Research Article

Xinyu Zhuang, Ruilin Zheng, Zhiying He, and Mingliang Fang*

Some results on value distribution concerning Hayman's alternative

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Abstract: In this article, we study the value distribution of meromorphic functions concerning Hayman's alternative. We extend and improve some results due to Zhu [*A far-reaching form of Hayman's inequality; fixed points of meromorphic functions and their derivatives*, Kexue Tongbao (Chinese) **31** (1986), no. 11, 801–804], Hua and Chuang [*On a conjecture of Hayman*, Acta Math. Sinica (N.S.) **7** (1991), no. 2, 119–126], and Charak and Singh [*A value distribution result related to Hayman's alternative*, Commun. Korean Math. Soc. **34** (2019), no. 2, 495–506].

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1 Introduction and main results

In this article, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory [1–4]. In the following, a meromorphic function always means meromorphic in the whole complex plane. By $S(r,f)$, we denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possible outside of an exceptional set E with finite measure.

Let f be a meromorphic function, and let k be a positive integer. We denote by $N_{k)}(r,f)$ the counting function for poles of f with multiplicity $\le k$, counting multiplicity and by $\overline{N}_{k)}(r,f)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,f)$ be the counting function for poles of f with multiplicity $\ge k$, counting multiplicity and by $\overline{N}_{(k}(r,f)$ be the corresponding one for which multiplicity is not counted.

Let f be a nonconstant meromorphic function, let $a_1, a_2, \ldots, a_k$ be small functions of f, and let $n_0, n_1, n_2, \ldots, n_k$ be nonnegative integers. We define the differential monomial of f as follows:

$$M(f) = f^{n_0}(f')^{n_1} \ldots (f^{(k)})^{n_k},$$

and its degree $\gamma_M = n_0 + n_1 + \ldots + n_k$. Let $M_1(f), M_2(f), \ldots, M_n(f)$ be differential monomials. We define the differential polynomial of f as follows:

$$H(f) = a_1M_1(f) + a_2M_2(f) + \ldots + a_nM_n(f), \tag{1.1}$$

and define $\gamma_H = \max\{\gamma_{M_1}, \gamma_{M_2}, \ldots, \gamma_{M_n}\}$, $\underline{\gamma}_H = \min\{\gamma_{M_1}, \gamma_{M_2}, \ldots, \gamma_{M_n}\}$ by the degree and the lower degree of H, respectively.

* **Corresponding author: Mingliang Fang,** Department of Engineering, Shenzhen MSU-BIT University, Shenzhen 518172, China, e-mail: mlfang@hdu.edu.cn
Xinyu Zhuang: Department of Engineering, Shenzhen MSU-BIT University, Shenzhen 518172, China, e-mail: zhuangxyu@163.com
Ruilin Zheng: Department of Mathematics and Informatics, South China Agricultural University, Guangzhou, 510642, China, e-mail: 747684117@qq.com
Zhiying He: Department of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, China, e-mail: zhiyinggood@163.com

In 1986, Zhu [5] proved the following result.

Theorem 1.1. *Let f be a transcendental meromorphic function, and let φ ($\not\equiv 0$) be a small function of f. Then*

$$T(r,f) \le 8N\left(r,\frac{1}{f}\right) + 8\overline{N}\left(r,\frac{1}{f'-\varphi}\right) + S(r,f).$$

In this article, we improve Theorem 1.1 as follows.

Theorem 1.2. *Let f be a transcendental meromorphic function, and let φ ($\not\equiv 0$) be a small function of f. Then*

$$T(r,f) \le 5N\left(r,\frac{1}{f}\right) + 5\overline{N}\left(r,\frac{1}{f'-\varphi}\right) + S(r,f).$$

In 1991, Hua and Chuang [6] proved the following result.

Theorem 1.3. *Let f be a nonconstant meromorphic function, and let n, m be two positive integers. Assume that $Q(f) = f^m H(f)$, where $H(f)$ ($\not\equiv 0$) is a differential polynomial defined by* (1.1). *Then for any nonzero complex number b*:

(1) *If $n \ge 3$, then* $T(r,f) \le \frac{2}{m}\overline{N}\left(r,\frac{1}{Q^nQ'-b}\right) + S(r,f)$;

(2) *If $n = 2$, then* $T(r,f) \le \frac{1}{m}\overline{N}(r,f) + \frac{2}{m}\overline{N}\left(r,\frac{1}{Q^2Q'-b}\right) + S(r,f)$.

