Research Article

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Blow-up of solutions for Euler-Bernoulli equation with nonlinear time delay

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Abstract: We study the Euler-Bernoulli equations with time delay:

$$u_{tt} + \Delta^2 u - g_1 * \Delta^2 u + g_2 * \Delta u + \mu_1 u_t(x,t)|u_t(x,t)|^{m-2} + \mu_2 u_t(x,t-\tau)|u_t(x,t-\tau)|^{m-2} = f(u),$$

where τ represents the time delay. We exhibit the blow-up behavior of solutions with both positive and nonpositive initial energy for the Euler-Bernoulli equations involving time delay.

Keywords: Euler-Bernoulli equation, blow-up, nonlinear time delay

MSC 2020: 35L05, 35B44, 93D15

1 Introduction

In this article, we are concerned with the following delayed nonlinear Euler-Bernoulli problem:

$$\begin{cases} u_{tt} + \Delta^2 u - g_1 * \Delta^2 u + g_2 * \Delta u + \mu_1 u_t(x,t)|u_t(x,t)|^{m-2} \\ \quad + \mu_2 u_t(x,t-\tau)|u_t(x,t-\tau)|^{m-2} = f(u), \quad (x,t) \in \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \quad (x,t) \in \Omega \times (0,\tau), \\ u(x,t) = 0, \Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $\mu_1 \geq |\mu_2| > 0$, $m \geq 2$, g_1, g_2, and $f(u)$ are functions satisfying some conditions to be specified later, and

$$g_i * v(t) = \int_0^t g_i(t-s)v(s)\mathrm{d}s, \quad i = 1, 2.$$

The Euler-Bernoulli type equation is formulated as follows:

$$u_{tt} + \Delta^2 u = g(x,t,u,u_t). \tag{1.2}$$

In addition, it finds extensive a applications in numerous branches of physics, including nuclear physics, optics, geophysics, and ocean acoustics [1,2]. Therefore, an increasing number of references have investigated

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the global existence, energy decay, and blow-up phenomena of solutions for equation (1.2) [1–14]. When $g_1(\cdot) = g_2(\cdot) = 0, \mu_1 = 0$, Messaoudi [4] considered a more simple equation

$$u_{tt} + \Delta^2 u + au_t|u_t|^{m-2} = bu|u|^{p-2}, \quad x \in \Omega, t > 0. \tag{1.3}$$

The authors demonstrated the existence of a local weak solution that exhibits finite-time blowing-up behavior when $p > m$ and the energy is negative. The authors additionally found that the solution remains global if $m \geq p$. Subsequently, Chen and Zhou [5] demonstrated that the solution experiences finite-time blowing-up with a positive initial energy. When equation (1.3) contains a viscoelastic term ($g_1(\cdot) \neq 0$), Li and Gao [6] established that the solution with upper-bounded initial energy experiences finite-time blowing-up. Moreover, Liu et al. [7] demonstrated that the solution blows up in finite time under strong damping and with initial energy $E(0) = R$ for any given $R \geq 0$. In [8], Ye examined the following initial boundary value problem concerning the higher-order nonlinear viscoelastic wave equation

$$u_{tt} + (-\Delta)^m u - \int_0^t g(t-s)(-\Delta)^m u(s)\mathrm{d}s = |u|^{p-2}u, \quad (x,t) \in \Omega \times R^+,$$

where $m \geq 1$ denotes a natural number, and $p > 2$ represents a real number. By using the Galerkin method, Ye demonstrated the existence of global weak solutions. Simultaneously, it is established that the solution blows up in finite time under both positive and nonpositive initial energy, and lifetime estimates for the solutions are provided. In the absence of memory term for (1.1), Benaissa and Messaoudi [9] consider the wave equation in a bounded domain with a delay term in the nonlinear internal feedback

$$u_{tt} - \Delta u + \mu_1\sigma(t)g_1(u_t) + \mu_2\sigma(t)g_2(u_t(x, t-\tau(t))) = 0, \quad (x,t) \in \Omega \times R^+. \tag{1.4}$$

The authors demonstrated the global existence of solutions by employing the energy method in conjunction with the Faedo-Galerkin procedure, and investigated their asymptotic behavior utilizing a perturbed energy method. When the nonlinearity and delay of equation (1.4) are variable-exponent, Kafini and Messaoudi [10] established a global nonexistence result and exponential decay. Numerous researchers have examined both the delay term and the memory term for references [11–13].

In the absence of a time-delayed nonlinear term and with $\mu_2 = 0$, Mellah and Hakem [14] investigated the global existence and uniqueness of a solution to an initial boundary value problem for the Euler-Bernoulli viscoelastic equation:

$$u_{tt} + \Delta^2 u - g_1 * \Delta^2 u + g_2 * \Delta u + u_t = 0, \quad (x,t) \in \Omega \times R^+.$$

The authors also demonstrated an exponential decay. In [3], Feng et al. examined the extendable viscoelastic plate equation with a nonlinear time-varying delay feedback and a nonlinear source term

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \int_0^t h(t-s)\Delta^2 u(s)\mathrm{d}s + \mu_1 g_1(u_t(t)) + \mu_2 g_2(u_t(t-\tau(t))) + f(u(t)) = 0, \quad (x,t) \in \Omega \times (0,+\infty).$$

Under suitable assumptions regarding the relaxation function, nonlinear internal delay feedback, and source term, the authors demonstrated the global existence of solutions and the general decay of energy by employing the multiplier method.

