtracting (19) from (7), we obtain

$$E_{k+1} = \frac{1}{\Gamma(\alpha)}(\acute{S}_1 + \acute{S}_2 + \acute{S}_3).$$

By Lemma 4.1, one obtains:

$$|\acute{S}_1| \le \begin{cases} \zeta M^{-r(\varrho+\alpha)}, & \text{if } r(\varrho+\alpha) < 2, \\ \zeta M^{-2}\log M, & \text{if } r(\varrho+\alpha) = 2, \\ \zeta M^{-2}, & \text{if } r(\varrho+\alpha) > 2. \end{cases}$$

Applying the Lipschitz condition of $\mathbf{F}$ and by Lemma 4.2, there is a constant $\mathcal{L} > 0$, such that

$$|\acute{S}_2| = \left|\sum_{j=0}^{k} a_{k+1,j}^{r,\alpha,\Psi}(\mathbf{F}(t_j, \mathbf{Y}(t_j)) - \mathbf{F}_j)\right| \le \sum_{j=0}^{k} a_{k+1,j}^{r,\alpha,\Psi}|(\mathbf{F}(t_j, \mathbf{Y}(t_j)) - \mathbf{F}_j)| \le \mathcal{L}\sum_{j=0}^{k} a_{k+1,j}^{r,\alpha,\Psi}|\mathbf{Y}(t_j) - \mathbf{Y}_j|.$$

To estimate the term of $\acute{S}_3$, we have

$$\mathbf{Y}(t_{k+1}) - \mathbf{Y}_{k+1}^{P} = \frac{1}{\Gamma(\alpha)}\left\{\int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}(\mathbf{F}(s, y(s)) - P_0(s))\mathrm{d}\Psi(s) + \sum_{j=0}^{k} b_{k+1,j}^{r,\alpha,\Psi}(\mathbf{F}(t_j, \mathbf{Y}(t_j)) - \mathbf{F}_j)\right\}.$$

Denote that

$$\mathbb{D}_1 := \zeta a_{k+1,k+1}^{r,\alpha,\Psi}\mathcal{L}\int_{t_0}^{t_{k+1}} (\Psi_{k+1} - \Psi(s))^{\alpha-1}|\mathbf{F}(s, y(s)) - P_0(s)|\mathrm{d}\Psi(s)$$

and

$$\mathbb{D}_2 := \zeta a_{k+1,k+1}^{r,\alpha,\Psi}\mathcal{L}\sum_{j=0}^{k} b_{k+1,j}^{r,\alpha,\Psi}|\mathbf{F}(t_j, \mathbf{Y}(t_j)) - \mathbf{F}_j|.$$

By Lemma 4.3, the term $\mathbb{D}_1$ is estimated. Applying Lemma 4.2 for the case of $\mathbb{D}_2$, we obtain

$$\begin{aligned}\mathbb{D}_2 &\leq \zeta a^{r,\alpha,\Psi}_{k+1,k+1}\mathcal{L}\sum_{j=0}^{k} b^{r,\alpha,\Psi}_{k+1,j}|\mathbf{Y}(t_j)-\mathbf{Y}_j|\\ &\leq (\zeta M^{-r\alpha}k^{(r-1)\alpha})\sum_{j=0}^{k} b^{r,\alpha,\Psi}_{k+1,j}|\mathbf{Y}(t_j)-\mathbf{Y}_j|\\ &\leq \zeta(k/M)^{(r-1)\alpha}M^{-\alpha}\sum_{j=0}^{k} b^{r,\alpha,\Psi}_{k+1,j}|\mathbf{Y}(t_j)-\mathbf{Y}_j|\\ &\leq \zeta M^{-\alpha}\sum_{j=0}^{k} b^{r,\alpha,\Psi}_{k+1,j}|\mathbf{Y}(t_j)-\mathbf{Y}_j|.\end{aligned}$$

Similar to the proof of term $\acute{S}_2$, we apply Lemma 4.2 and the Lipschitz condition of $\mathbf{F}$ for the proof of term $\acute{S}_3$.

$$|\acute{S}_3| = |a^{r,\alpha,\Psi}_{k+1,k+1}(\mathbf{F}(t_{k+1},\mathbf{Y}(t_{k+1})) - \mathbf{F}(t_{k+1},\mathbf{Y}^P_{k+1}))| \leq a^{r,\alpha,\Psi}_{k+1,k+1}\mathcal{L}|\mathbf{Y}(t_{k+1})-\mathbf{Y}^P_{k+1}| = \mathbb{D}_1 + \mathbb{D}_2.$$

