Research Article

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# A note on the global existence and boundedness of an *N*-dimensional parabolic-elliptic predator-prey system with indirect pursuit-evasion interaction

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**Abstract:** We investigate the two-species chemotaxis predator-prey system given by the following system:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla w) + u(\lambda_1 - \mu_1 u^{r_1-1} + av), & x\in\Omega,\ t>0,\\ v_t = \Delta v + \xi\nabla\cdot(v\nabla z) + v(\lambda_2 - \mu_2 v^{r_2-1} - bu), & x\in\Omega,\ t>0,\\ 0 = \Delta w - w + v, & x\in\Omega,\ t>0,\\ 0 = \Delta z - z + u, & x\in\Omega,\ t>0, \end{cases}$$

in a bounded domain $\Omega\subset\mathbb{R}^N (N\ge 1)$ with smooth boundary, where parameters $\chi,\xi,\lambda_i,\mu_i>0$, and $r_i>1$ $(i=1,2)$. Under appropriate conditions, utilizing suitable *a priori* estimates, we demonstrate that if $\frac{(N-2)_+}{N} < \max\left\{(r_1-1)(r_2-1), \frac{4}{N^2}, (r_1-1)\frac{2}{N}, (r_2-1)\frac{2}{N}\right\}$, then the system admits a unique, uniformly bounded global classical solution. This finding extends the results of several previous studies.

**Keywords:** chemotaxis, boundedness, global existence, Pursuit-evasion

**MSC 2020:** 35K20, 35K55, 92C17

## 1 Introduction

In nature, the existence of a purely single population is impossible. One of the most illustrative relationships among populations is the predator-prey relationship. We discuss the following predator-prey model:

$$\begin{cases} u_t = \Delta u + f(u,v), & x\in\Omega, t>0,\\ v_t = \Delta v + g(u,v), & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

where $u=u(x,t)$ and $v=v(x,t)$ represent the densities of predators and prey, respectively. The functions $f$ and $g$ describe the local interactions between predators and prey. Specifically, if $f(u,v)=u(r_1(t)-a_1(t)u-b_1(t)v)$ and $g(u,v)=v(r_2(t)-a_2(t)u-b_2(t)v)$ with $r_i$, $a_i$, and $b_i$ being $T$-periodic continuous functions, [1] demonstrated the existence and stability of pulsating waves. Furthermore, Du et al. [2] proved the existence of entire solutions and their qualitative properties. Subsequently, Bao et al. [3] obtained the uniqueness and asymptotic stability of the traveling wave front of the time periodic pyramid in the three-dimensional whole space.

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The biological chemotaxis model is not only employed to describe biological movement processes at the microscale but also utilized to study population dynamics at the macro scale. The classical chemotaxis model first emerged in [4]. More precisely, we will initially consider the model:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0. \end{cases} \tag{1.2}$$

Over the past five decades, considerable attention has been devoted to this classical chemotaxis model. In the one-dimensional setting, Osaki and Yagi [5] proved that system (1.2) possesses a global bounded classical solution. Additionally, this system has been shown to exhibit finite/infinite-time blow-up under certain conditions in two- and higher-dimensional domains [6–12].

Second, we consider the model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + au - \mu u^2, & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \tag{1.3}$$

where $\tau \in \{0, 1\}$. If $\tau = 1$, for any $\mu > 0$, the solution of (1.3) never blows up when $N = 1$ (see Osaki and Yagi [5]) or $N = 2$ (see Osaki et al. [13]). If $N \geq 3$ and $\mu$ is large enough, then the model (1.3) admits a bounded global classical solution [14]. When the logistic source in (1.3) is replaced by the term $au - \mu u^r$ with $r > 1$, under the conditions that $N \in \{3, 4\}$ and $r < \frac{7}{6}$ or $N \geq 5$ and $r < 1 + \frac{1}{2(N-1)}$, the solution of (1.3) with $\tau = 0$ blows up in finite time [15].