For [14], global existence and energy decay are investigated in the absence of time-delay and nonlinearity terms. In [3], the author demonstrated energy decay with $g_2(\cdot) = 0$. In this article, we expand upon the results from [13] to [3]. When $g_1(\cdot)$ and $g_2(\cdot)$ satisfy the condition $\int_0^{+\infty} g_1(s)\mathrm{d}s + \int_0^{+\infty} g_2(s)\mathrm{d}s \leq \frac{(p-pa-1)^2-1}{(p-pa-1)^2}$ for every $0 < a < \frac{p-2}{p}$, we obtain the blow-up of the solution. Our work is structured as follows. In Section 2, we present some lemmas and a local existence theorem. In Section 3, we prove several lemmas related to blow-up. Moreover, we demonstrate the existence of finite-time blow-up when $E(0) < E_1$ and $E(0) < 0$.

2 Preliminaries

In this section, we will provide the necessary materials for the proof of our main results. We employ the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^2(\Omega)$ along with their conventional products and norms $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$. In particular, $\|\cdot\|_2$ is shorthanded for $\|\cdot\|$ when $p = 2$. The constant λ_1 is the embedding constant $\lambda_1{}^2\|u\|^2 \le \lambda_1\|\nabla u\|^2 \le \|\Delta u\|^2$ for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Now, we introduce a new variable, as in [10],

$$z(x,\rho,t) = u_t(x, t - \tau\rho), \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,+\infty).$$

Thus, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,+\infty).$$

Then, problem (1.1) assumes the following form:

$$\begin{cases} u_{tt} + \Delta^2 u - g_1 * \Delta^2 u + g_2 * \Delta u + \mu_1 u_t(x,t)|u_t(x,t)|^{m-2} \\ \quad + \mu_2 z(x,1,t)|z(x,1,t)|^{m-2} = f(u), \quad (x,t) \in \Omega \times (0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau), \quad (x,\rho) \in \Omega \times (0,1), \\ u(x,t) = 0, \Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty). \end{cases} \tag{2.1}$$

First, to state and prove our result, we require several assumptions.

(H_0) $g_1 : R^+ \to R^+$ is a bounded function, which satisfies

$$g_1(t) \in C^2(R^+) \cap L^1(R^+),$$

and there exist positive constants α_1, α_2, and α_3 such that

$$-\alpha_1 g_1(t) \le g'_1(t) \le -\alpha_2 g_1(t), \quad \forall t \ge 0,$$
$$0 \le g''_1(t) \le \alpha_3 g_1(t), \quad \forall t \ge 0.$$

(H_1) $g_2 : R^+ \to R^+$ is a bounded function, which satisfies

$$g_2(t) \in C^1(R^+) \cap L^1(R^+),$$

and there exist positive constants β_1, β_2, and β_3 such that

$$-\beta_1 g_2(t) \le g'_2(t) \le -\beta_2 g_2(t), \quad \forall t \ge 0,$$
$$1 - \int_0^{+\infty} g_1(s)\mathrm{d}s - \frac{1}{\lambda_1}\int_0^{+\infty} g_2(s)\mathrm{d}s = l > 0.$$

(H_2) The source term $f(u)$ is a nonlinear function that $f(0) = 0$, and

$$k_0\,|u|^p \le pF(u) \le f(u)u \le k_1\,|u|^p,$$

where k_0, k_1 are positive constants, and

$$F(z) = \int_0^z f(s)\mathrm{d}s,$$

$$2 < p < +\infty \quad \text{if } n \le 4 \quad \text{and} \quad 2 < p \le \frac{2(n-2)}{n-4} \quad \text{if } n \ge 5.$$

(H_3) Under the assumptions of g_1, g_2, and (H_2), concurrently, we also assume the relaxation functions satisfy:

$$\int_0^{+\infty} g_1(s)\mathrm{d}s + \frac{1}{\lambda_1}\int_0^{+\infty} g_2(s)\mathrm{d}s \le \frac{(p - pa - 1)^2 - 1}{(p - pa - 1)^2},$$

where $0 < a < \frac{p-2}{p}$ is a fixed number.

Next, we present several lemmas and a local existence theorem throughout the text.

Lemma 2.1. [15] (Sobolev-Poincaré inequality): *Let s be a number with*

$$2 \le s < +\infty \ (n \le 4) \quad or \quad 2 \le s \le \frac{2n}{n-4}.$$

Then there exists a positive constant B depending Ω *and s such that*

$$\|u\|_s \le B\|\Delta u\|, \quad \forall u \in H_0^2(\Omega).$$

Lemma 2.2. *Assume that psatisfies* (H_2). *Then there exists a positive constant C such that*

$$\|u(t)\|_p^s \le C(\|\Delta u(t)\|^2 + \|u(t)\|_p^p)$$

for all $t \in [0, T)$, $2 \le s \le p$.

