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Research Article

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On a generalized Krasnoselskii fixed point theorem

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Abstract: This study concerns a Krasnoselskii-type fixed point theorem for the sum of two operators A, B in a Banach space E, where B is a Reich-type contractive mapping and A is a k-set contractive mapping. We introduce a class of operators $\theta : X \times X \to [1, +\infty)$ satisfying some axioms and use it as a *new metric* to prove a fixed point theorem in the spirit of Azam et al. [*Reich-Krasnoselskii-type fixed point results with applications in integral equations*, J. Inequal. Appl. **2023** (2023), 131].

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1 Introduction and preliminaries

Fixed point theory plays a very important role in proving the existence, uniqueness, and properties of solution of various types of functional-differential-integral equations [1–4].

The authors have studied the problem by reduction to a fixed point problem of the mapping T, that is a point x such that

$$T(x) = x.$$

The foundation of fixed point theory is the well-known Banach's fixed point theorem which states that if T is a contractive mapping, i.e.,

$$d(Tx, Ty) \leq qd(x, y)$$

for some $q \in (0, 1)$, acting in a complete space (X, d), then it has a unique fixed point. A simple proof of Banach's fixed point theorem can be found in [5]. While the Banach theorem is well-known for the uniqueness results, the Schauder theorem is famous for the existence results.

Theorem 1.1. [6] *Let M be a non-empty bounded convex subset of a normed space X and* $T : M \to M$ *be a compact mapping.*

Then, T has a fixed point in M.

In the next theorem, Krasnoselskii proved the existence of fixed point result for the sum of compact and contractive mappings. This combines the Banach and Schauder fixed point theorems.

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Theorem 1.2. [6] *Let M be a non-empty convex closed subset of a Banach space E, A and B be mappings $M \to E$. Assume that*

i. for all $x, y \in M$, $Ax + By \in M$;
ii. A is compact and continuous;
iii. B is contractive.

Then, the sum $A + B$ has a fixed point in M.

The Krasnoselskii's theorem generalizes the Banach theorem and Schauder theorem, as we can see by taking $A = O$ and $B = O$, respectively. For generalizations of Krasnoselskii's theorem, refer [7]. We next introduce the Reich fixed point theorem, which is a generalization of the Banach fixed point theorem.

Theorem 1.3. [8] *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping with the following property:*

$$d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where a_1, a_2, a_3 are nonnegative numbers which satisfy $a_1 + a_2 + a_3 < 1$.

Then, T has a unique fixed point in X.

The notion of Kuratowski measure of non-compactness was introduced by Kuratowski, which is defined by [1], for each bounded subset $\Omega \subset E$,

$$\alpha(\Omega) = \inf\{d > 0 : \Omega \text{ is covered by a finite family of subsets with diameter less than } d\}.$$

By introducing the notation of Kuratowski measure of non-compactness and k-set contractive mappings, Darbo generalized Schauder fixed point theorem in the following theorem.

Theorem 1.4. [4] *Let M be a non-empty bounded closed convex subset of a Banach space E and $T : M \to M$ be a k-set contractive mappings with respect to a measure of non-compactness α in E, namely, T is continuous and*

$$\alpha(T(\Omega)) \leq k\alpha(\Omega), \quad \forall \Omega \subset M, \quad \alpha(\Omega) > 0, \tag{1.2}$$

and $k \in (0, 1)$. Then, T has a fixed point theorem.

More generally, the authors have defined the measure of non-compactness in a Banach space as follows.

Definition 1.5. [1] Let $\mathcal{M}$ be the family of bounded subsets of a Banach E. A mapping $\mu : \mathcal{M} \to [0, +\infty)$ is called a measure of non-compactness if the following conditions hold for all $\Omega, \Omega_1, \Omega_2 \in \mathcal{M}$:

i. $\mu(\Omega) = \mu(\overline{\Omega})$.
ii. $\mu(\Omega_1 \cup \Omega_2) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}$.
iii. $\mu(\Omega) = 0 \Leftrightarrow \Omega$ is precompact.

Based on the measure of non-compactness defined by the above definition (with some suitable changing), the authors have proved a number of generalizations of Darbo fixed point theorem which can be seen in [9,10]. Also, the class of k-set contractive mappings has been generalized to be condensing mappings class in [10–12].

In 2023, Azam et al. combined the Reich theorem and Darbo theorem in type of Krasnoselskii's theorem and obtained the following theorem.

