Research Article

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A note on weighted measure-theoretic pressure

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Abstract: In this article, we investigate the relations between various types of weighted measure-theoretic pressures and different versions of weighted topological pressures. We show that various types of weighted measure-theoretic pressures for an ergodic measure with respect to a potential function are equal to the sum of measure-theoretic entropy of this measure and the integral of the potential function.

Keywords: measure-theoretic pressure, weighted pressure, packing topological pressure

MSC 2020: 37A05, 37G05

1 Introductions

Entropies serve as fundamental invariants in characterizing the complexity of dynamical systems. Among their extensions, topological pressure stands out as a non-trivial and natural generalization of topological entropy. The study of these concepts traces back to Kolmogorov, who introduced measure-theoretic entropy as an isomorphic invariant for measure-preserving dynamical systems [1,2]. Shortly afterward, Adler et al. defined topological entropy via open covers as a conjugate invariant [3], while Bowen [4] and Dinaburg [5] independently provided equivalent formulations using separated and spanning sets.

Building on ideas from statistical mechanics, Ruelle [6] introduced topological pressure for continuous functions under $\mathbb{Z}^+$-actions on compact spaces, establishing a variational principle under expansivity and the specification property. Walters [7] later generalized this result to conditions without such constraints. Further developments by Bowen extended topological entropy to arbitrary sets in topological dynamical systems using a Hausdorff dimension-like approach [8]. Pesin and Pitskel' [9] subsequently generalized this to noncompact sets, proving a variational principle under additional conditions. These concepts – topological pressure, variational principles, and equilibrium states – play a pivotal role in statistical mechanics, ergodic theory, and dynamical systems [10].

As key components of thermodynamic formalism [11], topological pressure and its associated variational principle and equilibrium measures are indispensable in the dimension theory of dynamical systems. They provide essential tools for analyzing the dimension of invariant sets and measures in conformal dynamics [10,12,13]. Recent work by Feng and Huang [14] introduced weighted topological pressure with a corresponding variational principle, while Tsukamoto [15] proposed alternative definitions of weighted topological entropy and pressure. The equivalence between these frameworks remains non-trivial and can be viewed as a topological generalization of the Bedford-McMullen carpet dimension formula.

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In this article, inspired by Feng and Huang [14], we introduce a generalized measure-theoretic pressure for factor maps between topological dynamical systems, extending the work of Pesin and Pitskel [9]. Adopting their approach, we define a weighted measure-theoretic pressure analogous to Hausdorff and packing measures, aiming to establish connections between Pesin-Pitskel pressure, packing-weighted pressure, and measure-theoretic entropy.

2 Preliminaries

Let $k \geq 2$. Assume that (X_i, d_i), $i = 1, \ldots, k$, are compact metric spaces, and (X_i, T_i) are topological dynamical systems. Moreover, assume that for each $1 \leq i \leq k-1$, (X_{i+1}, T_{i+1}) is a factor of (X_i, T_i) with a factor map $\pi_i : X_i \to X_{i+1}$; in other words, $\pi_1, \ldots, \pi_{k-1}$ are continuous maps such that the following diagrams commute.

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\pi_1} & X_2 & \xrightarrow{\pi_2} & \cdots & \xrightarrow{\pi_{k-1}} & X_k \\ {\scriptstyle T_1}\downarrow & & {\scriptstyle T_2}\downarrow & & & & {\scriptstyle T_k}\downarrow \\ X_1 & \xrightarrow{\pi_1} & X_2 & \xrightarrow{\pi_2} & \cdots & \xrightarrow{\pi_{k-1}} & X_k \end{array}$$

For convenience, we use π_0 as the identity map on X_1. Define $\tau_i : X_1 \to X_{i+1}$ by $\tau_i = \pi_i \circ \pi_{i-1} \circ \ldots \circ \pi_0$, for $i = 0, 1, \ldots, k-1$. Let $M(X_i, T_i)$ denote the set of all T_i-invariant Borel probability measures on X_i and $E(X_i, T_i)$ denote the set of ergodic measures. Fix $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$. For $\mu \in M(X_1, T_1)$, we call

$$h_\mu^{\mathbf{a}}(T_1) := \sum_{i=1}^{k} a_i h_{\mu \circ \tau_{i-1}^{-1}}(T_i)$$

the $\mathbf{a}$-weighted measure-theoretic entropy of μ with respect to T_1, or simply, the $\mathbf{a}$-weighted entropy of μ, where $h_{\mu \circ \tau_{i-1}^{-1}}(T_i)$ denotes the measure-theoretic entropy of $\mu \circ \tau_{i-1}^{-1}$ with respect to T_i. Let $C(X_1, \mathbb{R})$ denote the set of all continuous functions of X_1, and let $P_\mu^{\mathbf{a},B}(f)$ and $P_\mu^{\mathbf{a},KB}(f)$ denote, respectively, the Pesin-Pitskel pressure of μ (see Definition 2.4) and the Pesin-Pitskel pressure of μ in the sense of Katok (see Definition 2.5). Thus, we try to find relationships between these notions of different weighted pressure.

Definition 2.1. [14] ($\mathbf{a}$-Weighted Bowen ball) For $x \in X_1$, $n \in \mathbb{N}$, $\varepsilon > 0$, denote

$$B_n^{\mathbf{a}}(x, \varepsilon) = \{y \in X_1 : d_i(T_i^j \tau_{i-1} x, T_i^j \tau_{i-1} y) < \varepsilon \ \text{ for } \ 0 \leq j \leq \lceil (a_1 + \ldots + a_i)n \rceil - 1, i = 1, \ldots, k\},$$

where $\lceil u \rceil$ denotes the least integer $\geq u$. We call $B_n^{\mathbf{a}}(x, \varepsilon)$ the n-th $\mathbf{a}$-weighted Bowen ball of radius ε centered at x.

2.1 Weighted topological pressure

Definition 2.2. Let $Z \subset X_1$ be a nonempty set. Given $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\varepsilon > 0$, and $f \in C(X_1, \mathbb{R})$, define

$$M^{\mathbf{a}}(N, \alpha, \varepsilon, Z, f) = \inf\left\{\sum_i \exp\left(-\alpha n_i + \frac{1}{a_1} S_{\lceil a_1 n_i \rceil} f(x_i)\right) : Z \subset \bigcup_i B_{n_i}^{\mathbf{a}}(x_i, \varepsilon)\right\},$$

where the infimum is taken over all finite or countable collections of $\{B_{n_i}^{\mathbf{a}}(x_i, \varepsilon)\}_i$ such that $x_i \in X$, $n_i \geq N$, and $\cup_i B_{n_i}^{\mathbf{a}}(x_i, \varepsilon) \supset Z$. Likewise, we define

$$R^{\mathbf{a}}(n, \alpha, \varepsilon, Z, f) = \inf\left\{\sum_i \exp\left(-\alpha n + \frac{1}{a_1} S_{\lceil a_1 n \rceil} f(x_i)\right) : Z \subset \bigcup_i B_n^{\mathbf{a}}(x_i, \varepsilon)\right\},$$

where the infimum is taken over all finite or countable collections of $\{B_n^{\mathbf{a}}(x_i,\varepsilon)\}_i$ such that $x_i \in X$, $n \in \mathbb{N}$, and $\cup_i B_n^{\mathbf{a}}(x_i,\varepsilon) \supset Z$. Define

$$M^{\mathbf{a},P}(N,\alpha,\varepsilon,Z,f) = \sup\left\{\sum_i \exp\left(-\alpha n + \frac{1}{a_1} S_{\lceil a_1 n_i \rceil} f(x_i)\right)\right\},$$

where the supremum is taken over all finite or countable pairwise disjoint families $\{\overline{B}_{n_i}^{\mathbf{a}}(x_i,\varepsilon)\}$ such that $x_i \in Z$, $n_i \ge N$ for all i, where

$$\overline{B}_{n_i}^{\mathbf{a}}(x_i,\varepsilon) = \{y \in X_1 : d_{n_i}(T_i^j \tau_{i-1} x, T_i^j \tau_{i-1} y) \le \varepsilon \text{ for } 0 \le j \le \lceil (a_1 + \ldots + a_i) n \rceil - 1, i = 1, \ldots, k\}.$$