Proof. By employing continuous embedding inequalities and Sobolev-Poincaré inequality, we can establish the conclusion parallel to Lemma 2.5 in [6]. □

The localized existence results for problem (2.1) are as follows:

Theorem 2.3. (Local existence) *Assume the assumptions* (H_0)–(H_2) *hold. Let*

$$(u_0, u_1, f_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega \times (0,1)) \tag{2.2}$$

satisfy the compatibility condition $f_0(\cdot, 0) = u_t(\cdot, 0) = u_1$. *Then problem* (2.1) *possesses a unique weak solution satisfying*

$$\begin{aligned} u &\in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)),\\ u_t &\in C([0,T]; L^2(\Omega)) \cap L^m(\Omega \times (0,T)),\\ z &\in C([0,T); L^m(\Omega \times (0,1))). \end{aligned}$$

Remark 2.4. Although the sign of the nonlinear source term in [3] is opposite to that of our nonlinear source term, local well-posedness can be further established by combining the methods in [3, 14] with the contraction mapping theorem. A similar approach can be referred to in reference [16].

3 Blow up result

To state and prove our main results, we introduce the following notation and the associated energy functional related to (2.1).

$$B_1 := \frac{\sqrt[p]{k_1}B}{\sqrt{l}}, \quad \Phi(\alpha(t)) := \frac{1}{2}\alpha^2(t) - \frac{B_1{}^p}{p}\alpha^p(t), \tag{3.1}$$

where

$$\alpha(t) = \left(l\,\|\Delta u\|^2 + (g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t) + \frac{2\xi}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x \right)^{\frac{1}{2}}.$$

We can define energy function of (2.1):

$$\begin{aligned} E(t) := {}& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g_1(s)\mathrm{d}s\right)\|\Delta u\|^2 - \frac{1}{2}\int_0^t g_2(s)\mathrm{d}s\|\nabla u\|^2 \\ &+ \frac{1}{2}[(g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)] + \frac{\xi}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x, \end{aligned} \tag{3.2}$$

where

$$\tau|\mu_2|(m-1) < \xi < \tau(\mu_1 m - |\mu_2|), \tag{3.3}$$

$$(g_1 \circ \Delta u)(t) = \int_0^t g_1(t-s)\|\Delta u(t) - \Delta u(s)\|^2 \mathrm{d}s,$$

$$(g_2 \circ \nabla u)(t) = \int_0^t g_2(t-s)\|\nabla u(t) - \nabla u(s)\|^2 \mathrm{d}s.$$

Lemma 3.1. *Let* (u, z) *be a weak solution of* (2.1), *and assume* (H_0)–(H_1), (3.3) *hold. Then, for some* $C_0 > 0$,

$$\frac{\mathrm{d}}{\mathrm{dt}}E(t) \le -C_0(\|u_t(t)\|_m^m + \|z(\cdot,1,t)\|_m^m) \le 0,$$

where

$$0 < C_0 \le \min\left\{\mu_1 - \frac{|\mu_2|}{m} - \frac{\xi}{\tau m}, \frac{\xi}{\tau m} - \frac{|\mu_2|(m-1)}{m}\right\}.$$

Proof. After multiplying the first equation within (2.1) by factor u_t and performing integration across region Ω, and subsequently multiplying the second equation from (2.1) by factor $\frac{\xi}{\tau}z|z|^{m-2}$ with integration carried out over region $\Omega \times (0,1)$, we sum these integrated outcomes to derive the following result.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{\xi}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x\right] \\
&\quad - \int_0^t g_1(t-s)\int_\Omega \Delta u(s)\Delta u_t(t)\mathrm{d}x\mathrm{d}s - \int_0^t g_2(t-s)\int_\Omega \nabla u(s)\nabla u_t(t)\mathrm{d}x\mathrm{d}s \\
&= -\mu_1\|u_t(t)\|_m^m - \mu_2\int_\Omega u_t(x,t)z(x,1,t)|z(x,1,t)|^{m-2}\mathrm{d}x \\
&\quad - \frac{\xi}{\tau}\int_\Omega\int_0^1 |z(x,\rho,t)|^{m-2}zz_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x.
\end{aligned} \tag{3.4}$$

For the last two terms on the left side of (3.4), a direct calculation reveals that

$$\begin{aligned}
&-\int_0^t g_1(t-s)\int_\Omega \Delta u(s)\Delta u_t(t)\mathrm{d}x\mathrm{d}s \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left[(g_1 \circ \Delta u)(t) - \left(\int_0^t g_1(s)\mathrm{d}s\right)\|\Delta u\|^2\right] + \frac{1}{2}g_1(t)\|\Delta u\|^2 - \frac{1}{2}(g_1' \circ \Delta u)(t),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&-\int_0^t g_2(t-s)\int_\Omega \nabla u(s)\nabla u_t(t)\mathrm{d}x\mathrm{d}s \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left[(g_2 \circ \nabla u)(t) - \left(\int_0^t g_2(s)\mathrm{d}s\right)\|\nabla u\|^2\right] + \frac{1}{2}g_2(t)\|\nabla u\|^2 - \frac{1}{2}(g_2' \circ \nabla u)(t).
\end{aligned} \tag{3.6}$$