The parabolic-elliptic predator-prey system with logistic source can be formulated as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla w) + f(u, v), & x \in \Omega, t > 0, \\ v_t = \Delta v + \xi \nabla \cdot (v \nabla z) + g(u, v), & x \in \Omega, t > 0, \\ \tau w_t = \Delta w - w + v, & x \in \Omega, t > 0, \\ \tau z_t = \Delta z - z + u, & x \in \Omega, t > 0, \end{cases} \tag{1.4}$$

where $\tau \in \{0, 1\}$. This system also includes the concentrations of the chemicals discharged, represented by $w = w(x, t)$ and $z = z(x, t)$. The existence and boundedness of global weak solutions to (1.4) were shown in [16,17]. Indeed, in the crucial scenario where $\tau = 0$ (the parabolic-elliptic case), it is fundamental to note that $f$ is defined as $u(\lambda - u + av)$ and $g$ as $g = v(\mu - v - bu)$, as outlined in [18,19]. Within this context, the existence of global solutions and their behavior over extended periods have been rigorously constructed within a confined interval in one-dimensional space. We firmly believe that these findings offer a pivotal advancement in the understanding of predator-prey systems, marking a significant leap forward in this domain. If $\tau = 0$, $f = u(\lambda - u + av)$, and $g = v(\mu - v - bu)$, Li et al. [20] demonstrated that the solutions of system (1.4) are global and bounded when $N \leq 3$. Very recently, this result was improved in [21], where the existence of classical solutions and their long-time behavior were proved for $N \geq 1$. Zheng and Zhang [22] extended this model and obtained the global existence and boundedness of the classical solution. If $\tau = 1$, $f = u(\mu - u + av)$, and $g = v(\lambda - v - bu)$, for arbitrary $N \geq 1$, problem (1.4) possesses a global bounded classical solution when $a < 2$ and $\frac{N(2-a)}{2\left(C_{\frac{N}{2}+1}\right)^{\frac{1}{\frac{N}{2}+1}}(N-2)_+} > \max\{\chi, \xi\}$ [23], where $(N-2)_+ = \max\{0, N-2\}$.

In this article, we primarily focus on the following indirect pursuit-evasion system, which is formulated as follows:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla w) + u(\lambda_1 - \mu_1 u^{r_1 - 1} + av), & x \in \Omega, t > 0, \\ v_t = \Delta v + \xi \nabla \cdot (v \nabla z) + v(\lambda_2 - \mu_2 v^{r_2 - 1} - bu), & x \in \Omega, t > 0, \\ 0 = \Delta w - w + v, & x \in \Omega, t > 0, \\ 0 = \Delta z - z + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where $\Omega$ represents the habitat of both species, which is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial\Omega$. The notation $\frac{\partial}{\partial \nu}$ denotes the derivative taken with respect to the outer normal of $\partial\Omega$. The parameters $\chi, \xi, a, b, \mu_i$, and $\lambda_i$ are all positive, and $r_i > 1$ for $i = 1, 2$. The variables $u, v, w, z, \chi$, and $\xi$ are defined as previously mentioned.

The organization of the remainder of this article is as follows. In the subsequent section, we will gather basic facts that will be utilized later on. Section 3 presents some estimates related to the solution of model (1.5). Finally, in Section 4, we specifically provide the proof of Theorem 2.1.

## 2 Preliminaries and main results

In this section, we recall some results that will be used in our proof. In order to express the main results, for the initial data $(u_0, v_0)$, we assume that

$$\begin{aligned} &u_0 \in C^0(\bar{\Omega}), \quad \text{with } u_0 \geq 0 \text{ in } \bar{\Omega}, \\ &v_0 \in C^0(\bar{\Omega}), \quad \text{with } v_0 \geq 0 \text{ in } \bar{\Omega}. \end{aligned} \tag{2.1}$$

Under these aforementioned assumptions, we are going to introduce our main result.

**Theorem 2.1.** *Suppose $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary. Assume that $\chi, \xi, \lambda_1$, $\lambda_2, \mu_1$, and $\mu_2$ are positive and $r_i > 1$ ($i = 1, 2$) as well as $\frac{(N-2)_+}{N} < \max\left\{(r_2 - 1)(r_1 - 1), \frac{4}{N^2}, (r_1 - 1)\frac{2}{N}, (r_2 - 1)\frac{2}{N}\right\}$. Then, for any choice of $u_0$ and $v_0$ fulfilling (2.1), problem (1.5) has a globally classical solution $(u, v, w, z)$ that is unique within the class of functions satisfying*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ v \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ w \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,0}(\bar{\Omega} \times (0, \infty)), \\ z \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,0}(\bar{\Omega} \times (0, \infty)). \end{cases} \tag{2.2}$$

*Moreover, the solution is bounded in the sense that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C, \quad \text{for all } t > 0,$$

*is valid with some $C > 0$.*

**Remark 2.1.**

(i) For any $N \geq 1$, if $r_1 = r_2 = 2$, then $(r_2 - 1)(r_1 - 1) = 1$, and it is evident that $1 > \frac{(N-2)_+}{N}$. Consequently, Theorem 2.1 generalizes previous results found in [20,21].