Let

$$\begin{aligned} M^{\mathbf{a}}(\alpha,\varepsilon,Z,f) &= \lim_{N\to\infty} M^{\mathbf{a}}(N,\alpha,\varepsilon,Z,f), \\ \underline{R}^{\mathbf{a}}(\alpha,\varepsilon,Z,f) &= \liminf_{N\to\infty} R^{\mathbf{a}}(N,\alpha,\varepsilon,Z,f), \\ \overline{R}^{\mathbf{a}}(\alpha,\varepsilon,Z,f) &= \limsup_{N\to\infty} R^{\mathbf{a}}(N,\alpha,\varepsilon,Z,f), \\ M^{\mathbf{a},P}(\alpha,\varepsilon,Z,f) &= \lim_{N\to\infty} M^{\mathbf{a},P}(N,\alpha,\varepsilon,Z,f). \end{aligned}$$

Define

$$M^{\mathbf{a},\mathcal{P}}(\alpha,\varepsilon,Z,f) = \inf\left\{\sum_{i=1}^{\infty} M^{\mathbf{a},P}(\alpha,\varepsilon,Z_i,f) : Z \subset \bigcup_{i=1}^{\infty} Z_i\right\}.$$

It is routine to check that when α goes from $-\infty$ to $+\infty$, the quantities

$$M^{\mathbf{a}}(\alpha,\varepsilon,Z,f), \quad \underline{M}^{\mathbf{a}}(\alpha,\varepsilon,Z,f), \quad \overline{M}^{\mathbf{a}}(\alpha,\varepsilon,Z,f), \quad M^{\mathbf{a},\mathcal{P}}(\alpha,\varepsilon,Z,f)$$

jump from $+\infty$ to 0 at unique critical values, respectively. Hence, we can define the numbers

$$\begin{aligned} P^{\mathbf{a},B}(\varepsilon,Z,f) &= \sup\{\alpha : M^{\mathbf{a}}(\alpha,\varepsilon,Z,f) = +\infty\} = \inf\{\alpha : M^{\mathbf{a}}(\alpha,\varepsilon,Z,f) = 0\}, \\ \underline{CP}^{\mathbf{a}}(\varepsilon,Z,f) &= \sup\{\alpha : \underline{R}^{\mathbf{a}}(\alpha,\varepsilon,Z,f) = +\infty\} = \inf\{\alpha : \underline{R}^{\mathbf{a}}(\alpha,\varepsilon,Z,f) = 0\}, \\ \overline{CP}^{\mathbf{a}}(\varepsilon,Z,f) &= \sup\{\alpha : \overline{R}^{\mathbf{a}}(\alpha,\varepsilon,Z,f) = +\infty\} = \inf\{\alpha : \overline{R}^{\mathbf{a}}(\alpha,\varepsilon,Z,f) = 0\}, \\ P^{\mathbf{a},P}(\varepsilon,Z,f) &= \sup\{\alpha : M^{\mathbf{a},\mathcal{P}}(\alpha,\varepsilon,Z,f) = +\infty\} = \inf\{\alpha : M^{\mathbf{a},\mathcal{P}}(\alpha,\varepsilon,Z,f) = 0\}. \end{aligned}$$

Definition 2.3. We call the following quantities:

$$\begin{aligned} P^{\mathbf{a},B}(Z,f) &= \lim_{\varepsilon\to 0} P^{\mathbf{a},B}(\varepsilon,Z,f), \\ \underline{CP}^{\mathbf{a}}(Z,f) &= \lim_{\varepsilon\to 0} \underline{CP}^{\mathbf{a}}(\varepsilon,Z,f), \\ \overline{CP}^{\mathbf{a}}(Z,f) &= \lim_{\varepsilon\to 0} \overline{CP}^{\mathbf{a}}(\varepsilon,Z,f), \\ P^{\mathbf{a},P}(Z,f) &= \lim_{\varepsilon\to 0} P^{\mathbf{a},P}(\varepsilon,Z,f), \end{aligned}$$

weighted Pesin-Pitskel, weighted lower capacity, weighted upper capacity, and weighted packing topological pressures of T_1 on the set Z with respect to f, respectively.

2.2 Weighted measure-theoretic pressure

Definition 2.4. We call the following quantities:

$$\begin{aligned}
P_\mu^{\mathbf{a},B}(f) &:= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf\{P^{\mathbf{a},B}(\varepsilon,Z,f) : \mu(Z)\ge 1-\delta\}\\
&= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf\{P^{\mathbf{a},B'}(\varepsilon,Z,f) : \mu(Z)\ge 1-\delta\},\\
\underline{CP}_\mu^{\mathbf{a}}(f) &:= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf\{\underline{CP}^{\mathbf{a}}(\varepsilon,Z,f) : \mu(Z)\ge 1-\delta\}\\
&= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf\{\underline{CP}^{\mathbf{a},'}(\varepsilon,Z,f) : \mu(Z)\ge 1-\delta\},\\
\overline{CP}_\mu^{\mathbf{a}}(f) &:= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf\{\overline{CP}^{\mathbf{a}}(\varepsilon,Z,f) : \mu(Z)\ge 1-\delta\}\\
&= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf\{\overline{CP}^{\mathbf{a},'}(\varepsilon,Z,f) : \mu(Z)\ge 1-\delta\},\\
P_\mu^{\mathbf{a},P}(f) &:= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf\{P^{\mathbf{a},P}(\varepsilon,Z,f) : \mu(Z)\ge 1-\delta\}\\
&= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf\{P^{\mathbf{a},P'}(\varepsilon,Z,f) : \mu(Z)\ge 1-\delta\}.
\end{aligned}$$

Definition 2.5. Let $Z\subset X$ be a nonempty set. Given $\mu\in\mathcal{M}(X), n\in\mathbb{N}, \alpha\in\mathbb{R}, \varepsilon>0, 0<\delta<1$, and $f\in C(X_1,\mathbb{R})$, define

$$M_\mu^{\mathbf{a}}(N,\alpha,\varepsilon,\delta,f) = \inf\left\{\sum_i \exp\left(-\alpha n_i + \frac{1}{a_1}S_{\lceil a_1 n_i\rceil}f(x_i)\right) : \mu\left(\bigcup_i B_{n_i}^{\mathbf{a}}(x_i,\varepsilon)\right)\ge 1-\delta\right\},$$

where the infimum is taken over all finite or countable collections of $\{B_{n_i}^{\mathbf{a}}(x_i,\varepsilon)\}_i$ such that $x_i\in X, n_i\ge N$, and $\mu(\cup_i B_{n_i}^{\mathbf{a}}(x_i,\varepsilon))\ge 1-\delta$. Likewise, we define

$$R_\mu^{\mathbf{a}}(n,\alpha,\varepsilon,\delta,f) = \inf\left\{\sum_i \exp\left(-\alpha n + \frac{1}{a_1}S_{\lceil a_1 n\rceil}f(x_i)\right) : \mu\left(\bigcup_i B_n(x_i,\varepsilon)\right)\ge 1-\delta\right\},$$

where the infimum is taken over all finite or countable collections of $\{B_n^{\mathbf{a}}(x_i,\varepsilon)\}_i$ such that $x_i\in X$, $n\in\mathbb{N}$ and $\mu(\cup_i B_n^{\mathbf{a}}(x_i,\varepsilon))\ge 1-\delta$. Let

$$\begin{aligned}
M_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f) &= \lim_{N\to\infty} M_\mu^{\mathbf{a}}(N,\alpha,\varepsilon,\delta,f),\\
\underline{M}_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f) &= \liminf_{N\to\infty} R_\mu^{\mathbf{a}}(N,\alpha,\varepsilon,\delta,f),\\
\overline{M}_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,T,f) &= \limsup_{N\to\infty} R_\mu^{\mathbf{a}}(N,\alpha,\varepsilon,\delta,f).
\end{aligned}$$