By Young's inequality for the second term on the right side of (3.4), we can obtain

$$\begin{aligned}
&-\mu_2\int_\Omega u_t(x,t)z(x,1,t)|z(x,1,t)|^{m-2}\mathrm{d}x \le |\mu_2|\int_\Omega u_t(x,t)|z(x,1,t)|^{m-1}\mathrm{d}x \\
&\le \frac{|\mu_2|}{m}\|u_t(t)\|_m^m + \frac{|\mu_2|(m-1)}{m}\|z(\cdot,1,t)\|_m^m.
\end{aligned} \tag{3.7}$$

For the last term on the right side of (3.4), a direct computation shows that

$$\begin{aligned}
-\frac{\xi}{\tau}\int_\Omega\int_0^1 |z(x,\rho,t)|^{m-2}zz_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x &= -\frac{\xi}{\tau m}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}|z(x,\rho,t)|^m\mathrm{d}\rho\mathrm{d}x\\
&= -\frac{\xi}{\tau m}\int_\Omega (|z(x,1,t)|^m - |z(x,0,t)|^m)\mathrm{d}x \\
&= \frac{\xi}{\tau m}\|u_t(t)\|_m^m - \frac{\xi}{\tau m}\|z(\cdot,1,t)\|_m^m.
\end{aligned}\tag{3.8}$$

Inserting (3.5)–(3.8) into (3.4), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le& -\frac{1}{2}g_1(t)\|\Delta u\|^2 - \frac{1}{2}g_2(t)\|\nabla u\| + \frac{1}{2}(g_1' \circ \Delta u)(t) + \frac{1}{2}(g_2' \circ \nabla u)(t)\\
&- \left(\mu_1 - \frac{|\mu_2|}{m} - \frac{\xi}{\tau m}\right)\|u_t(t)\|_m^m - \left[\frac{\xi}{\tau m} - \frac{|\mu_2|(m-1)}{m}\right]\|z(\cdot,1,t)\|_m^m.
\end{aligned}$$

According to (H_0)–(H_1) and (3.3), then

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le -\left(\mu_1 - \frac{|\mu_2|}{m} - \frac{\xi}{\tau m}\right)\|u_t(t)\|_m^m - \left[\frac{\xi}{\tau m} - \frac{|\mu_2|(m-1)}{m}\right]\|z(\cdot,1,t)\|_m^m < 0.$$

Therefore, we can obtain the desired result. □

Lemma 3.2. *If the assumptions (H_0)–(H_2) hold, then*

$$E(t) \ge \frac{1}{2}\alpha^2(t) - \frac{B_1{}^p}{p}\alpha^p(t)$$

and $\Phi(\alpha)$ has a maximum value $\Phi(\alpha_1) = \left(\frac{1}{2} - \frac{1}{p}\right)\alpha_1{}^2 := E_1$ when $\alpha(t) = B_1{}^{-\frac{p}{p-2}} := \alpha_1$, where $\alpha(t)$ is given in (3.1).

Proof. By (H_0)–(H_2), (3.1) and applying Sobolev-Poincaré inequality, we have

$$\begin{aligned}
E(t) \ge& \frac{1}{2}\left(1 - \int_0^t g_1(s)\mathrm{d}s\right)\|\Delta u\|^2 - \frac{1}{2}\int_0^t g_2(s)\mathrm{d}s\|\nabla u\|^2\\
&+ \frac{1}{2}[(g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)] + \frac{\xi}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m\mathrm{d}\rho\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x\\
\ge& \frac{1}{2}\left(1 - \int_0^t g_1(s)\mathrm{d}s - \frac{1}{\lambda_1}\int_0^t g_2(s)\mathrm{d}s\right)\|\Delta u\|^2 + \frac{1}{2}[(g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)]\\
&+ \frac{\xi}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m\mathrm{d}\rho\mathrm{d}x - \frac{k_1}{p}\|u\|_p^p\\
\ge& \frac{1}{2}l\|\Delta u\|^2 + \frac{1}{2}[(g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)] + \frac{\xi}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m\mathrm{d}\rho\mathrm{d}x - \frac{k_1B^p}{p}\|\Delta u\|^p\\
\ge& \frac{1}{2}\alpha^2(t) - \frac{B_1{}^p}{p}\alpha^p(t).\\
=& \Phi(\alpha(t)).
\end{aligned}$$

Furthermore, according to the derivative of $\Phi(\alpha(t))$, we have easy access to $\Phi(\alpha)$ has a maximum value E_1 at $\alpha(t) = B_1{}^{-\frac{p}{p-2}}$. □

Lemma 3.3. *Assume that* (H_0)–(H_2) *and* (3.3) *hold. Let* (u, z) *be a solution of* (2.2) *with initial data satisfying*

$$E(0) < E_1, \quad \alpha(0) > \alpha_1.$$

Then there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$\alpha(t) = \left(l\, \|\Delta u\|^2 + (g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t) + \frac{2\xi}{m} \int_\Omega \int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho \mathrm{d}x \right)^{\frac{1}{2}} \geq \alpha_2, \tag{3.9}$$

and

$$\|u\|_p \geq \frac{B_1}{\sqrt[p]{k_1}} \alpha_2. \tag{3.10}$$

Proof. From the definition of $\Phi(\alpha)$ and (H_2), we can easily obtain

$$\Phi'(\alpha) = \alpha(1 - B_1{}^p \alpha^{p-2}) \begin{cases} > 0, & \text{if } \alpha \in (0, \alpha_1), \\ < 0, & \text{if } \alpha \in (\alpha_1, +\infty), \end{cases}$$

which implies that

$$\begin{cases} \Phi(\alpha) \text{ is strictly increasing in } (0, \alpha_1), \\ \Phi(\alpha) \text{ is strictly decreasing in } (\alpha_1, +\infty), \\ \Phi(\alpha) \to -\infty \text{ as } \alpha \to +\infty. \end{cases}$$

Similar to the proof of Lemma 2.4 in [6], we easily obtain (3.9) holds.