(ii) Notably, our result $\frac{(N-2)_+}{N} < \max\{(r_2 - 1)(r_1 - 1), \frac{4}{N^2}, (r_1 - 1)\frac{2}{N}, (r_2 - 1)\frac{2}{N}\}$ constitutes an improvement upon the findings in [22]. Specifically, as $r_1, r_2 \to 1$, $(r_1 - 1)(r_2 - 1) \to 0$. At this juncture, the inequality $\frac{(N-2)_+}{N} < (r_2 - 1)(r_1 - 1)$ no longer holds, indicating that our result is marginally weaker than that of [22] in this specific limit.

(iii) According to Theorem 2.1, we deduce that the conclusion holds for any $r_1 > 1$ and $r_2 > 1$ when $N \leq 3$. This significantly extends the applicability of our result compared to [20].

(iv) We are unable to directly apply the methodologies employed in [20–22] to tackle the problem under investigation, as we consider it under substantially weaker conditions.

Without loss of generality, we shall assume that $\frac{(N-2)_+}{N} < \max\left\{(r_2-1)(r_1-1), \frac{4}{N^2}, (r_1-1)\frac{2}{N}, (r_2-1)\frac{2}{N}\right\} = \max\left\{\frac{4}{N^2}, (r_1-1)\frac{2}{N}, (r_2-1)\frac{2}{N}\right\}$ in the sequel, since the case $\frac{(N-2)_+}{N} < (r_2-1)(r_1-1)$ has been covered [22].

The local existence and uniqueness of (1.5) have already been proven in [24], which is stated as follows.

**Lemma 2.1.** *Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Suppose that $u_0$ and $v_0$ comply with* (2.1). *Then, there exist $T_{\max} > 0$ and a uniquely determined quadruple $(u, v, w, z)$ of nonnegative functions satisfying*:

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ v \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ w \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,0}(\bar{\Omega} \times (0, T_{\max})), \\ z \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,0}(\bar{\Omega} \times (0, T_{\max})), \end{cases} \tag{2.3}$$

*solving* (1.5) *in the classical sense in $\Omega \times (0, T_{\max})$. Furthermore, if $T_{\max} < +\infty$, then*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}.$$

## 3 *A priori* estimates

This section is devoted to proving Theorem 2.1 by deriving some *a priori* estimates. In view of [22], we can obtain the following lemma.

**Lemma 3.1.** *Assume that the conditions of* Lemma 2.1 *hold. Then, there exists a positive constant C such that the solution of* (1.5) *satisfies*

$$\int_\Omega u + \int_\Omega v \leq C, \quad \text{for all } t \in (0, T_{\max}). \tag{3.1}$$

*Moreover, for any $l_0 \in [1, \frac{N}{(N-2)_+})$, there is $C > 0$ such that*

$$\int_\Omega z^{l_0} + \int_\Omega w^{l_0} \leq C, \quad \text{for all } t \in (0, T_{\max}). \tag{3.2}$$

**Proof.** The proofs of (3.1) and (3.2) can be found in Lemmas 3.1 and 3.2 of [22], respectively. To avoid repetition, we omit giving details here. □

In the subsequent analysis, we will embark on a novel approach distinct from those employed in [20,21] to derive the crucial $L^p(\Omega)$-estimate. Notably, owing to the weakened nature of our assumption, we are confronted with the necessity of estimating the terms $\int_\Omega u^q$ and $\int_\Omega v^p$ individually when tackling the strong coupling term, as opposed to directly handling $\int_\Omega u^p$ and $\int_\Omega v^p$. Fortunately, by leveraging the inherent symmetry between $u$ and $v$, along with the seminal findings presented in [22], we can streamline our proof by focusing solely on demonstrating the validity of the two pivotal lemmas outlined below. This strategy not only simplifies the complexity of our derivation but also underscores the elegance and power of the proposed methodology.