Define

$$M_\mu^{\mathbf{a},\mathcal{P}}(\alpha,\varepsilon,\delta,f) = \inf\left\{\sum_{i=1}^{\infty} M^{\mathbf{a},P}(\alpha,\varepsilon,Z_i,f) : \mu\left(\bigcup_{i=1}^{\infty} Z_i\right)\ge 1-\delta\right\}.$$

Thus, when α goes from $-\infty$ to $+\infty$, the quantities

$$M_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f),\quad \underline{M}_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f),\quad \overline{M}_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f),\quad \text{and}\quad M_\mu^{\mathbf{a},\mathcal{P}}(\alpha,\varepsilon,\delta,f)$$

jump from $+\infty$ to 0 at unique critical values, respectively. Hence, we can define the numbers

$$\begin{aligned}
P_\mu^{\mathbf{a},KB}(\varepsilon,\delta,f) &= \sup\{\alpha : M_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f)=+\infty\} = \inf\{\alpha : M_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f)=0\},\\
\underline{CP}_\mu^{\mathbf{a},K}(\varepsilon,\delta,f) &= \sup\{\alpha : \underline{R}_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f)=+\infty\} = \inf\{\alpha : \underline{R}_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f)=0\},\\
\overline{CP}_\mu^{\mathbf{a},K}(\varepsilon,\delta,f) &= \sup\{\alpha : \overline{R}_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f)=+\infty\} = \inf\{\alpha : \overline{R}_\mu^{\mathbf{a}}(\alpha,\varepsilon,\delta,f)=0\},\\
P_\mu^{\mathbf{a},KP}(\varepsilon,\delta,f) &= \sup\{\alpha : M_\mu^{\mathbf{a},\mathcal{P}}(\alpha,\varepsilon,\delta,f)=+\infty\} = \inf\{\alpha : M_\mu^{\mathbf{a},\mathcal{P}}(\alpha,\varepsilon,\delta,f)=0\}.
\end{aligned}$$

Definition 2.6. We call the following quantities:

$$P_\mu^{\mathbf{a},KB}(f) = \lim_{\varepsilon\to0}\lim_{\delta\to0} P_\mu^{\mathbf{a},KB}(\varepsilon,\delta,f),$$
$$\underline{CP}_\mu^{\mathbf{a},K}(f) = \lim_{\varepsilon\to0}\lim_{\delta\to0} \underline{CP}_\mu^{\mathbf{a},K}(\varepsilon,\delta,f),$$
$$\overline{CP}_\mu^{\mathbf{a},K}(f) = \lim_{\varepsilon\to0}\lim_{\delta\to0} \overline{CP}_\mu^{\mathbf{a},K}(\varepsilon,\delta,f),$$
$$P_\mu^{\mathbf{a},KP}(f) = \lim_{\varepsilon\to0}\lim_{\delta\to0} P_\mu^{\mathbf{a},KP}(\varepsilon,\delta,f),$$

weighted Pesin-Pitskel, weighted lower capacity, weighted upper capacity, and weighted packing pressures of μ in the sense of Katok with respect to f, respectively.

Let $f \in C(X_1,\mathbb{R})$ and $\mu \in \mathcal{M}(X)$. The measure-theoretic lower and upper local pressures of $x \in X_1$ with respect to μ and f are defined by

$$\underline{P}_\mu^{\mathbf{a}}(x,f) := \lim_{\varepsilon\to0}\liminf_{n\to\infty}\frac{-\log\mu(B_n^{\mathbf{a}}(x,\varepsilon)) + \frac{1}{a_1}S_{\lceil a_1 n\rceil}f(x)}{n},$$
$$\overline{P}_\mu^{\mathbf{a}}(x,f) := \lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{-\log\mu(B_n^{\mathbf{a}}(x,\varepsilon)) + \frac{1}{a_1}S_{\lceil a_1 n\rceil}f(x)}{n}.$$

Definition 2.7. The measure-theoretic lower and upper local pressures of μ with respect to f are defined as

$$\underline{P}_\mu^{\mathbf{a}}(f) := \int \underline{P}_\mu^{\mathbf{a}}(x,f)\mathrm{d}\mu,$$
$$\overline{P}_\mu^{\mathbf{a}}(f) := \int \overline{P}_\mu^{\mathbf{a}}(x,f)\mathrm{d}\mu.$$

Now, we state our main results as follows.

Theorem 2.1. *Let $f \in C(X_1,\mathbb{R})$ and μ be a non-atomic Borel ergodic measure on X_1. Then,*

$$\begin{aligned}P_\mu^{\mathbf{a},KB}(f) &= \underline{CP}_\mu^{\mathbf{a},K}(f) = \overline{CP}_\mu^{\mathbf{a},K}(f) = P_\mu^{\mathbf{a},KP}(f)\\ &= P_\mu^{\mathbf{a},B}(f) = \underline{CP}_\mu^{\mathbf{a}}(f) = \overline{CP}_\mu^{\mathbf{a}}(f) = P_\mu^{\mathbf{a},P}(f)\\ &= h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} f\mathrm{d}\mu.\end{aligned}$$

3 Proof of Theorem 2.1

To prove the main results, we first give a weighted topological pressure inequality as follows.

Proposition 3.1. *For any $f \in C(X_1,\mathbb{R})$ and any subset $Z \subset X_1$,*

$$P^{\mathbf{a},B}(Z,f) \le P^{\mathbf{a},P}(Z,f) \le \overline{CP}^{\mathbf{a}}(Z,f).$$

Proof. We first show that $P^{\mathbf{a},B}(Z,f) \le P^{\mathbf{a},P}(Z,f)$. Suppose that $P^{\mathbf{a},B}(Z,f) > s > -\infty$. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, let

$$\mathcal{F}_{n,\varepsilon}^{\mathbf{a}} = \{\mathcal{F}^{\mathbf{a}} : \mathcal{F}^{\mathbf{a}} = \{\overline{B}_n^{\mathbf{a}}(x_i,\varepsilon)\},\ x_i \in Z, \text{ and } \mathcal{F}^{\mathbf{a}} \text{ is a disjoint family}\}.$$

Take $\mathcal{F}^{\mathbf{a}}(N,\varepsilon,Z)\in\mathcal{F}^{\mathbf{a}}_{N,\varepsilon}$ such that $|\mathcal{F}^{\mathbf{a}}(N,\varepsilon,Z)|=\max_{\mathcal{F}^{\mathbf{a}}\in\mathcal{F}^{\mathbf{a}}_{N,\varepsilon}}\{|\mathcal{F}^{\mathbf{a}}|\}$, where $|\mathcal{F}^{\mathbf{a}}|$ denotes the cardinality of $\mathcal{F}$. We denote $\mathcal{F}^{\mathbf{a}}(N,\varepsilon,Z)=\{\overline{B}^{\mathbf{a}}_N(x_i,\varepsilon),\ i=1,\ldots,|\mathcal{F}^{\mathbf{a}}(N,\varepsilon,Z)|\}$. It is easy to check that

$$Z\subset\bigcup_{i=1}^{|\mathcal{F}^{\mathbf{a}}(N,\varepsilon,Z)|}B^{\mathbf{a}}_N(x_i,2\varepsilon+\delta),\quad\forall\delta>0.$$