Theorem 1.6. [13] *Let M be a nonempty convex closed subset of a Banach space $(E, ||.||)$, A and B be mappings $M \to E$. Assume that there are non-negative numbers a_1, a_2, a_3, $a_1 + a_2 + a_3 < 1$ such that*

i. $||Bx - By|| \leq a_1||Az - (I - B)x|| + a_2||Az - (I - B)y|| + a_3||x - y||$ *for all $x, y, z \in M$.*
ii. A is strictly k-set contractive, namely,

$$\mu(A(\Omega)) < k\mu(\Omega), \quad \forall \Omega \subset M,$$

where $0 < k \leq (1 - a_3)/(1 + a_2)$ *and μ is a measure of non-compactness defined by definition* (1.5).
iii. For all $x, y \in M$, $Ax + By \in M$.

Then, the sum $A + B$ has a fixed point in M.

In this work, we introduce a class of operators $\theta : X \times X \to [1, +\infty)$ satisfying some axioms and use it as a *new metric* in a Banach space E to prove a fixed point theorem of sum of a contractive operator and a condensing operator (Theorem (2.12)). Moreover, the contractive condition of the mapping B is more general. The main difficulties in studying our problem are proving the properties of the *new measure of non-compactness based on* θ and the fixed point theorem for *condensing operators with respect to this measure of non-compactness*. To overcome this challenge, we need to prove additional properties beyond the usual ones (refer Definition 2.8 and Proposition 2.9).

In Section 2, our results are organized as follows. We first present the class of operators θ, which is used as a *new metric* in a metric space and prove the Reich-type fixed point theorem. We then present more axioms for θ in a Banach space and prove the generalized Reich-type fixed point theorem for the sum $T = A + B$. Finally, we present the notation *measure of non-compactness based on* θ and prove the a Krasnoselskii fixed point theorem in type of Azam [13]. Also, by making use of the proved theorem, we consider the existence of solution of an integral equation.

For the reader's convenience, we denoted by (X, d) a metric space and by $(E, ||.||)$ a Banach space with the null element 0_X and 0_E, respectively.

2 Results

2.1 Reich theorem

Definition 2.1. Let X be a topological space, we denote by Θ_X the set of the functions $\theta : X \times X \to [1, +\infty)$ which satisfies the following properties:

θ1. for all $\{x_n\}_n \subset X$

$$\lim_{n\to\infty} \theta(x_n, x) = 1 \Leftrightarrow \lim_{n\to\infty} x_n = x;$$

θ2. $\theta(x, y) \leq \theta(x, z) \times \theta(z, y)$ for all $x, y, z \in X$.

Example. Let (X, d) be a metric space and

$$\theta(x, y) := a^{d(x,y)}$$

for some $a > 1$. Then, $\theta \in \Theta_X$.

Moreover, if the metric d is complete, then (X, θ) is complete in the sense of:

$$\lim_{m>n\to\infty} \theta(x_n, x_m) = 1 \quad \text{if and only if} \quad \text{the sequence } \{x_n\}_n \text{ is convergent.} \tag{2.1}$$

In the opposite case, suppose that $\theta \in \Theta_X$. We define $a(x, y)$ to be $\ln(\theta(x, y))$ for all $x, y \in X$, then $a(x, y) > 0$ for all $x \neq y$ and a satisfies the triangle inequality $a(x, y) \leq a(x, z) + a(z, y)$. However, the symmetry property is not guaranteed.

Lemma 2.2. *Let X be a topological space, $T : X \to X$ be a mapping, and $\theta \in \Theta_X$. Assume that there exists a number* $0 < \alpha < 1$ *such that*

$$\theta(Tx, Ty) \leq [\theta(x, y)]^\alpha. \tag{2.2}$$

Moreover, X is complete in the sense of (2.1).

Then, T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $x_{n+1} = Tx_n$. Then (2.2) shows that

$$\begin{aligned} 1 \leq \theta(x_n, x_{n+1}) &\leq [\theta(x_{n-1}, x_n)]^\alpha \leq [\theta(x_{n-2}, x_{n-1})]^{\alpha^2} \\ &\cdots \\ &\leq [\theta(x_0, x_1)]^{\alpha^n}, \quad \forall n \geq 1. \end{aligned}$$

Let $m > n$, the property $\theta 2$ yields

$$\begin{aligned}1 \le \theta(x_n, x_m) &\le \theta(x_n, x_{n+1}) \times \theta(x_{n+1}, x_{n+2}) \times \ldots \times \theta(x_{m-1}, x_m)\\ &\le [\theta(x_0, x_1)]^{\alpha^n} \times [\theta(x_0, x_1)]^{\alpha^{n+1}} \times \ldots \times [\theta(x_0, x_1)]^{\alpha^{m-1}}\\ &\le [\theta(x_0, x_1)]^{\alpha^n/(1-\alpha)}.\end{aligned}$$

By taking the limit as $n \to \infty$, since $\alpha \in (0, 1)$, we see that the sequence $\{x_n\}_n$ is convergent, set

$$\lim_{n\to\infty} x_n = z.$$

By (2.2) we have

$$1 \le \theta(Tx_n, Tz) \le [\theta(x_n, z)]^\alpha.$$

By taking the limit as $n \to \infty$, the property $\theta 1$ shows that

$$\lim_{n\to\infty} \theta(x_{n+1}, Tz) = \lim_{n\to\infty} \theta(Tx_n, Tz) = 1.$$

Hence, $Tz = \lim_{n\to\infty} x_{n+1} = z$.