Therefore, we conclude that

$$|\mathbf{Y}(t_{k+1})-\mathbf{Y}_{k+1}| \leq \zeta|\acute{S}_1| + \zeta\sum_{j=0}^{k} a^{r,\alpha,\Psi}_{k+1,j}|\mathbf{Y}(t_j)-\mathbf{Y}_j| + \zeta|\mathbb{D}_2| + \zeta M^{-\alpha}\sum_{j=0}^{k} b^{r,\alpha,\Psi}_{k+1,j}|\mathbf{Y}(t_j)-\mathbf{Y}_j|.$$

This completes the proof. □

5 Numerical examples

To support Theorem 4.5 in Section 4, we present some numerical examples in this section. The Ψ-Caputo fractional differential systems, both linear and nonlinear type, can be solved by using our numerical scheme. In the examples, we will choose distinct functions Ψ and different values of r to investigate the effectiveness of the error estimation.

Example 1. Let α and $\beta \in (0,1)$, we consider the following nonlinear Ψ-Caputo fractional differential equations:

$$\begin{cases} {}^C\mathbb{D}^{\alpha,\Psi}_{t_0} y(t) = f(t,y(t)), & t\in[1,2],\\ y(1)=0, \end{cases} \tag{36}$$

where the function f in this case is nonsmooth and nonlinear as

$$f(t,y(t)) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}(\Psi(t)-\Psi_0)^{\beta-\alpha} + (\Psi(t)-\Psi_0)^{2\beta} - y^2$$

with two kernels $\Psi_1(t) \coloneqq \log t$ and $\Psi_2(t) \coloneqq \sqrt{t-1}$. Moreover, it is well known that the exact solution in this example is provided as $y = (\Psi(t)-\Psi_0)^\beta$ and ${}^C\mathbb{D}^{\alpha,\Psi}_{t_0} y(t) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}(\Psi(t)-\Psi_0)^{\beta-\alpha}$. By Assumption 1 and Theorem 4.5, the error estimation is obtained as follows:

$$\max_{0\leq j\leq M}|E_j| \leq \begin{cases} \zeta M^{-r\beta}, & \text{if } r < \dfrac{1+\alpha}{\beta},\\ \zeta M^{-r\beta}\log M, & \text{if } r = \dfrac{1+\alpha}{\beta},\\ \zeta M^{-(1+\alpha)}, & \text{if } r > \dfrac{1+\alpha}{\beta}, \end{cases} \tag{37}$$

where $\varrho = \beta - \alpha$.

For this example, the value of β can be fixed at 0.9. In Tables 1–3, the maximum absolute errors of our numerical example are presented by varying the order α and the values of r at $r = 1$ and $r = \frac{1+\alpha}{\beta}$, which are uniform mesh and graded mesh, respectively. We found that the maximum absolute errors in the case of graded mesh give higher accuracy than the case uniform mesh. In particular, the numerical solutions of both

Table 1: Maximum absolute error with $\Psi(t) = \log t$

N	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$r = 1$	$r = \frac{1+\alpha}{\beta}$	$r = 1$	$r = \frac{1+\alpha}{\beta}$	$r = 1$	$r = \frac{1+\alpha}{\beta}$
10	9.17×10^{-3}	6.38×10^{-3}	2.15×10^{-2}	4.51×10^{-3}	3.54×10^{-2}	0.00452
20	5.63×10^{-3}	2.18×10^{-3}	1.19×10^{-2}	1.49×10^{-3}	1.92×10^{-2}	0.0013
40	3.18×10^{-3}	7.66×10^{-4}	6.47×10^{-3}	4.92×10^{-4}	1.03×10^{-2}	0.000373
80	1.74×10^{-3}	2.75×10^{-4}	3.48×10^{-3}	1.62×10^{-4}	5.53×10^{-3}	0.000107
160	9.38×10^{-4}	1.00×10^{-4}	1.87×10^{-3}	5.36×10^{-5}	2.96×10^{-3}	3.08×10^{-5}
320	5.04×10^{-4}	3.69×10^{-5}	1.00×10^{-3}	1.77×10^{-5}	1.59×10^{-3}	8.84×10^{-6}
640	2.71×10^{-4}	1.37×10^{-5}	5.37×10^{-4}	5.83×10^{-6}	8.51×10^{-4}	2.54×10^{-6}
1,280	1.45×10^{-4}	5.08×10^{-6}	2.88×10^{-4}	1.92×10^{-6}	4.56×10^{-4}	7.29×10^{-7}