**Lemma 3.2.** *Given that $p > 1$ and $q > 1$, under the condition that $\frac{4}{N} > (N-2)_+$, it is possible to identify a positive constant C that satisfies the following inequality for all $t \in (0, T_{\max})$*:

$$\int_\Omega u^q + \int_\Omega v^p \leq C. \tag{3.3}$$

**Proof.** Since

$$\frac{4}{N^2} > \frac{(N-2)_+}{N},$$

for any $p > 1$, one can choose $q > 1$ appropriately large and $l_0 \in \left(1, \frac{N}{(N-2)_+}\right)$, which is sufficiently close to $\frac{N}{(N-2)_+}$ such that

$$\frac{p + \frac{2}{N}}{\frac{2}{N}} < \left(q + \frac{2}{N}\right) \cdot l_0 \tag{3.4}$$

and

$$\frac{q + \frac{2}{N}}{\frac{2}{N}} < p + \frac{2}{N}. \tag{3.5}$$

Multiplying both sides of the third equation in (1.5) by $v^{p-1}$, integrating over Ω, and using the identity $\Delta z = z - u$, we deduce that

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + (p-1)\int_\Omega v^{p-2}|\nabla v|^2 + \int_\Omega v^p \\
&= \xi\int_\Omega \nabla\cdot(v\nabla z)v^{p-1} + \int_\Omega v^p(\lambda_2 - \mu_2 v^{r_2-1} - bu) + \int_\Omega v^p \\
&= \frac{p-1}{p}\xi\int_\Omega v^p\Delta z + \int_\Omega v^p(\lambda_2 - \mu_2 v^{r_2-1} - bu) + \int_\Omega v^p \\
&= \frac{p-1}{p}\xi\int_\Omega v^p(z-u) + \lambda_2\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} - b\int_\Omega uv^p + \int_\Omega v^p \\
&\le \frac{p-1}{p}\xi\int_\Omega v^p z + \lambda_2\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} - b\int_\Omega uv^p + \int_\Omega v^p,
\end{aligned} \tag{3.6}$$

for all $t \in (0, T_{\max})$, where, in the last step, we used the fact that $\frac{p-1}{p}\xi\int_\Omega v^p u \ge 0$. In light of the Gagliardo-Nirenberg inequality and Lemma 3.1, we have

$$\begin{aligned}
\int_\Omega v^{p+\frac{2}{N}} &= \left\|v^{\frac{p}{2}}\right\|_{L^{\frac{2(p+\frac{2}{N})}{p}}(\Omega)}^{\frac{2(p+\frac{2}{N})}{p}} \\
&\le C_1\left(\left\|\nabla v^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p+\frac{2}{N})}{p}\theta_1}\left\|v^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+\frac{2}{N})}{p}(1-\theta_1)} + \left\|v^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+\frac{2}{N})}{p}}\right) \\
&= C_1\left(\left\|\nabla v^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 \|v\|_{L^1(\Omega)}^{p+\frac{2}{N}-2} + \|v\|_{L^1(\Omega)}^{p+\frac{2}{N}}\right) \\
&\le C_2\left(\left\|\nabla v^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + 1\right), \quad \text{for all } t \in (0, T_{\max}),
\end{aligned} \tag{3.7}$$

where $\theta_1 = \frac{\frac{Np}{2} - \frac{Np}{2\left(p+\frac{2}{N}\right)}}{\frac{Np}{2} + 1 - \frac{N}{2}} \in (0,1)$ with some certain $C_1 > 0$ and $C_2 > 0$. A direct calculation yields

$$(p-1)\int_\Omega v^{p-2}|\nabla v|^2 \ge C_3\int_\Omega v^{p+\frac{2}{N}} - C_4, \tag{3.8}$$

for all $t \in (0, T_{\max})$ with positive constants $C_3$ and $C_4$. We multiply the first equation in (1.5) by $u^{q-1}$ and integrate by parts using the identity $\Delta w = w - v$ to derive that

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_\Omega u^q + (q-1)\int_\Omega u^{q-2}|\nabla u|^2 + \int_\Omega u^q \\
&= -\chi\int_\Omega \nabla\cdot(u\nabla w)u^{q-1} + \int_\Omega u^q(\lambda_1 - \mu_1 u^{r_1-1} + av) + \int_\Omega u^q \\
&= -\frac{q-1}{q}\chi\int_\Omega u^q\Delta w + \int_\Omega u^q(\lambda_1 - \mu_1 v^{r_1-1} + av) + \int_\Omega u^q \\
&= \frac{q-1}{q}\chi\int_\Omega u^q(v-w) + \lambda_1\int_\Omega u^q - \mu_1\int_\Omega u^{q+r_1-1} + a\int_\Omega vu^q + \int_\Omega u^q \\
&\le \frac{q-1}{q}\chi\int_\Omega u^q v + \lambda_1\int_\Omega u^q - \mu_1\int_\Omega u^{q+r_1-1} + a\int_\Omega vu^q + \int_\Omega u^q,
\end{aligned} \tag{3.9}$$

for all $t \in (0, T_{\max})$, where, in the last step, we used the fact that $\frac{q-1}{q}\chi\int_\Omega u^q w \ge 0$. Similar to the procedures of (3.7), we see that there exist some positive constants $C_5$ and $C_6$ such that

$$(q-1)\int_\Omega u^{q-2}|\nabla u|^2 \ge C_5\int_\Omega u^{q+\frac{2}{N}} - C_6, \quad \text{for all } t \in (0, T_{\max}). \tag{3.10}$$