If $Tx = x$ and $Ty = y$, then

$$\theta(x, y) = \theta(Tx, Ty) \le [\theta(x, y)]^\alpha.$$

As $\alpha \in (0, 1)$, then this is distraction unless $x = y$.

The proof is completed. □

Definition 2.3. The function $\theta \in \Theta_X$ is said to be symmetric if

$$\theta(x, y) = \theta(y, x), \quad \forall x, y \in X. \tag{2.3}$$

By the definition, if $\theta \in \Theta_X$ symmetric, then θ is continuous. Indeed, from the inequalities

$$\begin{aligned}\theta(x, y) &\le \theta(x, x_n) \times \theta(x_n, y_n) \times \theta(y_n, y)\\ &\le \theta(x, x_n) \times \theta(x_n, x) \times \theta(x, y) \times \theta(y, y_n) \times \theta(y_n, y),\end{aligned}$$

we infer that if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

$$\lim_{n\to\infty} \theta(x_n, y_n) = \theta(x, y).$$

Theorem 2.4. *Let (X, d) be a metric space and T be a mapping from X to X. Assume that there are nonnegative numbers a_i, $i = 1, \ldots, 5$ and $\theta \in \Theta_X$ such that*

$$\theta(Tx, Ty) \le [\theta(x, Tx)]^{a_1} \times [\theta(y, Ty)]^{a_2} \times [\theta(x, Ty)]^{a_3} \times [\theta(y, Tx)]^{a_4} \times [\theta(x, y)]^{a_5}, \tag{2.4}$$

where $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, θ is symmetric, and (X, θ) is complete in the sense of (2.1).

Then, T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $x_{n+1} = Tx_n$.

Step 1: There is a number $\alpha \in (0, 1)$ such that

$$\theta(x_1, x_2) \le [\theta(x_0, x_1)]^\alpha. \tag{2.5}$$

Indeed, setting $x = x_0$ and $y = x_1$ in (2.4), we have

$$\theta(x_1, x_2) \le [\theta(x_0, x_1)]^{a_1} \times [\theta(x_1, x_2)]^{a_2} \times [\theta(x_0, x_2)]^{a_3} \times [\theta(x_1, x_1)]^{a_4} \times [\theta(x_0, x_1)]^{a_5}.$$

By simplifying with the properties of θ, we have

$$[\theta(x_1, x_2)]^{1-a_2} \le [\theta(x_0, x_1)]^{a_1+a_5}[\theta(x_0, x_2)]^{a_3} \le [\theta(x_0, x_1)]^{a_1+a_5}[\theta(x_0, x_1) \times \theta(x_1, x_2)]^{a_3}.$$

Hence,

$$[\theta(x_1, x_2)] \le [\theta(x_0, x_1)]^{(a_1+a_3+a_5)/(1-a_2-a_3)}.$$

If $a_3 \le a_4$, then $a_1 + a_2 + 2a_3 + a_5 \le a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and we have

$$\alpha = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} < 1$$

and the step 1 is proved.

If $a_3 > a_4$, then we set $x = x_1$ and $y = x_0$, the result is

$$\alpha = \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} < 1.$$

Step 2: T has a unique fixed point in X.

From step 1, we see at once that

$$\theta(x_n, x_{n+1}) \le [\theta(x_{n-1}, x_n)]^\alpha \le \ldots \le [\theta(x_0, x_1)]^{\alpha^n}, \quad \forall n \ge 1.$$

Analysis similar to that in the proof of Lemma 2.2 shows that

$$\lim_{n\to\infty} x_n = z.$$

From (2.4), we have

$$\begin{aligned} 1 &\le \theta(x_{n+1}, Tz) = \theta(Tx_n, Tz) \\ &\le [\theta(x_n, x_{n+1})]^{a_1} \times [\theta(z, Tz)]^{a_2} \times [\theta(x_n, Tz)]^{a_3} \times [\theta(z, x_{n+1})]^{a_4} \times [\theta(x_n, z)]^{a_5}. \end{aligned}$$

From the continuity of θ, we have (as $n \to \infty$)

$$\begin{aligned} &\theta(x_n, x_{n+1}), \theta(x_n, z) \to 1; \\ &\theta(x_{n+1}, Tz), \theta(x_n, Tz) \to \theta(z, Tz). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\theta(z, Tz) \le [\theta(z, Tz)]^{a_2+a_3},$$

which is impossible unless $\theta(z, Tz) = 1$. Therefore, z is a fixed point of T.