Table 2: Maximum absolute error with $\Psi(t) = \sqrt{t-1}$

N	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$r = 1$	$r = \frac{1+\alpha}{\beta}$	$r = 1$	$r = \frac{1+\alpha}{\beta}$	$r = 1$	$r = \frac{1+\alpha}{\beta}$
10	1.08×10^{-2}	9.29×10^{-3}	2.87×10^{-2}	6.25×10^{-3}	4.85×10^{-2}	6.29×10^{-3}
20	7.42×10^{-3}	3.13×10^{-3}	1.64×10^{-2}	2.07×10^{-3}	2.65×10^{-2}	1.81×10^{-3}
40	4.33×10^{-3}	1.09×10^{-3}	8.95×10^{-3}	6.84×10^{-4}	1.43×10^{-2}	5.19×10^{-4}
80	2.40×10^{-3}	3.89×10^{-4}	4.83×10^{-3}	2.26×10^{-4}	7.68×10^{-3}	1.49×10^{-4}
160	1.30×10^{-3}	1.41×10^{-4}	2.60×10^{-3}	7.45×10^{-5}	4.12×10^{-3}	4.28×10^{-5}
320	7.00×10^{-4}	5.17×10^{-5}	1.39×10^{-3}	2.46×10^{-5}	2.21×10^{-3}	1.23×10^{-5}
640	3.76×10^{-4}	1.91×10^{-5}	7.46×10^{-4}	8.11×10^{-6}	1.18×10^{-3}	3.53×10^{-6}
1,280	2.02×10^{-4}	7.11×10^{-6}	4.00×10^{-4}	2.67×10^{-6}	6.34×10^{-4}	1.01×10^{-6}

Table 3: Maximum absolute error with $\Psi(t) = \cos\frac{\pi t}{2}$

N	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$r = 1$	$r = \frac{1+\alpha}{\beta}$	$r = 1$	$r = \frac{1+\alpha}{\beta}$	$r = 1$	$r = \frac{1+\alpha}{\beta}$
10	1.61×10^{-1}	1.58×10^{-1}	3.16×10^{-2}	2.46×10^{-2}	4.74×10^{-2}	2.46×10^{-2}
20	5.53×10^{-2}	6.34×10^{-2}	1.69×10^{-2}	6.47×10^{-3}	2.56×10^{-2}	1.72×10^{-3}
40	1.77×10^{-2}	2.14×10^{-2}	9.05×10^{-3}	1.82×10^{-3}	1.38×10^{-2}	4.89×10^{-4}
80	6.00×10^{-3}	7.29×10^{-3}	4.85×10^{-3}	5.45×10^{-4}	7.36×10^{-3}	1.39×10^{-4}
160	2.12×10^{-3}	2.57×10^{-3}	2.60×10^{-3}	1.70×10^{-4}	3.93×10^{-3}	3.95×10^{-5}
320	7.63×10^{-4}	9.24×10^{-4}	1.39×10^{-3}	5.41×10^{-5}	2.10×10^{-3}	1.12×10^{-5}
640	3.77×10^{-4}	3.36×10^{-4}	7.47×10^{-4}	1.75×10^{-5}	1.12×10^{-3}	3.18×10^{-6}
1,280	2.02×10^{-4}	1.23×10^{-4}	4.00×10^{-4}	5.70×10^{-6}	5.99×10^{-4}	8.97×10^{-7}

uniform and graded meshes are awfully close to the exact solution when N is increased. Moreover, the maximum absolute errors in Table 1 correspond to the result in [25].