Then, for any $s\in\mathbb{R}$,

$$M^{\mathbf{a}}(N,s,2\varepsilon+\delta,Z,f)\le e^{-sN}\sum_{i=1}^{|\mathcal{F}(N,\varepsilon,Z)|}\exp\left(\frac{1}{a_1}S_{\lceil a_1N\rceil}f(x_i)\right)\le M^{\mathbf{a},P}(N,s,\varepsilon,Z,f).$$

It thus follows that $M^{\mathbf{a}}(s,2\varepsilon+\delta,Z,f)\le M^{\mathbf{a},P}(s,\varepsilon,Z,f)$. By Definition 3.1, we can obtain that $M^{\mathbf{a},P}(\alpha,\varepsilon,Z,f)\le M^{\mathbf{a},\mathcal{P}}(s,\varepsilon,Z,f)$; thus, we have $M^{\mathbf{a}}(s,2\varepsilon+\delta,Z,f)\le M^{\mathbf{a},\mathcal{P}}(s,\varepsilon,Z,f)$. Since $P^{\mathbf{a},B}(Z,f)>s>-\infty$, $M^{\mathbf{a}}(s,2\varepsilon+\delta,Z,f)\ge1$ when ε and δ are small enough. Thus, $M^{\mathbf{a},\mathcal{P}}(s,\varepsilon,Z,f)\ge1$. This implies that $P^{\mathbf{a},P}(\varepsilon,Z,f)\ge s$ for ε small enough. Hence, $P^{\mathbf{a},P}(Z,f)\ge s$ and $P^{\mathbf{a},B}(Z,f)\le P^{\mathbf{a},P}(Z,f)$.

Next, we shall show $P^{\mathbf{a},P}(Z,f)\le\overline{CP}^{\mathbf{a}}(Z,f)$.

Without generality, we assume $P^{\mathbf{a},P}(Z,f)>-\infty$. Choose $-\infty<t<s<P^{\mathbf{a},P}(Z,f)$. Then, there exists $\delta>0$, such that for any $\varepsilon\in(0,\delta)$, $P^{\mathbf{a},P}(\varepsilon,Z,f)>s$ and $M^{\mathbf{a},P}(s,\varepsilon,Z,f)\ge M^{\mathbf{a},\mathcal{P}}(s,\varepsilon,Z,f)=\infty$. Hence, for any $N\in\mathbb{N}$, there exists a countable pairwise disjoint family $\{\overline{B}^{\mathbf{a}}_{n_i}(x_i,\varepsilon)\}$ such that $x_i\in Z, n_i\ge N$ for all i, and $\sum_i\exp\left(-n_is+\frac{1}{a_1}S_{\lceil a_1n_i\rceil}f_{n_i}(x)\right)>1$. For each k, let

$$m_k=\{x_i:n_i=k\}.$$

Then,

$$\sum_{k=N}^{\infty}\sum_{x\in m_k}\exp\left(\frac{1}{a_1}S_{\lceil a_1n_k\rceil}f(x)\right)e^{-ks}>1.$$

It is easy to check that there exists $k\ge N$ such that

$$\sum_{x\in m_k}\exp\left(\frac{1}{a_1}S_{\lceil a_1n_k\rceil}f(x)\right)e^{-kt}\ge1-e^{t-s}$$

(otherwise, $\sum_{k=N}^{\infty}\sum_{x\in m_k}\exp\left(\frac{1}{a_1}S_{\lceil a_1n_k\rceil}f(x)\right)e^{-ks}\le1$). Fixing a collection $\left\{B^{\mathbf{a}}_k\left(y_i,\frac{\varepsilon}{2}\right)\right\}_{i\in I}$ such that $Z\subset\bigcup_{i\in I}B^{\mathbf{a}}_k\left(y_i,\frac{\varepsilon}{2}\right)$, where I is at most countable, it is not difficult to check that for any $x_1,x_2\in m_k$ there exists different y_1 and y_2 such that $x_i\in B^{\mathbf{a}}\left(y_i,\frac{\varepsilon}{2}\right)$, $i=1,2$. Then,

$$R^{\mathbf{a}}\left(k,t,\frac{\varepsilon}{2},Z,f\right)\ge\sum_{x\in m_k}\exp\left(\frac{1}{a_1}S_{\lceil a_1n_i\rceil}f_k(x)\right)e^{-kt}\ge1-e^{t-s}.$$

Hence,

$$\overline{R}^{\mathbf{a}}\left(t,\frac{\varepsilon}{2},Z,f\right)=\limsup_{k\to\infty}R^{\mathbf{a}}\left(k,t,\frac{\varepsilon}{2},Z,f\right)\ge1-e^{t-s}>0.$$

Thus, $\overline{CP}^{\mathbf{a}}\left(\frac{\varepsilon}{2},Z,f\right)\ge t$. Letting $\varepsilon\to0$ yields $\overline{CP}^{\mathbf{a}}(Z,f)\ge t$. Since $t\in(-\infty,P^{\mathbf{a},P}(Z,f))$, it follows that $P^{\mathbf{a},P}(Z,f)\le\overline{CP}^{\mathbf{a}}(Z,f)$. □

Proposition 3.2. *Let $\mu\in\mathcal{M}(X_1)$ and $f\in C(X_1,\mathbb{R})$. Then,*

$$P^{\mathbf{a},KB}_{\mu}(f)=P^{\mathbf{a},B}_{\mu}(f),\quad\underline{CP}^{\mathbf{a},K}_{\mu}(f)=\underline{CP}^{\mathbf{a}}_{\mu}(f),$$
$$\overline{CP}^{\mathbf{a},K}_{\mu}(f)\le\overline{CP}^{\mathbf{a}}_{\mu}(f),\quad P^{\mathbf{a},KP}_{\mu}(f)=P^{\mathbf{a},P}_{\mu}(f).$$

Proof. We shall show that $P_\mu^{\mathbf{a},KB}(f) \le P_\mu^{\mathbf{a},B}(f)$. For any $N \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\varepsilon > 0$, $0 < \delta < 1$, and Z with $\mu(Z) \ge 1 - \delta$,

$$M_\mu^{\mathbf{a}}(N, \alpha, \varepsilon, \delta, f) \le M^{\mathbf{a}}(N, \alpha, \varepsilon, Z, f).$$

Letting $N \to \infty$ yields

$$M_\mu^{\mathbf{a}}(\alpha, \varepsilon, \delta, f) \le M^{\mathbf{a}}(\alpha, \varepsilon, Z, f).$$

This shows that

$$P_\mu^{\mathbf{a},KB}(\varepsilon, \delta, f) \le P^{\mathbf{a},B}(\varepsilon, Z, f),$$

and consequently,

$$P_\mu^{\mathbf{a},KB}(\varepsilon, \delta, f) \le \inf\{P^{\mathbf{a},B}(\varepsilon, Z, f) : \mu(Z) \ge 1 - \delta\}.$$

Letting $\delta \to 0$ and $\varepsilon \to 0$, the desired inequality follows. We can prove similarly $\underline{CP}_\mu^{\mathbf{a},K}(f) \le \underline{CP}_\mu^{\mathbf{a}}(f)$ and $\overline{CP}_\mu^{\mathbf{a},K}(f) \le \overline{CP}_\mu^{\mathbf{a}}(f)$.