We proceed to show that the fixed point of T is unique. Assume that $x = Tx$ and $y = Ty$, in view of

$$\theta(x, Ty) \le \theta(x, y) \times \theta(y, Ty) \quad \text{and} \quad \theta(y, Tx) \le \theta(y, x) \times \theta(x, Tx),$$

we conclude from (2.4) that

$$\theta(x, y) = \theta(Tx, Ty) \le [\theta(x, y)]^{a_3+a_4+a_5},$$

hence that $\theta(x, y) = 1$ and finally $x = y$. □

2.2 Generalized Reich theorem for the sum of two operators

In this section, we will present a fixed point result for the sum of two mappings in a Banach space.

We first introduce the class Θ_E where E is a Banach space.

Definition 2.5. Let E be a Banach space and $\theta \in \Theta_E$.

(i) θ is said to be transitive if

$$\theta(x, y) = \theta(x - z, y - z), \quad \forall x, y, z \in E. \tag{2.6}$$

(ii) θ is said to be positive homogeneous if

$$\theta(\lambda x, \lambda y) = [\theta(x, y)]^\lambda, \quad \forall x, y \in E, \ \lambda \ge 0. \tag{2.7}$$

Lemma 2.6. *Let E be a Banach space and $\theta \in \Theta_E$. If θ is transitive, then for all $x, y, z_1, z_2 \in E$, we have*
a. $\theta(x, y) = \theta(-y, -x)$;
b. $\theta(x + z_1, y + z_2) \leq \theta(x, y) \times \theta(z_1, z_2)$.

Proof. From the transitiveness and Definition 2.1, we have

$$\theta(x, y) = \theta(x - y, 0_E) = \theta(-y - (-x), 0_E) = \theta(-y, -x).$$

And

$$\theta(x + z_1, y + z_2) = \theta(x - y, z_2 - z_1) \leq \theta(x - y, 0_E) \times \theta(0_E, z_2 - z_1) = \theta(x, y) \times \theta(z_1, z_2). \qquad \square$$

Example. Let $(E, \|.\|)$ be a Banach space and

$$\theta(x, y) := e^{d(x,y)}.$$

Then, $\theta \in \Theta_E$, θ is symmetric, transitive, and homogeneous.

In the opposite case, suppose that $\theta \in \Theta_X$. We define $|x|$ to be $\ln(\theta(x, 0_E))$ $\forall x \in E$. Then, $|x| > 0$ for all $x \neq 0_E$ and if θ is transitive, then the triangle inequality $|x + y| \leq |x| + |y|$ holds true. However, the absolute homogeneity is not guaranteed even when θ is positive homogeneous.

Theorem 2.7. *Let E be a Banach space and $\theta \in \Theta_E$. Assume that θ is transitive, symmetric, and complete in the sense of* (2.1).

Consider the mappings $A, B : M \subset E \to E$ such that for all $x, y, z \in M$:

(*i*) *There are nonnegative numbers a_i, $i = 1, \ldots, 5$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and*

$$\theta(Bx, By) \leq [\theta(Az, x - Bx)]^{a_1} \times [\theta(Az, y - By)]^{a_2} \times [\theta(Az, x - By)]^{a_3}[\theta(Az, y - Bx)]^{a_4} \times [\theta(x, y)]^{a_5}; \tag{2.8}$$

(*ii*) *There is a number $k \geq 0$ with $k < \frac{1 - a_3 - a_4 - a_5}{1 + a_2 - a_3}$ and*

$$\theta(Ax, Ay) \leq [\theta(x, y)]^k; \tag{2.9}$$

(*iii*) $A + B : M \to M$.

Then, the sum $A + B$ has a unique fixed point.

Proof. Fixed $z \in E$ and set $Hx = Az + Bx$, $x \in E$. We thus obtain from assumption (2.8) and the transitiveness of θ that

$$\begin{aligned}\theta(Hx, Hy) = \theta(Bx, By) &\leq [\theta(Hx - Bx, x - Bx)]^{a_1} \times [\theta(Hy - By, y - By)]^{a_2} \\ &\quad \times [\theta(Hy - By, x - By)]^{a_3} \times [\theta(Hx - Bx, y - Bx)]^{a_4} \times [\theta(x, y)]^{a_5} \\ &\leq [\theta(Hx, x)]^{a_1} \times [\theta(Hy, y)]^{a_2} \times [\theta(Hy, x)]^{a_3} \times [\theta(Hx, y)]^{a_4} \times [\theta(x, y)]^{a_5}.\end{aligned}$$

This and the properties of θ show that H satisfies (2.4). By Theorem 2.4, we denote by $G(z)$ the unique fixed point of H. This means

$$G(z) = Az + BG(z), \quad z \in E.$$

We proceed to show that G satisfies condition (2.2).