Example 2. The linear Ψ-Caputo fractional differential system is defined as

$$\begin{cases} {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}\mathbf{Y}(t) = A\mathbf{Y}(t), & t \in [0,1], \\ \mathbf{Y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \end{cases} \tag{38}$$

where

$$A \coloneqq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

From [16], the exact solution of this example is given by

$$\mathbf{Y}(t) = \mathbf{Y}(0)E_\alpha(A(\Psi(t) - \Psi(0))^\alpha),$$

where

$$E_\alpha(A) \coloneqq \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha + 1)} = I + \frac{A}{\Gamma(\alpha + 1)} + \frac{A^2}{\Gamma(2\alpha + 1)} + \ldots$$

is the matrix Mittag-Leffler function for a square matrix A. Because $\mathbf{Y}(t)$ is smooth, this system shows that $\varrho = \alpha$ in Assumption 1. Theorem 4.5 is defined as

$$\max_{0\le j\le M}|E_j| \le \begin{cases} \zeta M^{-2r\alpha}, & \text{if } r < \dfrac{1+\alpha}{2\alpha}, \\ \zeta M^{-2r\alpha}\log M, & \text{if } r = \dfrac{1+\alpha}{2\alpha}, \\ \zeta M^{-(1+\alpha)}, & \text{if } r > \dfrac{1+\alpha}{2\alpha}, \end{cases} \tag{39}$$

In this example, we proposed the results with two kernels $\Psi_1(t) \coloneqq \sqrt{t-1}$ and $\Psi_2(t) \coloneqq \cos(\frac{\pi t}{2})$.

Similar the previous example, the maximum absolute errors of our numerical example in Tables 4 and 5 are shown by varying the order α and the values of r at $r = 1$ and $r = \frac{1+\alpha}{2\alpha}$, which are uniform mesh and graded mesh, respectively. The maximum absolute errors in case of both the uniform and graded meshes give higher accuracy. Because we are aware of specific details about the exact solutions in the examples above, the graded

Table 4: Maximum absolute error with $\Psi(t) = \sqrt{t-1}$

N	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$r = 1$	$r = \frac{1+\alpha}{2\alpha}$	$r = 1$	$r = \frac{1+\alpha}{2\alpha}$	$r = 1$	$r = \frac{1+\alpha}{2\alpha}$
10	5.89×10^{-1}	1.73×10^{-1}	3.63×10^{-1}	3.39×10^{-1}	4.96×10^{-1}	5.00×10^{-1}
20	2.55×10^{-1}	4.45×10^{-2}	7.68×10^{-2}	7.22×10^{-2}	1.06×10^{-1}	1.10×10^{-1}
40	1.24×10^{-1}	1.46×10^{-2}	2.16×10^{-2}	1.92×10^{-2}	2.53×10^{-2}	2.65×10^{-2}
80	5.94×10^{-2}	5.08×10^{-3}	7.13×10^{-3}	5.66×10^{-3}	6.60×10^{-3}	6.92×10^{-3}
160	2.83×10^{-2}	1.82×10^{-3}	2.63×10^{-3}	1.75×10^{-3}	1.80×10^{-3}	1.90×10^{-3}
320	1.35×10^{-2}	6.61×10^{-4}	1.23×10^{-3}	5.58×10^{-4}	5.05×10^{-4}	5.34×10^{-4}
640	6.48×10^{-3}	2.43×10^{-4}	5.59×10^{-4}	1.80×10^{-4}	1.43×10^{-4}	1.52×10^{-4}
1,280	3.17×10^{-3}	8.99×10^{-5}	2.50×10^{-4}	5.88×10^{-5}	4.09×10^{-5}	4.36×10^{-5}

Table 5: Maximum absolute error with $\Psi(t) = \cos(\frac{\pi t}{2})$

N	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$r = 1$	$r = \frac{1+\alpha}{2\alpha}$	$r = 1$	$r = \frac{1+\alpha}{2\alpha}$	$r = 1$	$r = \frac{1+\alpha}{2\alpha}$
20	6.24×10^{-2}	1.31×10^{-2}	7.67×10^{-3}	5.97×10^{-3}	2.14×10^{-3}	2.47×10^{-3}
40	2.97×10^{-2}	4.56×10^{-3}	2.76×10^{-3}	1.03×10^{-3}	6.02×10^{-4}	6.97×10^{-4}
80	1.42×10^{-2}	1.63×10^{-3}	1.30×10^{-3}	6.48×10^{-4}	2.89×10^{-4}	2.00×10^{-4}
160	6.80×10^{-3}	5.95×10^{-4}	5.90×10^{-4}	2.15×10^{-4}	1.72×10^{-4}	5.79×10^{-5}
320	3.31×10^{-3}	2.19×10^{-4}	2.64×10^{-4}	7.16×10^{-5}	4.97×10^{-5}	1.68×10^{-5}
640	2.00×10^{-3}	8.10×10^{-5}	1.17×10^{-4}	2.38×10^{-5}	1.44×10^{-5}	4.89×10^{-6}
1,280	1.20×10^{-3}	3.02×10^{-5}	5.13×10^{-5}	7.91×10^{-6}	1.28×10^{-6}	1.42×10^{-6}

mesh is divided based on the exact solution. However, choosing the value of r in a real problem depended on the real data because we cannot usually find the exact solution.