In view of the Young inequality, we can find $C_7 > 0$ such that

$$\frac{p-1}{p}\xi\int_\Omega v^p z \le \frac{C_3}{2}\int_\Omega v^{p+\frac{2}{N}} + C_7\int_\Omega z^{\frac{p+\frac{2}{N}}{\frac{2}{N}}}, \quad \text{for all } t \in (0, T_{\max}). \tag{3.11}$$

In the same way, we have

$$\left(\frac{q-1}{q}\chi + a\right)\int_\Omega u^q v \le \frac{C_5}{2}\int_\Omega u^{q+\frac{2}{N}} + C_8\int_\Omega v^{\frac{q+\frac{2}{N}}{\frac{2}{N}}}, \quad \text{for all } t \in (0, T_{\max}), \tag{3.12}$$

with $C_8 > 0$ being a constant. Plugging (3.8) and (3.11) into (3.6), we see that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p + \int_\Omega v^p \le -\frac{C_3}{2}\int_\Omega v^{p+\frac{2}{N}} + C_7\int_\Omega z^{\frac{p+\frac{2}{N}}{\frac{2}{N}}} + (\lambda_2+1)\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} - b\int_\Omega uv^p + C_4, \tag{3.13}$$

for all $t \in (0, T_{\max})$. Inserting (3.10) and (3.12) back into (3.9), we easily obtain

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u^q + \int_\Omega u^q \le -\frac{C_5}{2}\int_\Omega u^{q+\frac{2}{N}} + C_8\int_\Omega v^{\frac{q+\frac{2}{N}}{\frac{2}{N}}} + (\lambda_1+1)\int_\Omega u^q - \mu_1\int_\Omega u^{q+r_1-1} + C_6, \tag{3.14}$$

for all $t \in (0, T_{\max})$. Since $l_0 \in \left(1, \frac{N}{(N-2)_+}\right)$, using Lemma 3.1, we can find that $C_9$ satisfies

$$\|z\|_{L^{l_0}(\Omega)} \le C_9. \tag{3.15}$$

Without losing generality, we may assume that $N > 2$, since by using (3.15), one can estimate the term $\int_\Omega z^{\frac{p+\frac{2}{N}}{\frac{2}{N}}}$ on the right-hand side of (3.13) very easily. With the help of the Gagliardo-Nirenberg inequality, (3.15), and the $L^p$ theory of elliptic equation, we can find positive constants $C_{10}$, $C_{11}$, and $C_{12}$ such that

$$\int_\Omega z^{\frac{p+\frac{2}{N}}{\frac{2}{N}}} \le C_{10}\left(\|\Delta z\|_{L^{q+\frac{2}{N}}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}\theta_2}\|z\|_{L^{l_0}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}(1-\theta_2)} + \|z\|_{L^{l_0}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}}\right)$$

$$\begin{aligned} &\le C_{11}\left(\|\Delta z\|_{L^{q+\frac{2}{N}}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}\theta_2}+1\right)\\ &\le C_{12}\left(\|u\|_{L^{q+\frac{2}{N}}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}\theta_2}+1\right), \quad \text{for all } t\in(0,T_{\max}), \end{aligned} \tag{3.16}$$

where $\theta_2 = \frac{1-\frac{2}{N}-\frac{\frac{2}{N}}{p+\frac{2}{N}}}{1-\frac{1}{q+\frac{2}{N}}} \in (0,1)$. Altogether, (3.4) and (3.5) provide

$$\frac{p+\frac{2}{N}}{\frac{2}{N}}\cdot\frac{1-\frac{2}{N}-\frac{\frac{2}{N}}{p+\frac{2}{N}}}{1-\frac{1}{q+\frac{2}{N}}} < q+\frac{2}{N}. \tag{3.17}$$