Next, we will prove (3.10) holds. By (3.2) and (H_2), we have

$$\begin{aligned} \frac{k_1}{p} \|u\|_p^p \geq \int_\Omega F(u)\mathrm{d}x &\geq \frac{1}{2}\left(1 - \int_0^t g_1(s)\mathrm{d}s\right) \|\Delta u\|^2 - \frac{1}{2}\int_0^t g_2(s)\mathrm{d}s \|\nabla u\|^2 \\ &+ \frac{1}{2}[(g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)] + \frac{\xi}{m} \int_\Omega \int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho \mathrm{d}x - E(t) \\ &\geq \frac{1}{2}\left[l\|\Delta u\|^2 + (g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t) + \frac{2\xi}{m} \int_\Omega \int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho \mathrm{d}x \right] - E(0) \\ &\geq \frac{1}{2}\alpha_2{}^2 - \Phi(\alpha_2) = \frac{B_1{}^p \alpha_2{}^p}{p}. \end{aligned}$$

Therefore, (3.10) holds. □

Theorem 3.4. *Assume that hypothesis conditions* (H_0)–(H_3), (2.2), *and* (3.3) *hold. Then the local solution of problem* (2.1) *with* $p > m$ *and with initial conditions satisfying*

(*i*) $E(0) < E_1$, $\alpha(0) > \alpha_1$;

(*ii*) $E(0) < 0$.

Then any solution of (2.1) *blows up in finite time* T^*.

Proof. We will prove the theorem in two parts based on conditions (i) and (ii).

Part (i): Setting $H(t) = E_1 - E(t)$, by Lemma 3.1 and (3.2), we obtain

$$H'(t) = -E'(t) \geq C_0(\|u_t(t)\|_m^m + \|z(\cdot, 1, t)\|_m^m).$$

By using (H_2), (3.2), and Lemma 3.3, then

$$\begin{aligned} H(t) &= E_1 - \frac{1}{2}\|u_t\|^2 - \frac{1}{2}\left(1 - \int_0^t g_1(s)\mathrm{d}s\right)\|\Delta u\|^2 + \frac{1}{2}\int_0^t g_2(s)\mathrm{d}s\|\nabla u\|^2 \\ &\quad - \frac{1}{2}[(g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)] - \frac{\xi}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x + \int_\Omega F(u)\mathrm{d}x \\ &\le E_1 - \frac{1}{2}\left(1 - \int_0^t g_1(s)\mathrm{d}s - \frac{1}{\lambda_1}\int_0^t g_2(s)\mathrm{d}s\right)\|\Delta u\|^2 \\ &\quad - \frac{1}{2}[(g_1 \circ \Delta u)(t) + (g_2 \circ \nabla u)(t)] - \frac{\xi}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x + \frac{k_1}{p}\|u\|_p^p \\ &\le E_1 - \frac{1}{2}\alpha^2(t) + \frac{k_1}{p}\|u\|_p^p \\ &\le E_1 - \frac{1}{2}\alpha_1{}^2 + \frac{k_1}{p}\|u\|_p^p \\ &= -\frac{1}{p}\alpha_1{}^2 + \frac{k_1}{p}\|u\|_p^p. \end{aligned}$$

So, for $t \in [0, T)$ we can obtain

$$0 < E_1 - E(0) = H(0) \le H(t) \le -\frac{1}{p}\alpha_1{}^2 + \frac{k_1}{p}\|u\|_p^p \le \frac{k_1}{p}\|u\|_p^p. \tag{3.11}$$

Therefore, we define

$$L(t) = H^{1-\theta}(t) + \varepsilon M(t) + KE_1 t, \quad \forall t \in [0, T),$$

where $\varepsilon, K > 0$ are positive constants to be specified later, and

$$M(t) = \int_\Omega u u_t \mathrm{d}x, \quad 0 < \theta \le \min\left\{\frac{p-2}{2p}, \frac{p-m}{p(m-1)}\right\}.$$

Then,

$$L'(t) = (1-\theta)H^{-\theta}(t)H'(t) + \varepsilon M'(t) + KE_1, \quad \forall t \in [0, T). \tag{3.12}$$