To prove $P_\mu^{\mathbf{a},KB}(f) \ge P_\mu^{\mathbf{a},B}(f)$, let $a = P_\mu^{\mathbf{a},KB}(f)$. For any $s > 0$, there exists $\varepsilon' > 0$ such that

$$\lim_{\delta \to 0} P_\mu^{\mathbf{a},KB}(\varepsilon, \delta, f) < a + s, \quad \forall \varepsilon < \varepsilon'.$$

It follows that for any $\varepsilon \in (0, \varepsilon')$, there exists δ_ε so that

$$P_\mu^{\mathbf{a},KB}(\varepsilon, \delta, f) < a + s, \quad \forall \delta < \delta_\varepsilon.$$

This implies that $\lim_{n\to\infty} M_\mu^{\mathbf{a}}(n, a + s, \varepsilon, \delta, f) = 0$. For any $N \in \mathbb{N}$, we can find a sequence of $\delta_{N,m}$ with $\lim_{m\to 0} \delta_{N,m} = 0$ and a collection of $\{B_{n_i}^{\mathbf{a}}(x_i, \varepsilon)\}_{i\in I_{N,m}}$ such that $x_i \in X$, $n_i \ge N$, $\mu(\bigcup_{i\in I_{N,m}} B_{n_i}^{\mathbf{a}}(x_i, \varepsilon)) \ge 1 - \delta_{N,m}$, and

$$\sum_{i\in I_{N,m}} \exp\left(-(a + s)n_i + \frac{1}{a_1} S_{\lceil a_1 n_i \rceil} f(x)\right) \le \frac{1}{2^m}.$$

Let

$$Z_N = \bigcup_{m\in\mathbb{N}} \bigcup_{i\in I_{N,m}} B_{n_i}^{\mathbf{a}}(x_i, \varepsilon).$$

Then, $\mu(Z_N) = 1$ and

$$M^{\mathbf{a}}(N, a + s, \varepsilon, Z_N, f) \le 1.$$

Let $Z_\varepsilon = \bigcap_{N\in\mathbb{N}} Z_N$. Thus, $\mu(Z_\varepsilon) = 1$ and

$$M^{\mathbf{a}}(N, a + s, \varepsilon, Z_\varepsilon, f) \le M^{\mathbf{a}}(N, a + s, \varepsilon, Z_N, f) \le 1, \quad \forall N \in \mathbb{N}.$$

It follows that

$$P^{\mathbf{a},B}(\varepsilon, Z_\varepsilon, f) \le a + s.$$

Therefore,

$$P_\mu^{\mathbf{a},B}(f) = \lim_{\varepsilon\to 0} \liminf_{\delta\to 0}\{P^{\mathbf{a},B}(\varepsilon, Z, f) : \mu(Z) \ge 1 - \delta\} \le a + s.$$

The arbitrariness of s then implies the desired inequality. To prove $\underline{CP}_\mu^{\mathbf{a},K}(f) \ge \underline{CP}^{\mathbf{a}}(f)$, let $a = \underline{CP}_\mu^{\mathbf{a},K}(f)$. For any $s > 0$, there exists $\varepsilon' > 0$ such that for any $\varepsilon \in (0, \varepsilon')$, there exists δ_ε so that

$$\liminf_{N\to\infty} R_\mu^{\mathbf{a}}(N, a + s, \varepsilon, \delta, f) = 0, \quad \forall \delta < \delta_\varepsilon.$$

Fix $\delta \in (0, \delta_\varepsilon)$. For any $m \in \mathbb{N}$, we have

$$\liminf_{N\to\infty} R_\mu^{\mathbf{a}}\left(N, a + s, \varepsilon, \frac{\delta}{2^m}, f\right) = 0.$$

Then, for every $m \in \mathbb{N}$, there exists a family $\{B^{\mathbf{a}}_{k_m}(x_i,\varepsilon)\}_{i\in I_m}$ with $\mu(\cup_{i\in I_m} B^{\mathbf{a}}_{k_m}(x_i,\varepsilon)) \geq 1 - \frac{\delta}{2^m}$ such that

$$\sum_{i\in I_m} e^{-(a+s)k_m + \frac{1}{a_1} S_{\lceil a_1 k_m\rceil} f(x_i)} \leq 1.$$

Let $Z_\delta = \cap_{m\in\mathbb{N}} \cup_{i\in I_m} B^{\mathbf{a}}_{k_m}(x_i,\varepsilon)$. Then, $\mu(Z_\delta) \geq 1 - \delta$. It is easy to check that

$$\liminf_{N\to\infty} R^{\mathbf{a}}(N, a+s, \varepsilon, Z_\delta, f) \leq 1.$$

Thus,

$$\underline{CP}^{\mathbf{a}}(\varepsilon, Z_\delta, f) \leq a + s.$$

This implies that $\underline{CP}^{\mathbf{a}}_{\mu}(f) \leq a + s$, and the desired inequality follows from the arbitrariness of s.

Similarly, we can obtain

$$\overline{CP}^{\mathbf{a}}(f) \leq \overline{CP}^{\mathbf{a},K}(f).$$

We now show the fourth equality. We first prove that $P^{\mathbf{a},P}_{\mu}(f) \geq P^{\mathbf{a},KP}_{\mu}(f)$. For any $s < P^{\mathbf{a},KP}_{\mu}(f)$, there exists $\varepsilon', \delta' > 0$ such that

$$P^{\mathbf{a},KP}_{\mu}(\varepsilon,\delta,f) > s, \quad \forall \varepsilon \in (0,\varepsilon'), \delta \in (0,\delta').$$

Thus,

$$M^{\mathbf{a},\mathcal{P}}_{\mu}(s,\varepsilon,\delta,f) = \infty.$$

For any Z with $\mu(Z) \geq 1 - \delta$. If $Z \subset \cup_i Z_i$, then $\mu(\cup_i Z_i) \geq 1 - \delta$. It follows that

$$\sum_{i=1}^{\infty} M^{\mathbf{a},P}(s,\varepsilon,Z_i,f) = \infty,$$

which implies that $M^{\mathbf{a},\mathcal{P}}(s,\varepsilon,Z,f) = \infty$. Hence, $P^{\mathbf{a},P}(\varepsilon,Z,f) \geq s$ and $P^{\mathbf{a},P}_{\mu}(f) \geq s$. This shows that $P^{\mathbf{a},P}_{\mu}(f) \geq P^{\mathbf{a},KP}_{\mu}(f)$.

We shall show the inverse inequality. If $s < P^{\mathbf{a},P}_{\mu}(f)$, then there exists $\varepsilon', \delta' > 0$ such that

$$\inf\{P^{\mathbf{a},P}(\varepsilon,Z,f) : \mu(Z) \geq 1 - \delta\} > s, \quad \forall \varepsilon \in (0,\varepsilon'), \delta \in (0,\delta').$$

For any family $\{Z_i\}_{i\geq 1}$ with $\mu(\cup_i Z_i) \geq 1 - \delta$, we have

$$P^{\mathbf{a},P}\left(\varepsilon, \bigcup_i Z_i, f\right) > s.$$

This implies that

$$M^{\mathbf{a},\mathcal{P}}\left(s,\varepsilon, \bigcup_i Z_i, f\right) = \infty.$$