From (2.8), we have for $z_1, z_2 \in E$:

$$\begin{aligned}\theta(BG(z_1), BG(z_2)) &\leq [\theta(Az_1, G(z_1) - BG(z_1))]^{a_1} \times [\theta(Az_1, G(z_2) - BG(z_2))]^{a_2} \\ &\quad \times [\theta(Az_1, G(z_1) - BG(z_2))]^{a_3} \times [\theta(Az_1, G(z_2) - BG(z_1))]^{a_4} \times [\theta(G(z_1), G(z_2))]^{a_5}.\end{aligned}$$

We see at once that

$$\theta(Az_1, G(z_1) - BG(z_1)) = \theta(Az_1, Az_1) = 1,$$

$$\begin{aligned}\theta(Az_1, G(z_1) - BG(z_2)) &\leq \theta(Az_1, G(z_1) - BG(z_1)) \times \theta(G(z_1) - BG(z_1), G(z_1) - BG(z_2))\\ &= 1 \times \theta(BG(z_1), BG(z_2)),\\ \theta(Az_1, G(z_2) - BG(z_1)) &\leq \theta(Az_1, G(z_1) - BG(z_1)) \times \theta(G(z_1) - BG(z_1), G(z_2) - BG(z_1))\\ &= 1 \times \theta(G(z_1), G(z_2)).\end{aligned}$$

It follows that

$$[\theta(BG(z_1), BG(z_2))]^{1-a_3} \leq [\theta(G(z_1), G(z_2))]^{a_4+a_5} \times [\theta(Az_1, Az_2)]^{a_2}. \tag{2.10}$$

We now apply Lemma 2.6 to see that

$$\theta(Gz_1, Gz_2) = \theta(Az_1 + BG(z_1), Az_2 + BG(z_2)) \leq \theta(Az_1, Az_2) \times \theta(BG(z_1), BG(z_2)).$$

Combining this with (2.10) shows that

$$\theta(Gz_1, Gz_2) \leq \theta(Az_1, Az_2) \times [\theta(G(z_1), G(z_2))]^{\frac{a_4+a_5}{1-a_3}} \times [\theta(Az_1, Az_2)]^{\frac{a_2}{1-a_3}}.$$

Hence,

$$\theta(Gz_1, Gz_2)^{\left(1-\frac{a_4+a_5}{1-a_3}\right)} \leq [\theta(Az_1, Az_2)]^{\left(1+\frac{a_2}{1-a_3}\right)}.$$

Finally, (2.9) is applied and we thus obtain

$$\theta(Gz_1, Gz_2) \leq [\theta(z_1, z_2)]^q,$$

where

$$q < k\left(1 + \frac{a_2}{1 - a_3}\right)\left(1 - \frac{a_4 + a_5}{1 - a_3}\right)^{-1} < \left(1 + \frac{a_2}{1 - a_3}\right)\left(\frac{1 - a_3}{1 - a_3 - a_4 - a_5}\right)\left(\frac{1 - a_3 - a_4 - a_5}{1 + a_2 - a_3}\right) = 1.$$

Therefore, G has a unique fixed point which is denoted by z_0 from Lemma 2.2. We then obtain

$$z_0 = G(z_0) = Az_0 + BG(z_0) = Az_0 + Bz_0.$$

We finally note that if x is a fixed point of $A + B$, then x is a fixed point of G. This means that the uniqueness of the fixed point of $A + B$ is yielded from the uniqueness of the fixed point of G. □

2.3 Generalized Krasnoselskii theorem

Based on the idea of Kuratowski measure of non-compactness, we introduce the following definition.

Definition 2.8. Let E be a Banach space, $\theta \in \Theta_E$, θ is symmetric, and $\mathcal{M}$ be a family of non-empty bounded subsets of E. We will consider a measure of $\Omega \in \mathcal{M}$ defined as follows.

$$\phi_\theta(\Omega) = \inf\left\{\delta > 1 : \Omega \subset \bigcup_{i=1}^{n}\Omega_i,\ \sup_{x,y\in\Omega_i} \theta(x, y) < \delta \text{ for all } i = 1, 2, \dots, n\right\}.$$

Since θ is symmetric, the continuity of θ and boundedness of $\Omega \in \mathcal{M}$ show that the definition is well-posed. From now on we write the measure ϕ_θ as ϕ for convenience. Next let us mention some important properties of these concepts.