Next, we estimate the function $M'(t)$. By (2.1) and taking the derivative of $M(t)$ with respect to time t, we obtain

$$\begin{aligned} M'(t) &= \|u_t\|^2 + (u, u_{tt}) \\ &= \|u_t\|^2 - \|\Delta u\|^2 + \int_0^t g_1(t-s)\int_\Omega \Delta u(t)\Delta u(s)\mathrm{d}x\mathrm{d}s + \int_0^t g_2(t-s)\int_\Omega \nabla u(t)\nabla u(s)\mathrm{d}x\mathrm{d}s \\ &\quad - \mu_1\int_\Omega u u_t|u_t|^{m-2}\mathrm{d}x - \mu_2\int_\Omega u z(x,1,t)|z(x,1,t)|^{m-2}\mathrm{d}x + \int_\Omega u f(u)\mathrm{d}x. \end{aligned} \tag{3.13}$$

Applying Young's inequality, for every $\eta, \delta > 0$, we have

$$\begin{aligned} &\int_0^t g_1(t-s)\int_\Omega \Delta u(t)\Delta u(s)\mathrm{d}x\mathrm{d}s \\ &= \int_0^t g_1(t-s)\int_\Omega \Delta u(t)(\Delta u(s) - \Delta u(t))\mathrm{d}x\mathrm{d}s + \int_0^t g_1(t-s)\mathrm{d}s\|\Delta u\|^2 \\ &\ge -\eta(g_1 \circ \Delta u)(t) + \left(1 - \frac{1}{4\eta}\right)\int_0^t g_1(t-s)\mathrm{d}s\|\Delta u\|^2, \end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\int_0^t g_2(t-s)\int_\Omega \nabla u(t)\nabla u(s)\mathrm{d}x\mathrm{d}s \\
&= \int_0^t g_2(t-s)\int_\Omega \nabla u(t)(\nabla u(s)-\nabla u(t))\mathrm{d}x\mathrm{d}s + \int_0^t g_2(t-s)\mathrm{d}s\|\nabla u\|^2 \\
&\geq -\eta(g_2\circ\nabla u)(t) + \left(1-\frac{1}{4\eta}\right)\int_0^t g_2(t-s)\mathrm{d}s\|\nabla u\|^2,
\end{aligned} \tag{3.15}$$

$$-\mu_1\int_\Omega uu_t|u_t|^{m-2}\mathrm{d}x \geq -\frac{\mu_1\delta^m}{m}\|u\|_m^m - \frac{\mu_1(m-1)\delta^{-\frac{m}{m-1}}}{m}\|u_t(t)\|_m^m, \tag{3.16}$$

$$-\mu_2\int_\Omega uz(x,1,t)|z(x,1,t)|^{m-2}\mathrm{d}x \geq -\frac{|\mu_2|\delta^m}{m}\|u\|_m^m - \frac{|\mu_2|(m-1)\delta^{-\frac{m}{m-1}}}{m}\|z(\cdot,1,t)\|_m^m. \tag{3.17}$$

According to the definition of $H(t)$, adding $(1-a)p(H(t)+E(t)-E_1)$ (for every $0<a<\frac{p-2}{p}$) to the right-hand side of inequality (3.13) and inserting (3.14)–(3.17) into (3.13), we arrive at

$$\begin{aligned}
M'(t) \geq& \left[1+\frac{(1-a)p}{2}\right]\|u_t\|^2 + \left[\left(1-\frac{1}{4\eta}-\frac{(1-a)p}{2}\right)\int_0^t g_1(s)\mathrm{d}s + \frac{(1-a)p}{2}-1\right]\|\Delta u\|^2 \\
&+\left(1-\frac{1}{4\eta}-\frac{(1-a)p}{2}\right)\int_0^t g_2(s)\mathrm{d}s\,\|\nabla u\|^2 + \left[\frac{(1-a)p}{2}-\eta\right][(g_1\circ\Delta u)(t)+(g_2\circ\nabla u)(t)] \\
&+\frac{p\xi(1-a)}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m\mathrm{d}\rho\mathrm{d}x + (1-a)pH(t) \\
&+\left[\int_\Omega uf(u)\mathrm{d}x - (1-a)p\int_\Omega F(u)\mathrm{d}x\right] - \left(\frac{\mu_1\delta^m}{m}+\frac{|\mu_2|\delta^m}{m}\right)\|u\|_m^m \\
&-\frac{\mu_1(m-1)\delta^{-\frac{m}{m-1}}}{m}\|u_t(t)\|_m^m - \frac{|\mu_2|(m-1)\delta^{-\frac{m}{m-1}}}{m}\|z(\cdot,1,t)\|_m^m - (1-a)pE_1.
\end{aligned} \tag{3.18}$$