Thus,

$$\sum_i M^{\mathbf{a},P}(s,\varepsilon,Z_i,f) = \infty.$$

Then,

$$M^{\mathbf{a},\mathcal{P}}_{\mu}(s,\varepsilon,\delta,f) = \infty.$$

Hence,

$$P^{\mathbf{a},KP}_{\mu}(\varepsilon,\delta,f) > s,$$

which yields the desired inequality. □

Definition 3.1. [16] For $1 \le i \le k$, we fix open covers $\{\mathcal{U}_i\}_{i=1}^k$, where $\mathcal{U}_i$ is a finite cover of X_i. For $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, we define the weighted string

$$\begin{aligned}\mathbf{U}_n^{\mathrm{a}} \coloneqq\ & U_1^1 \cap T_1^{-1}U_2^1 \cap T_1^{-2}U_3^1 \cap \ldots \cap T_1^{-\lceil a_1 n\rceil -1}U^1_{\lceil a_1 n\rceil} \cap \tau_1^{-1}U_1^2 \cap \tau_1^{-1}T_2^{-1}U_2^2 \cap \tau_1^{-1}T_2^{-2}U_3^2\\ & \cap \ldots \cap \tau_1^{-1}T_2^{-\lceil (a_1+a_2)n\rceil -1}U^2_{\lceil (a_1+a_2)n\rceil}\\ & \ldots\\ & \cap \tau_{k-1}^{-1}U_1^k \cap \tau_{k-1}^{-1}T_k^{-1}U_2^k \cap \tau_{k-1}^{-1}T_k^{-2}U_3^k \cap \ldots \cap \tau_{k-1}^{-1}T_k^{-\lceil (a_1+a_2+\ldots a_k)n\rceil -1}U^k_{\lceil (a_1+a_2+\ldots+a_k)n\rceil}\end{aligned}$$

where $U_j^i \in \mathcal{U}_i$, for all $1 \le i \le k, 1 \le j \le \lceil (a_1 + a_2 + \ldots + a_k)n\rceil$

Definition 3.2. Let μ be a Borel probability measure on X_1. Consider finite open covers $\{\mathcal{U}_i\}_{i=1}^k$. According to [10, Section 10], the C-structure $\tau = (\mathcal{S}, \mathcal{F}, \xi, \eta, \psi)$ on X_1 generates the Carathéodory dimension of μ and lower and upper Carathéodory capacities of μ specified by the covers $\{\mathcal{U}_i\}_{i=1}^k$ and the map f. Replace the $\mathbf{a}$-weighted Bowen ball $B_n^{\mathbf{a}}(x, \varepsilon)$ by the weighted string $\mathbf{U}_n^{\mathrm{a}}$; it is routine to give an equal definition of weighted Pesin-Pitskel, weighted lower capacity, and weighted upper capacity topological pressures as follows. We denote them by $P_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k)$, $\underline{CP}_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k)$, and $\overline{CP}_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k)$, respectively. We have that

$$\begin{aligned}P_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) &= \inf\{P_Z^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) : \mu(Z) = 1\},\\ \overline{CP}_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) &= \lim_{\delta\to 0}\inf\{\underline{CP}_Z^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) : \mu(Z) \ge 1-\delta\},\\ \overline{CP}_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) &= \lim_{\delta\to 0}\inf\{\overline{CP}_Z^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) : \mu(Z) \ge 1-\delta\}.\end{aligned} \tag{3.1}$$

It is routine to show that there exist the limits

$$\begin{aligned}P_\mu^{\mathbf{a}}(f) &\overset{\text{def}}{=} \lim_{\operatorname{diam}(\{\mathcal{U}_i\}_{i=1}^k)\to 0} P_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k),\\ \underline{CP}_\mu^{\mathbf{a}}(f) &\overset{\text{def}}{=} \lim_{\operatorname{diam}(\{\mathcal{U}_i\}_{i=1}^k)\to 0} \underline{CP}_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k),\\ \overline{CP}_\mu^{\mathbf{a}}(f) &\overset{\text{def}}{=} \lim_{\operatorname{diam}(\{\mathcal{U}_i\}_{i=1}^k)\to 0} \overline{CP}_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k).\end{aligned}$$

According to [10, Section 10], the C-structure $\tau = (\mathcal{S}, \mathcal{F}, \xi, \eta, \psi)$ on X_1, we use the weighted string to define the lower and upper α-Carathéodory pointwise dimensions of μ at x as follows.

Definition 3.3. Given $\alpha \in \mathbf{R}$ and $x \in X$, we define now the lower and upper α-Carathéodory pointwise dimensions of μ at x by

$$\begin{aligned}\underline{\mathcal{D}}_{C,\mu,\alpha}(x, f, \{\mathcal{U}_i\}_{i=1}^k) &= \varliminf_{N\to\infty}\inf_{\mathbf{U}_N^{\mathrm{a}}} \frac{\alpha\log\mu(\mathbf{U}_N^{\mathrm{a}})}{-N\alpha + \sup_{y\in\mathbf{U}_N^{\mathrm{a}}}\frac{1}{a_1}S_{\lceil a_1 N\rceil}f(y)},\\ \overline{\mathcal{D}}_{C,\mu,\alpha}(x, f, \{\mathcal{U}_i\}_{i=1}^k) &= \varlimsup_{N\to\infty}\sup_{\mathbf{U}_N^{\mathrm{a}}} \frac{\alpha\log\mu(\mathbf{U}_N^{\mathrm{a}})}{-N\alpha + \sup_{y\in\mathbf{U}_N^{\mathrm{a}}}\frac{1}{a_1}S_{\lceil a_1 N\rceil}f(y)},\end{aligned}$$

where the infimum and supremum are taken over all strings $\mathbf{U}_N^{\mathrm{a}}$.

Also, we have the following theorems to estimate the dimension of measure.

Theorem 3.1. [10] *Assume that there are a number $\beta \ne 0$ and an interval $[\beta_1, \beta_2]$ such that $\beta \in (\beta_1, \beta_2)$ and for μ-almost every $x \in X$ and any $\alpha \in [\beta_1, \beta_2]$*

(1) *if $\beta > 0$, then $\underline{\mathcal{D}}_{C,\mu,\alpha}(x) \ge \beta$, and if $\beta < 0$, then $\overline{\mathcal{D}}_{C,\mu,\alpha}(x) \le \beta$;*

(2) *there exists $\varepsilon(x) > 0$ such that $e^{-N\alpha+\sup_{y\in\mathbf{U}_N^{\mathrm{a}}}\frac{1}{a_1}S_{\lceil a_1 N\rceil}f(y)} < 1$ for any set $U(x, \varepsilon) \in \mathcal{F}'$; moreover, the function $\varepsilon(x)$ is measurable.*

Then, $\dim_C\mu \ge \beta$.

Theorem 3.2. [10] *Assume that there are a number $\beta \neq 0$ and an interval $[\beta_1, \beta_2]$ such that $\beta \in (\beta_1, \beta_2)$ and for μ-almost every $x \in X$ and any $\alpha \in [\beta_1, \beta_2]$*

(1) *if $\beta > 0$, then $\overline{\mathcal{D}}_{C,\mu,\alpha}(x) \leq \beta$, and if $\beta < 0$, then $\underline{\mathcal{D}}_{C,\mu,\alpha}(x) \geq \beta$;*

(2) *there exists $\varepsilon(x) > 0$ such that $e^{-N\alpha + \sup_{y \in \mathbf{U}_N^{\mathrm{a}}} \frac{1}{a_1} S_{\lceil a_1 N \rceil} f(y)} < 1$ for any set $U(x, \varepsilon) \in \mathcal{F}'$; moreover, the function $\varepsilon(x)$ is measurable;*

Then, $\overline{\mathrm{Cap}}_C \mu \leq \beta$.

Theorem 3.3. [14] *For each ergodic measure $\mu \in \mathcal{M}(X_1, T_1)$, we have*

$$\lim_{\varepsilon \to 0} \liminf_{n \to +\infty} \frac{-\log \mu(B_n^{\mathbf{a}}(x, \varepsilon))}{n} = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{-\log \mu(B_n^{\mathbf{a}}(x, \varepsilon))}{n} = h_\mu^{\mathbf{a}}(T_1),$$

for μ-a.e. $x \in X_1$. When $\mathbf{a} = (1, 0, \ldots, 0)$, the aforementioned result reduces to the Brin-Katok theorem on local entropy [14].