Proposition 2.9. *The measure ϕ has the following properties for any $\Omega, \Omega_1, \Omega_2 \in \mathcal{M}$*

(1) *If $x \in E$, we have*

$$\phi(\Omega \cup x) = \phi(\Omega).$$

(2) $\phi(\Omega_1 \cup \Omega_2) \leq \max\{\phi(\Omega_1), \phi(\Omega_2)\}$ *and if* $\Omega_1 \subset \Omega_2$, *then* $\phi(\Omega_1) \leq \phi(\Omega_2)$.

(3)

$$\phi(\overline{\Omega}) = \phi(\Omega).$$

(4) *Let $B(\delta)$ be the ball centered at 0_E with the radius δ, then*

$$\lim_{\delta\to 0^+} \phi(\overline{B}(\delta)) = 1.$$

(5) *If θ is transitive, then*

$$\phi(\Omega_1 + \Omega_2) \le \phi(\Omega_1) \times \phi(\Omega_2).$$

(6) *If θ is positive homogeneous, then for all $\lambda \ge 0$, we have*

$$\phi(\lambda\Omega) \le [\phi(\Omega)]^{\lambda}.$$

(7) *If θ is transitive and positive homogeneous, then*

$$\phi(\overline{\text{conv}}(\Omega)) = \phi(\Omega),$$

where $\overline{\text{conv}}(\Omega)$ stands for the closed convex hull of Ω.

(8) *If E, θ is complete in the sense of* (2.1) *and $\phi(\Omega) = 1$, then $\overline{\Omega}$ is compact.*

Proof. It is easy to check (1) and (2) by means of Definition 2.8, (3) by the continuity of θ, (5) by Lemma 2.6, and (6) by the positive homogeneity (2.7).

To prove (4) we suppose, contrary to our claim, that

$$\lim_{\delta\to 0^+} \phi(\overline{B}(\delta)) = r > 1.$$

Then, there exist $r' \in (1, r)$ such that for all $n \in \mathbb{N}$ large enough, we find $x_n \in B(1/n)$ such that $\theta(x_n, 0_E) \ge r'$. This contradicts the first property in Definition 2.1.

By Definition 2.8, to prove (7), it is sufficient to show that

$$\phi(conv(\Omega_1 \cup \Omega_2)) \le \max\{\phi(\Omega_1), \phi(\Omega_2)\}, \tag{2.11}$$

where Ω_1, Ω_2 are convex and bounded.

Given $\delta > 0$, since $\Omega_1 - \Omega_2$ is bounded, we can find $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$conv(\Omega_1 \cup \Omega_2) \subset \bigcup_{0\le\lambda\le 1} (\lambda\Omega_1 + (1-\lambda)\Omega_2) \subset \bigcup_{i=1}^{m} (\lambda_i\Omega_1 + (1-\lambda_i)\Omega_2 + \overline{B}(\delta)),$$

where $B(\delta)$ is the ball centered at 0_E with radius δ.

From (2), (5), and (6) we find $k \in \{1, 2, \ldots, m\}$ such that

$$\phi(\text{conv}(\Omega_1 \cup \Omega_2))\phi(\lambda_i\Omega_1 + (1-\lambda_i)\Omega_2 + \overline{B}(\delta)) \le [\phi(\Omega_1)]^{\lambda_k}[\phi(\Omega_2)]^{1-\lambda_k}\phi(\overline{B}(\delta)) \le \max\{\phi(\Omega_1), \phi(\Omega_2)\}\phi(\overline{B}(\delta)).$$

From this and (4), we thus obtain (2.11).

We are now in a position to show the last property. Let $\{x_n\}_n$ be a sequence in Ω. From Definition 2.8, we can construct, by the induction method, sequences $\{x_n^k\}_n$, $k \in \mathbb{N}^+$ such that

$$\begin{aligned} &\{x_n^1\}_n \text{ is a subsequence of } \{x_n\}, \\ &\{x_n^{k+1}\}_n \text{ is a subsequence of } \{x_n^k\}, \quad k \ge 1, \\ &\theta(x_n^k, x_m^k) \le 1 + \frac{1}{k}, \quad k \ge 1. \end{aligned} \tag{2.12}$$

It follows that $\lim_{m>n\to\infty}\theta(x_n^n, x_m^m) = 1$, hence $\{x_n^n\}_n$ is convergent, and finally $\overline{\Omega}$ is compact. □

Definition 2.10. Let $D \in \mathcal{M}$. An operator $F : D \to X$ is said to be condensing with respect to ϕ_θ (or ϕ-condensing for short) if the followings hold true:

(i) $F(D) \in \mathcal{M}$;

(ii) for every $\Omega \subset D$, if $\phi(\Omega) > 1$, then

$$\phi(F(\Omega)) < \phi(\Omega).$$

Theorem 2.11. *Let $D \in \mathcal{M}$. If the operator $F : D \to D$ is condensing with respect to $\theta \in \Theta_E$ where θ is symmetric, transitive, positive homogeneous, and complete in the sense of* (2.1).