Since (H_2), then

$$\int_\Omega uf(u)\mathrm{d}x - (1-a)p\int_\Omega F(u)\mathrm{d}x \geq ak_0\|u\|_p^p,$$

and choosing $\delta^{-\frac{m}{m-1}} = NH^{-\theta}(t)$, where $N>0$ is a positive constant to be specified later, such that (3.18) becomes

$$\begin{aligned}
M'(t) \geq& \left[1+\frac{(1-a)p}{2}\right]\|u_t\|^2 + \left[\left(1-\frac{1}{4\eta}-\frac{(1-a)p}{2}\right)\int_0^t g_1(s)\mathrm{d}s + \frac{(1-a)p}{2}-1\right]\|\Delta u\|^2 \\
&+\left(1-\frac{1}{4\eta}-\frac{(1-a)p}{2}\right)\int_0^t g_2(s)\mathrm{d}s\,\|\nabla u\|^2 + \left[\frac{(1-a)p}{2}-\eta\right][(g_1\circ\Delta u)(t)+(g_2\circ\nabla u)(t)] \\
&+\frac{p\xi(1-a)}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m\mathrm{d}\rho\mathrm{d}x + (1-a)pH(t) + ak_0\|u\|_p^p - \frac{(\mu_1+|\mu_2|)}{mN^{m-1}}H^{\theta(m-1)}(t)\|u\|_m^m \\
&-\frac{(m-1)N}{m}H^{-\theta}(t)(\mu_1\|u_t(t)\|_m^m + |\mu_2|\|z(\cdot,1,t)\|_m^m) - (1-a)pE_1.
\end{aligned} \tag{3.19}$$

By using (3.11), $\theta \leq \frac{p-m}{p(m-1)}$ and Lemma 2.2, we obtain

$$-\frac{(\mu_1+|\mu_2|)}{mN^{m-1}}H^{\theta(m-1)}(t)\|u\|_m^m \geq -\frac{(\mu_1+|\mu_2|)k_1^{\theta(m-1)}}{mN^{m-1}p^{\theta(m-1)}}\|u\|_p^{\theta p(m-1)}\|u\|_m^m \tag{3.20}$$

$$
\begin{aligned}
&\geq -\frac{(\mu_1 + |\mu_2|)|\Omega|^{\frac{p-m}{p}} k_1^{\theta(m-1)}}{mN^{m-1}p^{\theta(m-1)}} \|u\|_p^{\theta p(m-1)+m} \\
&\geq -\frac{C_1}{mN^{m-1}}(\|\Delta u\|^2 + \|u\|_p^p),
\end{aligned}
$$

where $C_1 \coloneqq \frac{(\mu_1 + |\mu_2|)|\Omega|^{\frac{p-m}{p}} k_1^{\theta(m-1)} C}{p^{\theta(m-1)}}$.

Let $\eta = \frac{(1-a)p}{2}$, we now use Sobolev-Poincaré inequality, and by inserting (3.20) into (3.19), we obtain

$$
\begin{aligned}
M'(t) \geq &\left[1 + \frac{(1-a)p}{2}\right]\|u_t\|^2 + \left[\frac{(1-a)p}{2} - 1 + \left(1 - \frac{1}{2(1-a)p} - \frac{(1-a)p}{2}\right)\int_0^{+\infty} g_1(s)\mathrm{d}s\right. \\
&\left. + \frac{1}{\lambda_1}\left(1 - \frac{1}{2(1-a)p} - \frac{(1-a)p}{2}\right)\int_0^{+\infty} g_2(s)\mathrm{d}s\right]\|\Delta u\|^2 - \frac{C_1}{mN^{m-1}}\|\Delta u\|^2 \\
&+ \frac{p\xi(1-a)}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x + (1-a)pH(t) + \left(ak_0 - \frac{C_1}{mN^{m-1}}\right)\|u\|_p^p \\
&- \frac{(m-1)N}{m}H^{-\theta}(t)(\mu_1\|u_t(t)\|_m^m + |\mu_2|\|z(\cdot,1,t)\|_m^m) - (1-a)pE_1.
\end{aligned} \tag{3.21}
$$

Therefore, by substituting (3.21) into (3.12) and using Lemma 3.1, we have

$$
\begin{aligned}
L'(t) \geq &\left[(1-\theta)C_0 - \frac{(m-1)\mu_1 N\varepsilon}{m}\right]H^{-\theta}(t)\|u_t(t)\|_m^m \\
&+ \left[(1-\theta)C_0 - \frac{(m-1)|\mu_2|N\varepsilon}{m}\right]\|z(\cdot,1,t)\|_m^m H^{-\theta}(t) + \varepsilon\left[1 + \frac{(1-a)p}{2}\right]\|u_t\|^2 \\
&+ \varepsilon\left[\frac{(1-a)p}{2} - 1 - \frac{C_1}{mN^{m-1}} + \left(1 - \frac{1}{2(1-a)p} - \frac{(1-a)p}{2}\right)\int_0^{+\infty} g_1(s)\mathrm{d}s\right. \\
&\left. + \frac{1}{\lambda_1}\left(1 - \frac{1}{2(1-a)p} - \frac{(1-a)p}{2}\right)\int_0^{+\infty} g_2(s)\mathrm{d}s\right]\|\Delta u\|^2 \\
&+ \frac{\varepsilon p\xi(1-a)}{m}\int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x + \varepsilon(1-a)pH(t) \\
&+ \varepsilon\left(ak_0 - \frac{C_1}{mN^{m-1}}\right)\|u\|_p^p + (K - \varepsilon(1-a)p)E_1.
\end{aligned} \tag{3.22}
$$

At this point, we choose a small enough and using (H_3), such that

$$
\frac{(1-a)p}{2} - 1 + \left(1 - \frac{1}{2(1-a)p} - \frac{(1-a)p}{2}\right)\int_0^{+\infty} g_1(s)\mathrm{d}s + \frac{1}{\lambda_1}\left(1 - \frac{1}{2(1-a)p} - \frac{(1-a)p}{2}\right)\int_0^{+\infty} g_2(s)\mathrm{d}s > 0.
$$