Collecting (3.13), (3.16), (3.17) and applying the Young inequality, we deduce that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + \int_\Omega v^p \le -\frac{C_3}{2}\int_\Omega v^{p+\frac{2}{N}} + C_7C_{12}\int_\Omega u^{q+\frac{2}{N}} + (\lambda_2+1)\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} - b\int_\Omega uv^p + C_4 + C_{12}, \tag{3.18}$$

for all $t\in(0,T_{\max})$. Combining (3.14) with (3.18) and using Young's inequality, we can find some positive constants $C_{13}$ and $C_{14}$ satisfying

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + \int_\Omega v^p + \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q + \int_\Omega u^q\\ \le\ & -\frac{C_3}{2}\int_\Omega v^{p+\frac{2}{N}} + C_7C_{12}\int_\Omega u^{q+\frac{2}{N}} + (\lambda_2+1)\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} - b\int_\Omega uv^p + C_4 + C_{12}\\ & -\frac{C_5}{2}\int_\Omega u^{q+\frac{2}{N}} + C_8\int_\Omega v^{\frac{q+\frac{2}{N}}{\frac{2}{N}}} + (\lambda_1+1)\int_\Omega u^q - \mu_1\int_\Omega u^{q+r_1-1} + C_6\\ \le\ & -\frac{C_3}{4}\int_\Omega v^{p+\frac{2}{N}} - \frac{C_5}{4}\int_\Omega u^{q+\frac{2}{N}} - \frac{\mu_2}{2}\int_\Omega v^{p+r_2-1} - \frac{\mu_1}{2}\int_\Omega u^{q+r_1-1} + C_{13}\\ \le\ & C_{14}, \end{aligned} \tag{3.19}$$

for all $t\in(0,T_{\max})$, where we have used the fact that $\frac{q+\frac{2}{N}}{\frac{2}{N}} < p+\frac{2}{N}$ (see (3.5)). By the Gronwall inequality, (3.3) is obtained. Thereupon, the proof of this lemma is complete. □

**Lemma 3.3.** *Let $p>1$ and $q>1$. If $(r_1-1)\frac{2}{N} > \frac{(N-2)_+}{N}$ or $(r_2-1)\frac{2}{N} > \frac{(N-2)_+}{N}$, there exists $C>0$ such that*

$$\int_\Omega u^q + \int_\Omega v^p \le C, \quad \textit{for all } t\in(0,T_{\max}). \tag{3.20}$$

**Proof.** Based on the given conditions, we divide the proof into the following two cases for analysis. Case $(r_1-1)\frac{2}{N} > \frac{(N-2)_+}{N}$: Since

$$r_1>1 \quad\text{and}\quad (r_1-1)\frac{2}{N} > \frac{(N-2)_+}{N},$$

for any $p>1$, we pick $q>1$ appropriately large and $l_1\in\left(1,\frac{N}{(N-2)_+}\right)$, which is sufficiently close to $\frac{N}{(N-2)_+}$ such that

$$\frac{p+\frac{2}{N}}{\frac{2}{N}} \le \left(q+\frac{2}{N}\right)l_1 \tag{3.21}$$

and

$$\frac{q+r_1-1}{r_1-1} \le p + \frac{2}{N}. \tag{3.22}$$

We multiply the second equation of (1.5) by $v^{p-1}$ and integrate by parts using the identity $\Delta z = z - u$ to obtain

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + (p-1)\int_\Omega v^{p-2}|\nabla v|^2 + \int_\Omega v^p \\
&= \xi\int_\Omega \nabla\cdot(v\nabla z)v^{p-1} + \int_\Omega v^p(\lambda_2 - \mu_2 v^{r_2-1} - bu) + \int_\Omega v^p \\
&= \frac{p-1}{p}\xi\int_\Omega v^p\Delta z + \int_\Omega v^p(\lambda_2 - \mu_2 v^{r_2-1} - bu) + \int_\Omega v^p \\
&= \frac{p-1}{p}\xi\int_\Omega v^p(z-u) + \lambda_2\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} - b\int_\Omega uv^p + \int_\Omega v^p \\
&\le \frac{p-1}{p}\xi\int_\Omega v^p z + \lambda_2\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} - b\int_\Omega uv^p + \int_\Omega v^p,
\end{aligned} \tag{3.23}$$

for all $t \in (0, T_{\max})$, where, in the last step, we used the fact that $\frac{p-1}{p}\xi\int_\Omega v^p u \ge 0$. Using (3.7) and (3.8), there exist positive constants $C_1$ and $C_2$ such that

$$(p-1)\int_\Omega v^{p-2}|\nabla v|^2 \ge C_1\int_\Omega v^{p+\frac{2}{N}} - C_2. \tag{3.24}$$