Then, F has a fixed point in D.

Proof. Let us choose a point $x_0 \in \overline{\text{conv}}(F(D))$ and denote by Σ the class of all closed and convex subsets Ω of D such that $x_0 \in \Omega$ and $F(\Omega) \subset \Omega$. Also, set

$$C = \bigcap_{\Omega \in \Sigma} \Omega, \quad K = \overline{\text{conv}}(F(C) \cup \{x_0\}).$$

Obviously, $\overline{\text{conv}}(F(D)) \in \Sigma$. Furthermore, from $F(\Omega) \subset \Omega$, $\forall \Omega \in \Sigma$, it follows that $F(C) \subset C$. We now claim that $C = K$. Indeed, since $x_0 \in C$ and $F(C) \subset C$, it follows that $K \subset C$. This implies $F(K) \subset F(C) \subset K$, hence $K \in \Sigma$, and finally $C \subset K$.

Therefore,

$$\phi(C) = \phi(K) = \phi(F(C) \cup \{u\}) = \phi(F(C)).$$

Since F is ϕ-condensing, it follows that $\phi(C) = 1$ and that C is compact. Thus, from the Schauder theorem, we conclude that there is a fixed point for the operator $F : C \to C$. □

Theorem 2.12. *Let E be a Banach space, M be a bounded subset of E, and $\theta \in \Theta_E$. Assume that θ is transitive, symmetric, positive homogeneous, and complete in the sense of* (2.1).

Consider the mappings $A, B : M \to M$ such that for all $x, y, z \in M$,

(*i*) *There are nonnegative numbers a_i, $i = 1, \ldots, 5$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and*

$$\theta(Bx, By) \leq [\theta(Az, x - Bx)]^{a_1} \times [\theta(Az, y - By)]^{a_2} \times [\theta(Az, x - By)]^{a_3} \times [\theta(Az, y - Bx)]^{a_4} \times [\theta(x, y)]^{a_5}; \tag{2.13}$$

(*ii*) *There is a number $k \geq 0$ with $k < \frac{1 - a_3 - a_4 - a_5}{1 + a_2 - a_3}$ and A is k-set contractive mapping in the sense of*

$$\phi(A(\Omega)) \leq [\phi(\Omega)]^k, \quad \forall \Omega \subset M. \tag{2.14}$$

(*iii*) $A + B : M \to M$.

Then, the sum $A + B$ has a fixed point in M.

Proof. As in the proof of theorem (2.7), the mapping $G : M \to M$ defined by

$$G(z) = A(z) + B(G(z)), \quad z \in M$$

satisfies

$$\theta(Gz_1, Gz_2)^{\left(1 - \frac{a_4 + a_5}{1 - a_3}\right)} \leq [\theta(Az_1, Az_2)]^{\left(1 + \frac{a_2}{1 - a_3}\right)}, \quad \forall z_1, z_2 \in M.$$

According to Definition 2.8, we have

$$\phi(G(\Omega)) \leq [\phi(A(\Omega))]^p, \quad \forall \Omega \subset M,$$

where $p = \left(1 + \frac{a_2}{1 - a_3}\right)\left(1 - \frac{a_4 + a_5}{1 - a_3}\right)^{-1}$.

For all $\Omega \subset M$, by (2.14), we have

$$\phi(G(\Omega)) \leq [\phi(\Omega)]^{kp}.$$

Since by assumption *(ii)* we have $kp < 1$, hence if $\phi(\Omega) > 1$, then

$$\phi(G(\Omega)) \leq \phi(\Omega).$$

Therefore, G is ϕ-condensing and we conclude from Theorem 2.11 that G has a fixed point, which is denoted by z_0. We then obtain

$$z_0 = G(z_0) = Az_0 + BG(z_0) = Az_0 + Bz_0.$$

The proof is complete. □

2.4 Example

As an application of our results, we consider the following non-linear integral equation:

$$x(t) = \int_0^t f(t, s, x(s))\mathrm{d}s + \frac{c(t) - a}{1 - a}x(t), \quad t \in [0, 1], \tag{2.15}$$

where $a \in [0, 1]$, $f : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$, and $c : [0, 1] \to \mathbb{R}$ are continuous. Let $E = C([0, 1])$ be the Banach space of all continuous functions defined on $[0, 1]$ with $\|x\| = \max\{|x(t)| : t \in [0, 1]\}$.

We will define $\theta(x, y) = e^{\|x-y\|}$ for all $x, y \in E$, hence we first prove the conditions in Theorem 2.12 for the norm $\|.\|$.