Example 3. Brusselator system with nonlinear Ψ-Caputo fractional-order derivative is defined as

$$\begin{cases} {}^{C}\mathbb{D}_{t_0}^{\alpha,\Psi}\mathbf{Y}(t) = \mathbf{F}(t, \mathbf{Y}(t)), & t \in [0, 100], \\ \mathbf{Y}(0) = \begin{bmatrix} 1.2 \\ 2.8 \end{bmatrix}, \end{cases} \tag{40}$$

where

$$\mathbf{F}(t, \mathbf{Y}(t)) := \begin{bmatrix} 1 - 4y_1(t) + y_1(t)^2 y_2(t) \\ 3y_1(t) - y_1(t)^2 y_2(t) \end{bmatrix}.$$

In this example, we cannot know the exact solution of (40). Therefore, we choose the value of ϱ based on Assumption 1 and the value of r based on Theorem 4.5. In addition, we present the numerical simulations with different r and two kernels as $\Psi_1(t) := t$ and $\Psi_2(t) := \sqrt{t}$ with $\alpha = 0.7$.

For the case of $\Psi(t) = t$, Figures 1 and 2 represent the behavior of the numerical solution for the Brusselator system (40) with $r = 1$ and $r = 1.5$, respectively.

In this case, we find that the behavior of Figure 1 is in agreement with the work of Garrappa [27].

For the case of $\Psi(t) = \sqrt{t}$, Figures 3 and 4 represent the behavior of the numerical solution for the Brusselator system (40) with value of $r = 1$ and $r = 1.5$, respectively.

(a) **(b)**

Figure 1: Behavior of the numerical solution for the system (40) with $\Psi(t) = t$ and $r = 1$ in the $(t, y_1(t))$ and $(t, y_2(t))$ planes and in the phase plane, respectively.

Figure 2: Behavior of the numerical solution for the system (40) with $\Psi(t) = t$ and $r = 1.5$ in the $(t, y_1(t))$ and $(t, y_2(t))$ planes and in the phase plane, respectively.

Figure 3: Behavior of the numerical solution for the system (40) with $\Psi(t) = \sqrt{t}$ and $r = 1$ in the $(t, y_1(t))$ and $(t, y_2(t))$ planes and in the phase plane at $T = 1{,}000$, respectively.

Figure 4: Behavior of the numerical solution for the system (40) with $\Psi(t) = \sqrt{t}$ and $r = 1.5$ in the $(t, y_1(t))$ and $(t, y_2(t))$ planes and in the phase plane at $T = 1{,}000$, respectively.

6 Discussion and conclusion

In order to solve the nonlinear Ψ-Caputo fractional-order differential systems with order $0 < \alpha < 1$, the predictor–corrector scheme with graded mesh is proposed in this article. The smoothness properties of the solution to equation (6) are also reviewed and discussed to help the proof of error estimation. After that, the error estimation on the fractional rectangle and fractional trapezoidal schemes with uniform ($r = 1$) and graded meshes ($r \geq 1$) have been made. It is found that the error estimation of the proposed scheme depends on the order of fractional derivative, the partition size on graded mesh, and the value of r. Based on the various functions of Ψ and the different values of α, N, and r, the utility and accuracy of numerical solutions on Examples 1 and 2 was investigated to support the theoretical analysis of predictor–corrector scheme. We found that the truncation error of predictor–corrector scheme with graded mesh has better convergence than the case of uniform mesh. For the case $\Psi(t) = \log t$ in Example 1, the maximum absolute error is in agreement with the numerical results of [25]. Moreover, the behaviors of numerical solution for the case $\Psi(t) = \log t$ in Example 3 are also in agreement with the results of [27]. All tables and figures indicate that our suggested scheme performed particularly well. Additionally, choosing the value of r for graded mesh depends on the error estimation in Theorem 4.5, whereas the choice of function $\Psi(t)$ depends on nonlinear term and initial condition. A general way to determine these parameters is not known. The optimal choice for the parameter r and the function $\Psi(t)$ is still open for further investigation.

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