Simplified, we obtain

$$
\int_0^{+\infty} g_1(s)\mathrm{d}s + \frac{1}{\lambda_1}\int_0^{+\infty} g_2(s)\mathrm{d}s < 1 - \frac{1}{(p - pa - 1)^2},
$$

and N so large that

$$
ak_0 - \frac{C_1}{mN^{m-1}} > 0,
$$

$$
\frac{(1-a)p}{2} - 1 - \frac{C_1}{mN^{m-1}} + \left(1 - \frac{1}{2(1-a)p} - \frac{(1-a)p}{2}\right)\int_0^{+\infty} g_1(s)\mathrm{d}s + \frac{1}{\lambda_1}\left(1 - \frac{1}{2(1-a)p} - \frac{(1-a)p}{2}\right)\int_0^{+\infty} g_2(s)\mathrm{d}s > 0.
$$

Once a and N are fixed, we pick ε small enough that

$$(1-\theta)C_0 - \frac{(m-1)\mu_1 N\varepsilon}{m} > 0, \quad (1-\theta)C_0 - \frac{(m-1)|\mu_2| N\varepsilon}{m} > 0, \quad K - \varepsilon(1-a)p > 0$$

and

$$L(0) = H^{1-\theta}(0) + \varepsilon\int_\Omega u_0(x)u_1(x)\mathrm{d}x > 0.$$

Thus, (3.22) takes the form

$$L'(t) \geq \varepsilon C_2\left(H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x + \|u\|_p^p\right), \tag{3.23}$$

for a constant $C_2 > 0$. Consequently,

$$L(t) \geq L(0) > 0, \quad \forall t \in [0, T).$$

Next, according to inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ $(a > 0, b > 0, p \geq 1)$ and Lemma 3.3, we obtain that

$$\begin{aligned} L^{\frac{1}{1-\theta}}(t) &\leq C_2\left[H(t) + \left(\left|\int_\Omega u_t u \mathrm{d}x\right|\right)^{\frac{1}{1-\theta}} + (KtE_1)^{\frac{1}{1-\theta}}\right] \\ &\leq C_2\left[H(t) + \left(\left|\int_\Omega u_t u \mathrm{d}x\right|\right)^{\frac{1}{1-\theta}} + (KTE_1)^{\frac{1}{1-\theta}}\right] \\ &\leq C_2\left[H(t) + \left(\left|\int_\Omega u_t u \mathrm{d}x\right|\right)^{\frac{1}{1-\theta}} + \frac{k_1(KTE_1)^{\frac{1}{1-\theta}}}{(B_1\alpha_2)^p}\|u\|_p^p\right]. \end{aligned} \tag{3.24}$$

Similar to literature [10] and using Lemma 2.1, we know

$$\left(\left|\int_\Omega u_t u \mathrm{d}x\right|\right)^{\frac{1}{1-\theta}} \leq C_3(\|u_t\|^2 + \|\Delta u\|^2 + \|u\|_p^p),$$

which implies for a constant $C_4 > 0$, then

$$L^{\frac{1}{1-\theta}}(t) \leq C_4\left(H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \int_\Omega\int_0^1 |z(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x + \|u\|_p^p\right). \tag{3.25}$$

Combining (3.23) and (3.25), we deduce that for some $C_5 > 0$,

$$L'(t) \geq C_5 L^{\frac{1}{1-\theta}}(t).$$

Integration over $(0, t)$ yields

$$L(t) \geq \left(L^{-\frac{\theta}{1-\theta}}(0) - C_5\frac{\theta}{1-\theta}t\right)^{-\frac{1-\theta}{\theta}}. \tag{3.26}$$

Therefore, by $L(0) > 0$ and (3.26), we have $\lim_{t\to T^*} L(t) = +\infty$, where $T^* \leq \frac{1-\theta}{\theta C_5}L^{-\frac{\theta}{1-\theta}}(0)$.
Part (ii): Setting $H_1(t) = -E(t)$, by Lemma 3.1 and (3.2), we obtain

$$H'(t) = -E'(t) \geq C_0(\|u_t(t)\|_m^m + \|z(\cdot, 1, t)\|_m^m).$$

Thus, using (H_2), such that

$$0 < -E(0) = H_1(0) \le H_1(t) \le \frac{k_1}{p}\,\|u\|_p^p.$$

We then define

$$L_1(t) = H_1^{1-\theta_1}(t) + \varepsilon M_1(t), \quad \forall t \in [0, T),$$

where ε are positive constants to be specified later, and

$$M_1(t) = \int_\Omega u u_t \mathrm{d}x, \quad 0 < \theta_1 \le \min\left\{\frac{p-2}{2p}, \frac{p-m}{p(m-1)}\right\}.$$

Then,

$$L'(t) = (1-\theta_1)H_1^{-\theta_1}(t)H'(t) + \varepsilon M'(t), \quad \forall t \in [0, T).$$

Similar to (3.13)–(3.26), we also prove any solution of (2.1) blows up in finite time T^* for $E(0) < 0$. This completes the proof. □

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