The above integral equation has the form $x = Ax + Bx$, where the operators $A, B : E \to E$ defined by

$$Ax(t) = \int_0^t f(t, s, x(s))\mathrm{d}s$$

and

$$Bx(t) = \frac{c(t) - a}{1 - a}x(t).$$

This equation will be studied under the following assumptions:

(A1) For all $s \in [0, 1]$, $x \in \mathbb{R}$, the function $f(\cdot, s, x)$ is Lipschitz on $[0, 1]$. This means that there is a constant T such that

$$|f(t_1, s, x) - f(t_2, s, y)| \leq T|t_1 - t_2|, \ \ \forall t_1, t_2 \in [0, 1];$$

(A2) There is a constant $k \geq 0$ such that for all $0 \leq s \leq t \leq 1$ and $x, y \in \mathbb{R}$ then

$$|f(t, s, x) - f(t, s, y)| \leq k|x - y|;$$

(A3) $2a + c < 1$ and $k + \frac{c+a}{1-a} < 1$ where $c = \max\{|c(t)| : t \in [0, 1]\}$.

Theorem 2.13. *If the assumptions* (A1), (A2), *and* (A3) *hold, then equation* (2.15) *has a solution in* E.

Proof. Let $\mathbb{B}(R)$ be the ball centered at 0_E with radius R. We first prove that there is a number R such that $A + B : \mathbb{B}(R) \to \mathbb{B}(R)$.

Set $M = \max\{|f(t, s, 0)| : 0 \leq s \leq t \leq 1\}$, then for all $t \in [0, 1]$ and $x \in \mathbb{B}(R)$, we see that

$$|Ax(t) + Bx(t)| \leq \int_0^t |f(t, s, x(s)) - f(t, s, 0)|\mathrm{d}s + \int_0^t |f(t, s, 0)|\mathrm{d}s$$

$$\frac{|c(t)| + a}{1 - a}|x(t)| \leq k\int_0^t |x(s)|\mathrm{d}s + M + \frac{c + a}{1 - a}|x(t)| \leq \left[k + \frac{c + a}{1 - a}\right]R + M.$$

We conclude from (A3) that $|Ax(t) + Bx(t)| \leq R$, where R is large enough.

We next claim *(i)* of Theorem 2.12. For all $x, y, z \in \mathbb{B}(R)$, we see that

$$(1 - a)[Bx(t) - By(t)] = [c(t) - a]x(t) - [c(t) - a]y(t).$$

This gives

$$\begin{aligned} |Bx(t) - By(t)| &\leq |aBx(t) - ax(t) + Az(t)| + |ay(t) - aBy(t) - Az(t)| + |c(t)||x(t) - y(t)| \\ &\leq a|Az(t) - (I - B)x(t)| + a|Az(t) - (I - B)y(t)| + c|x(t) - y(t)| \\ &\leq a\|Az - (I - B)x\| + a\|Az - (I - B)y\| + c\|x - y\|. \end{aligned}$$

On account of above remark, $\theta(x, y) = e^{\|x-y\|}$, we thus obtain *(i)* of Theorem 2.12, where $a_1 = a_2 = a$ and $a_5 = c$.

We proceed to show that $A(\mathbb{B}(R))$ is relative compact in E. The results is $\phi(A(\mathbb{B}(R))) = 1$ and *(ii)* of Theorem 2.12 is proved. Indeed, the set

$$A(\mathbb{B}(R))(t) = \left\{ \int_0^t f(t,s,x(s))\mathrm{d}s | x \in \mathbb{B}(R) \right\} \subset \left\{ \int_0^t f(t,s,x)\mathrm{d}s||x| \le R \right\}.$$

It follows that $A(\mathbb{B}(R))(t)$ is bounded in $\mathbb{R}$, hence it is relative compact in $\mathbb{R}$ for all $t \in [0,1]$.

Consider $x \in \mathbb{B}(R)$ and $0 \le t_1 \le t_2 \le 1$, we see that

$$|Ax(t_2) - Ax(t_1)| \le \int_0^{t_1} |f(t_2,s,x(s)) - f(t_1,s,x(s))|\mathrm{d}s + \int_{t_1}^{t_2} |f(t_2,s,x(s))|\mathrm{d}s \le T|t_1 - t_2| + K|t_1 - t_2|,$$

where $K = \max\{|f(t,s,x)| 0 \le s \le t \le 1, |x| < R\}$.

Therefore, $A(\mathbb{B}(R))$ is equicontinuous and by the Azèla-Ascoli theorem, we conclude that $A(\mathbb{B}(R))$ is relative compact in E.

The proof is completed by Theorem 2.12. □

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