Employing Young's inequality and using (3.24), there exists $C_3 > 0$ such that

$$\frac{p-1}{p}\xi\int_\Omega v^p z \le \frac{C_1}{2}\int_\Omega v^{p+\frac{2}{N}} + C_3\int_\Omega z^{\frac{p+\frac{2}{N}}{\frac{2}{N}}}, \quad \text{for all } t \in (0, T_{\max}). \tag{3.25}$$

Plugging (3.25) and (3.24) into (3.23), we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + \int_\Omega v^p \le -\frac{C_1}{2}\int_\Omega v^{p+\frac{2}{N}} + C_3\int_\Omega z^{\frac{p+\frac{2}{N}}{\frac{2}{N}}} + (\lambda_2+1)\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} + C_4, \tag{3.26}$$

for all $t \in (0, T_{\max})$. Recall $l_1 \in \left(1, \frac{N}{(N-2)_+}\right)$. Thus, by using Lemma 3.1, we can find that $C_6$ satisfies

$$\|z\|_{L^{l_1}(\Omega)} \le C_6. \tag{3.27}$$

Applying the Gagliardo-Nirenberg inequality, (3.27), and the $L^p$ theory of elliptic equation, we can then derive that

$$\begin{aligned}
C_5\int_\Omega z^{\frac{p+\frac{2}{N}}{\frac{2}{N}}} &\le C_7\left(\|\Delta z\|_{L^{q+r_1-1}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}\theta_2}\|z\|_{L^{l_1}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}(1-\theta_2)} + \|z\|_{L^{l_1}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}}\right) \\
&\le C_8\left(\|\Delta z\|_{L^{q+r_1-1}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}\theta_2} + 1\right) \\
&\le C_9\left(\|u\|_{L^{q+r_1-1}(\Omega)}^{\frac{p+\frac{2}{N}}{\frac{2}{N}}\theta_2} + 1\right), \quad \text{for all } t \in (0, T_{\max}),
\end{aligned} \tag{3.28}$$

where $\theta_2 = \frac{1-\frac{2}{N}-\frac{\frac{2}{N}}{p+\frac{2}{N}}}{1-\frac{1}{q+r_1-1}} \in (0,1)$ with certain $C_7 > 0$, $C_8 > 0$ and $C_9 > 0$. Altogether, (3.21) and (3.22) provide

$$\frac{p+\frac{2}{N}}{\frac{2}{N}} \cdot \frac{1-\frac{2}{N}-\frac{\frac{2}{N}}{p+\frac{2}{N}}}{1-\frac{1}{q+r_1-1}} \le q + r_1 - 1. \tag{3.29}$$

Collecting (3.26), (3.28), (3.29) along with the Young's inequality, we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + \int_\Omega v^p \le -\frac{C_3}{2}\int_\Omega v^{p+\frac{2}{N}} + C_9\int_\Omega u^{q+r_1-1} + (\lambda_2+1)\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} + C_4 + C_9, \tag{3.30}$$

for all $t \in (0, T_{\max})$. We multiply the first equation in (1.5) by $u^{q-1}$ and integrate it by parts over $\Omega$ applying the identity $\Delta w = w - v$ to infer that

$$\begin{aligned}
&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q + (q-1)\int_\Omega u^{q-2}|\nabla u|^2 + \int_\Omega u^q \\
&= -\chi\int_\Omega \nabla\cdot(u\nabla w)u^{q-1} + \int_\Omega u^q(\lambda_1 - \mu_1 u^{r_1-1} + av) + \int_\Omega u^q \\
&= -\frac{q-1}{q}\chi\int_\Omega u^q\Delta w + \int_\Omega u^q(\lambda_1 - \mu_1 v^{r_1-1} + av) + \int_\Omega u^q \\
&= \frac{q-1}{q}\chi\int_\Omega u^q(v-w) + \lambda_1\int_\Omega u^q - \mu_1\int_\Omega u^{q+r_1-1} + a\int_\Omega vu^q + \int_\Omega u^q \\
&\le \frac{q-1}{q}\chi\int_\Omega u^q v + \lambda_1\int_\Omega u^q - \mu_1\int_\Omega u^{q+r_1-1} + a\int_\Omega vu^q + \int_\Omega u^q,
\end{aligned} \tag{3.31}$$

for all $t \in (0, T_{\max})$, where, in the last step, we used the fact that $\frac{q-1}{q}\chi\int_\Omega u^q w \ge 0$. Using Young's inequality once more, we arrive at

$$\left(\frac{q-1}{q}\chi + a\right)\int_\Omega u^q v \le \frac{\mu_1}{2}\int_\Omega u^{q+r_1-1} + C_{10}\int_\Omega v^{\frac{q+r_1-1}{r_1-1}}, \tag{3.32}$$

for all $t \in (0, T_{\max})$ with some constant $C_{10} > 0$. Then, we substitute (3.32) into (3.31) to discover