Proposition 3.3. *If μ is a Borel probability measure on X_1 invariant under the map f and ergodic, then for every $\alpha \in \mathbb{R}$ and μ-almost every $x \in X_1$,*

$$\lim_{\mathrm{diam}(\{\mathcal{U}_i\}_{i=1}^k) \to 0} \underline{\mathcal{D}}_{C,\mu,\alpha}(x, f, \{\mathcal{U}_i\}_{i=1}^k) = \lim_{\mathrm{diam}(\{\mathcal{U}_i\}_{i=1}^k) \to 0} \overline{\mathcal{D}}_{C,\mu,\alpha}(x, f, \{\mathcal{U}_i\}_{i=1}^k) = \frac{\alpha h_\mu^{\mathbf{a}}(T_1)}{\alpha - \int_{X_1} f \, \mathrm{d}\mu},$$

where $h_\mu^{\mathbf{a}}(T_1)$ is the measure-theoretic entropy of T_1.

Proof. Let $\{\mathcal{U}_i\}_{i=1}^k$ be finite open covers of X_i, $i = 1, 2, \ldots, N$ and $\delta(\{\mathcal{U}_i\}_{i=1}^k) \coloneqq \min_{1 \leq i \leq k} \delta(\mathcal{U}_i)$, where $\delta(\mathcal{U}_i)$ denotes its Lebesgue number. $\delta(\{\mathcal{U}_i\}_{i=1}^k) \to 0$ as $\mathrm{diam}(\{\mathcal{U}_i\}_{i=1}^k) \to 0$. It is easily seen that for every $x \in X_1$, if $x \in \mathbf{U}_N^{\mathrm{a}}$, then

$$B_n^{\mathbf{a}}\left(x, \frac{1}{2}\delta(\{\mathcal{U}_i\}_{i=1}^k)\right) \subset \mathbf{U}_n^{\mathbf{a}} \subset B_n^{\mathbf{a}}(x, 2(\{\mathcal{U}_i\}_{i=1}^k)).$$

Combining with Theorem 3.3,

$$h_\mu^{\mathbf{a}}(T_1) = \lim_{\mathrm{diam}(\{\mathcal{U}_i\}_{i=1}^k) \to 0} \underline{\lim}_{N \to \infty} \inf_{\mathbf{U}_N^{\mathrm{a}}} \frac{-\log \mu(\mathbf{U}_N^{\mathrm{a}})}{N} = \lim_{\mathrm{diam}(\{\mathcal{U}_i\}_{i=1}^k) \to 0} \overline{\lim}_{N \to \infty} \sup_{\mathbf{U}_N^{\mathrm{a}}} \frac{-\log \mu(\mathbf{U}_N^{\mathrm{a}})}{N}, \tag{3.2}$$

where the infimum and supremum are taken over all strings U for which $x \in \mathbf{U}_N^{\mathrm{a}}$. Let us fix a number $\varepsilon > 0$. Since f is continuous on X_1, there exists a number $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ for any two points $x, y \in X_1$ with $d_1(x, y) \leq \delta$. Therefore, if $\mathrm{diam}(\{\mathcal{U}_i\}_{i=1}^k) \leq \delta$, then by view of Birkhoff ergodic theorem, we obtain for μ-almost every $x \in X_1$ that

$$\left| \liminf_{N \to \infty} \inf_{\mathbf{U}_N^{\mathrm{a}}} \sup_{y \in \mathbf{U}_N^{\mathrm{a}}} \frac{1}{N} \frac{1}{a_1} S_{\lceil a_1 N \rceil} f(y) - \int_{X_1} f \, \mathrm{d}\mu \right| \leq \varepsilon,$$

$$\left| \limsup_{N \to \infty} \sup_{\mathbf{U}_N^{\mathrm{a}}} \sup_{y \in \mathbf{U}_N^{\mathrm{a}}} \frac{1}{N} \frac{1}{a_1} S_{\lceil a_1 N \rceil} f(y) - \int_{X_1} f \, \mathrm{d}\mu \right| \leq \varepsilon,$$

where the infimum and supremum are taken over all strings U for which $x \in \mathbf{U}_N^{\mathbf{a}}$. Since ε is arbitrary, this implies that

$$\begin{aligned} &\lim_{\mathrm{diam}(\{\mathcal{U}_i\}_{i=1}^k) \to 0} \liminf_{N \to \infty} \inf_{\mathbf{U}_N^{\mathrm{a}}} \sup_{y \in \mathbf{U}_N^{\mathrm{a}}} \frac{1}{N} \frac{1}{a_1} S_{\lceil a_1 N \rceil} f(y) \\ &= \lim_{\mathrm{diam}(\{\mathcal{U}_i\}_{i=1}^k) \to 0} \limsup_{N \to \infty} \sup_{\mathbf{U}_N^{\mathrm{a}}} \sup_{y \in \mathbf{U}_N^{\mathrm{a}}} \frac{1}{N} \frac{1}{a_1} S_{\lceil a_1 N \rceil} f(y) = \int_{X_1} f \, \mathrm{d}\mu. \end{aligned} \tag{3.3}$$

The desired result follows immediately from (3.2) and (3.3) . □

Proposition 3.4. *Let f be a continuous function of a compact metric space X_1 and μ a non-atomic Borel ergodic measure on X_1. Then,*

$$P_\mu^{\mathbf{a}}(f) = \underline{CP}_\mu^{\mathbf{a}}(f) = \overline{CP}_\mu^{\mathbf{a}}(f) = h_\mu^{\mathbf{a}}(T_1) + \int_{X_1} f \mathrm{d}\mu.$$

Proof. Set $h^{\mathbf{a}} = h_\mu^{\mathbf{a}}(f) \geq 0$ and $a = \int_X f \mathrm{d}\mu$. We first assume that $a > 0$. We wish to use Theorems 3.1 and 3.2 to obtain the proper lower bound for $P_\mu^{\mathbf{a}}(f)$ and upper bound for $\overline{CP}_\mu^{\mathbf{a}}(f)$. To do so, we need to find estimates of $\underline{\mathcal{D}}_{C,\mu,\alpha}(x, f, \{\mathcal{U}_i\}_{i=1}^k)$ and $\overline{\mathcal{D}}_{C,\mu,\alpha}(x, f, \{\mathcal{U}_i\}_{i=1}^k)$ from below and above, respectively, which do not depend on α.

Fix ε, $0 < \varepsilon < \frac{a}{2}$. By Theorem 3.3, one can choose $\delta > 0$ such that for μ-almost every $x \in X_1$,

$$\underline{\mathcal{D}}_{C,\mu,\alpha}(x, f, \{\mathcal{U}_i\}_{i=1}^k) \geq \frac{\alpha h}{\alpha - a} - \varepsilon.$$

Note that the function $g(\alpha) = \alpha h(\alpha - a)^{-1} - \varepsilon$ is decreasing. Assuming that α varies on the interval $[h + a - \varepsilon, h + a]$, we obtain that for μ-almost every $x \in X_1$,

$$\underline{\mathcal{D}}_{C,\mu,\alpha}(x, f, \{\mathcal{U}_i\}_{i=1}^k) \geq h + a - 2\varepsilon.$$

We conclude, using Theorem 3.1, that $P_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) \geq h + a - 2\varepsilon$, and hence, $P_\mu^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) \geq h + a$. Since this holds for every finite open covers $\{\mathcal{U}_i\}_{i=1}^k$, by (3.1), we obtain that $P_\mu^{\mathbf{a}}(f) \geq h + a$.

We now show that $\overline{CP}_\mu^{\mathbf{a}}(f) \leq h + a$. Fix $\varepsilon > 0$. We can choose $\xi_i = \{C_1^1, \ldots, C_{p_i}^1\}$ be a finite measurable partition of X_i for any $1 \leq i \leq k$ with

$$\left| \sum_{i=1}^k h_{\mu \circ \tau_{i-1}^{-1}}(T_i, \bigvee_{j=0}^{\lceil (a_1 + \ldots a_j)n \rceil - 1} T_i^{-j} \xi_i) - h \right| \leq \varepsilon$$

and $\mathcal{U}_i = \{U_1, \ldots, U_{p_i}\}$ a finite open cover of X_i of diameter $\leq \varepsilon$ for which $C_j^i \subset U_j^i$, $j = 1, \ldots, p_i$.

By the Birkhoff ergodic theorem for μ-almost every $x \in X_1$, there exists a number $N_1(x) > 0$ such that for any $n \geq N_1(x)$,

$$\left| \frac{1}{a_1 n} S_{\lceil a_1 N \rceil} f(x) - a \right| \leq \varepsilon. \tag{3.4}$$

By the proof of weighted Shannon-McMillan-Breiman theorem [14] for μ-almost every $x \in X_1$, there exists a number $N_2(x) > 0$ such that for any $n \geq N_2(x)$,

$$\left| \frac{1}{n} \log \mu \left(\bigvee_{j=0}^{\lceil (a_1 + \ldots a_j)n \rceil - 1} T_1^{-j} \tau_i^{-1} \xi_i(x) \right) + \sum_{i=1}^k a_i h_{\mu \circ \tau_{i-1}^{-1}} \left(T_i, \bigvee_{j=0}^{\lceil (a_1 + \ldots a_j)n \rceil - 1} T_i^{-j} \xi_i \right) \right| \leq \varepsilon. \tag{3.5}$$

Let Δ be the set of points for which (3.4) and (3.5) hold. Given $N > 0$, consider the set $\Delta_N = \{x \in \Delta : N_1(x) \leq N$ and $N_2(x) \leq N\}$. We have that $\Delta_N \subset \Delta_{N+1}$ and $\Delta = \cup_{N \geq 0} \Delta_N$ Therefore, given $\delta > 0$, one can find $N_0 > 0$ for which $\mu(\Delta_{N_0}) \geq 1 - \delta$. Fix a number $N \geq N_0$ and a point $x \in \Delta_N$. Let $U_N^{\mathbf{a}}$ be a string of length $m(U_N^{\mathbf{a}}) = N$ for which $x \in U_n^{\mathbf{a}}$. It follows from (3.4) that

$$\left| \sup_{y \in U_N^{\mathbf{a}}} \frac{1}{a_1 N} S_{\lceil a_1 N \rceil} f(y) - a \right| \leq \varepsilon + \gamma, \tag{3.6}$$

where $\gamma = \gamma(\mathcal{U}) := \max_{1 \leq i \leq k} \sup\{ |f(x) - f(y)| : x, y \in U_j^i, 1 \leq j \leq p_i\}$. Furthermore, using (3.5), we obtain that

$$\mu \left(\bigvee_{j=0}^{\lceil (a_1 + \ldots a_j)n \rceil - 1} T_1^{-j} \tau_i^{-1} \xi_i(x) \right) \geq \exp(-h - 2\varepsilon)N.$$

This implies that the number of elements of the partition $\vee_{j=0}^{\lceil (a_1 + \ldots a_j)n \rceil - 1} T_1^{-j} \tau_i^{-1} \xi_i$ that have non-empty intersection with the set Δ_N does not exceed $\exp(h + 2\varepsilon)N$.

To each element $C_{\vee_{j=0}^{\lceil (a_1+\ldots a_j)n\rceil-1} T_1^{-j}\tau_i^{-1}\xi_i}$ of the partition ξ_N, we associate a string $U_N^{\mathbf{a}}$ of length $m(U) = N$ for which $C_{\vee_{j=0}^{\lceil (a_1+\ldots a_j)n\rceil-1} T_1^{-j}\tau_i^{-1}\xi_i} \subset U_N^{\mathbf{a}}$. The collection of such strings consists of at most $\exp(h + 2\varepsilon)N$ elements that comprise a cover $\mathcal{G}$ of Δ_N. By (3.5) and (3.6), we obtain that

$$\Lambda(\Delta_N, f, \{\mathcal{U}_i\}_{i=1}^k, N) \le \sum_{U\in\mathcal{G}} \exp\left(\sup_{y\in \mathbf{U}_N^{\mathbf{a}}} \frac{1}{a_1} S_{\lceil a_1 N\rceil} f(y)\right) \le \exp(a + h + 3\varepsilon + \gamma)N.$$

Then,

$$\overline{CP}_{\Delta_N}^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) \le a + h + 3\varepsilon + \gamma.$$

This implies that $\overline{CP}_{\mu}^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) \le a + h + 3\varepsilon + \gamma$. Passing to the limit as $\operatorname{diam}\{\mathcal{U}_i\}_{i=1}^k \to 0$ yields that $\overline{CP}_{\mu}^{\mathbf{a}}(f) \le a + h + 3\varepsilon$. It remains to note that ε can be chosen arbitrarily small to conclude that $\overline{CP}_{\mu}^{\mathbf{a}}(f) \le a + h$.

In the case $a \le 0$, let us consider a function $\psi = f + C$, where C is chosen such that $\int_X \psi \mathrm{d}\mu > 0$. Note that $P_{\mu}^{\mathbf{a}}(\psi, \{\mathcal{U}_i\}_{i=1}^k) = P_{\mu}^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) + C$ and $\overline{CP}_{\mu}^{\mathbf{a}}(\psi, \{\mathcal{U}_i\}_{i=1}^k) = \overline{CP}_{\mu}^{\mathbf{a}}(f, \{\mathcal{U}_i\}_{i=1}^k) + C$, and the desired result follows. □

Now, we can prove the Theorem 2.1 as follows.

Proof. Employing Proposition 3.4, the following equalities follow

$$P_{\mu}^{\mathbf{a},B}(f) = \underline{CP}_{\mu}^{\mathbf{a}}(f) = \overline{CP}_{\mu}^{\mathbf{a}}(f) = h_{\mu}^{\mathbf{a}}(T_1) + \int f \mathrm{d}\mu. \tag{3.7}$$

From the proof of Proposition 3.1, it is easy to check that for any $Z \subset X$ and $\varepsilon > 0$,

$$P^{\mathbf{a},B}(\varepsilon, Z, f) \le P^{\mathbf{a},P}(\varepsilon, Z, f) \le \overline{CP}^{\mathbf{a}}\left(\frac{\varepsilon}{2}, Z, f\right).$$

Thus,

$$P_{\mu}^{\mathbf{a},B}(f) \le P_{\mu}^{\mathbf{a},P}(f) \le \overline{CP}_{\mu}^{\mathbf{a}}(f),$$

which together with (3.7) yields

$$P_{\mu}^{\mathbf{a},P}(f) = h_{\mu}^{\mathbf{a}}(T_1) + \int f \mathrm{d}\mu.$$

Using Proposition 3.2, we obtain

$$P_{\mu}^{\mathbf{a},KP}(f) = P_{\mu}^{\mathbf{a},P}(f) = h_{\mu}^{\mathbf{a}}(T_1) + \int f \mathrm{d}\mu$$

and

$$P_{\mu}^{\mathbf{a},B}(f) = P_{\mu}^{\mathbf{a},KB}(f) \le \underline{CP}_{\mu}^{\mathbf{a},K}(f) \le \overline{CP}_{\mu}^{\mathbf{a},K}(f) \le \overline{CP}_{\mu}^{\mathbf{a}}(f).$$

The equalities in Theorem 2.1 then follows. □

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