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q + (q-1)\int_\Omega u^{q-2}|\nabla u|^2 + \int_\Omega u^q \le (\lambda_1+1)\int_\Omega u^q - \frac{\mu_1}{2}\int_\Omega u^{q+r_1-1} + C_{10}\int_\Omega v^{\frac{q+r_1-1}{r_1-1}}, \quad \text{for all } t \in (0, T_{\max}). \tag{3.33}$$

Adding (3.30) and (3.33) gives

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + \int_\Omega v^p + \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q + (q-1)\int_\Omega u^{q-2}|\nabla u|^2 + \int_\Omega u^q \\
&\le -\frac{C_3}{2}\int_\Omega v^{p+\frac{2}{N}} + C_9\int_\Omega u^{q+r_1-1} + C_9 + (\lambda_2+1)\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} + C_4 \\
&\quad + (\lambda_1+1)\int_\Omega u^q - \frac{\mu_1}{2}\int_\Omega u^{q+r_1-1} + C_{10}\int_\Omega v^{\frac{q+r_1-1}{r_1-1}}, \quad \text{for all } t \in (0, T_{\max}).
\end{aligned} \tag{3.34}$$

According to (3.22), we employ Young's inequality to find $C_{11} > 0$ fulfilling

$$C_{10}\int_\Omega v^{\frac{q+r_1-1}{r_1-1}} \le \frac{C_3}{4}\int_\Omega v^{p+\frac{2}{N}} + C_{11}, \quad \text{for all } t \in (0, T_{\max}). \tag{3.35}$$

Let $C_8 := \frac{\mu_1}{4}$. Thus, a combination of (3.34) and (3.35) yields

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + \int_\Omega v^p + \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q + (q-1)\int_\Omega u^{q-2}|\nabla u|^2 + \int_\Omega u^q \\ &\le -\frac{C_3}{4}\int_\Omega v^{p+\frac{2}{N}} - \frac{\mu_1}{4}\int_\Omega u^{q+r_1-1} + (\lambda_2+1)\int_\Omega v^p - \mu_2\int_\Omega v^{p+r_2-1} + (\lambda_1+1)\int_\Omega u^q + C_{12} \\ &\le C_{13}, \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \tag{3.36}$$

with some positive constants $C_{12}$ and $C_{13}$. Here, we have used the fact that $\frac{q+r_1-1}{r_1-1} \le p + \frac{2}{N}$ (see (3.22)) and $q + r_1 - 1 > q$ by $r_1 > 1$. By means of the Gronwall inequality, we obtain the desired results.

By leveraging the principle of symmetry, case $(r_2-1)\frac{2}{N} > \frac{(N-2)_+}{N}$ can be proven using exactly the same steps. In order to avoid repetition, we omit it here. □

Now, we can gather all our previous findings and synthesize them to arrive at the primary and pivotal result of our investigation.

**Proof of Theorem 2.1.** In view of

$$\frac{(N-2)_+}{N} < \max\left\{(r_2-1)(r_1-1), \frac{4}{N^2}, (r_1-1)\frac{2}{N}, (r_2-1)\frac{2}{N}\right\} = \max\left\{\frac{4}{N^2}, (r_1-1)\frac{2}{N}, (r_2-1)\frac{2}{N}\right\},$$

given the considerations outlined in Lemmas 3.2 and 3.3, for any $p > 1$, there exists $C_1 > 0$ such that

$$\int_\Omega u^q + \int_\Omega v^p \le C_1, \quad \text{for all } t \in (0, T_{\max}).$$

Then, we may invoke Lemma A.1 in [25], which establishes the following estimate through a Moser-type iteration (or a series of standard semigroup arguments) and the $L^p$ theory of elliptic equations:

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_2, \quad \text{for all } t \in (0, T_{\max})$$

with some constant $C_2 > 0$. Consequently, we choose to omit the detailed proofs for these lemmas here, as they would largely replicate the arguments found in the aforementioned section. This decision serves to streamline our presentation and avoid unnecessary repetition. □

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