In this article, we prove the following result.

Theorem 1.4. *Let f be a nonconstant meromorphic function, and let n, m be two positive integers. Assume that $Q(f) = f^m H(f)$, where $H(f)(\not\equiv 0)$ is a differential polynomial defined by* (1.1). *Then for any nonzero complex number b,*

(1) *If $n \ge 3$, then* $T(r,f) \le \frac{1}{m}\overline{N}\left(r,\frac{1}{Q^nQ'-b}\right) + S(r,f)$;

(2) *If $n = 2$, then* $T(r,f) \le \frac{1}{m}\overline{N}(r,f) + \frac{1}{m}\overline{N}\left(r,\frac{1}{Q^2Q'-b}\right) + S(r,f)$.

In 2019, Charak and Singh [7] proved the following result.

Theorem 1.5. *Let f be a transcendental meromorphic function, let ϕ be a small function of f such that f and ϕ have no common poles, and let k be a positive integer. If $f \ne 0$ and $f^{(k)} \ne \phi$, then $f^{(k+1)} = \phi$ and $f^{(k+1)} = \phi'$ have infinitely many solutions.*

Remark 1.6. Theorem 1.5 is not valid by the following example.

Example 1.7. Let $f = e^z$, and let $\phi \equiv 0$. Obviously, $f \ne 0$ and $f^{(k)} \ne \phi$, but $f^{(k+1)} = \phi$ does not have infinitely many solutions.

Although Theorem 1.5 is not valid, we have the following result.

Theorem 1.8. *Let f be a transcendental meromorphic function, let ϕ be a small function of f, and let k be a positive integer. If $f \ne 0$ and $f^{(k)} \ne \phi$, then $\phi \equiv 0$. In addition, if $k \ge 2$, then $f = e^{az+b}$, where a ($\ne 0$), b are constants; if $k = 1$, then f'' has infinitely many zeros, except $f = e^{az+b}$, where a ($\ne 0$), b are constants.*

Remark 1.9. $f \ne 0$ refers to the fact that for any $z \in \mathbb{C}$, it holds that $f(z) \ne 0$. $f^{(k)} \ne \phi$ refers to the fact that for any $z \in \mathbb{C}$, it holds that $f^{(k)}(z) \ne \phi(z)$.

The following examples show that two cases occur in Theorem 1.8.

Example 1.10. Let $f(z) = e^z$, and let k be a positive integer. Obviously, $f \neq 0, f^{(k)} \neq 0$.

Example 1.11. Let $f(z) = e^{e^z}$. Obviously, $f \neq 0, f' \neq 0$. We have $f'' = e^{e^z} e^z(e^z + 1)$. Thus, f'' has infinitely many zeros.

2 Some lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1. [3] *Let f be a nonconstant meromorphic function, and let k be a positive integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

Lemma 2.2. [1,3] *Let f be a nonconstant meromorphic function, let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be distinct small functions of f. Then*

$$m\left(r, \frac{1}{f - a_1}\right) + \ldots + m\left(r, \frac{1}{f - a_n}\right) \leq m\left(r, \frac{1}{f - a_1} + \ldots + \frac{1}{f - a_n}\right) + S(r, f).$$

It follows from the theorem in [8, p. 247] the following result.

Lemma 2.3. *Let f be a transcendental meromorphic function. Then*

$$\frac{1}{4} \leq \frac{T(r, f)}{T(r, f')} \leq 3e + \frac{1}{4},$$

as $r \to \infty$ on a set of positive lower logarithmic density.

Lemma 2.4. [1,3] *Let f_1 and f_2 be two nonconstant meromorphic functions. Then*

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right).$$

Lemma 2.5. *Let f be a nonconstant meromorphic function, let b be a nonzero complex number, and let n be a positive integer. Then we have*

(1) $n \geq 3$, $T(r, f) \leq \overline{N}\left(r, \frac{1}{f^n f' - b}\right) + S(r, f)$;

(2) $n = 2$, $T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^2 f' - b}\right) + S(r, f)$.

Proof. By Lemmas 2.1 and 2.2 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} m\left(r, \frac{1}{f^{n+1}}\right) + m\left(r, \frac{1}{f^n f' - b}\right) &\leq m\left(r, \frac{f^n f'}{f^{n+1}}\right) + m\left(r, \frac{1}{f^n f'}\right) + m\left(r, \frac{1}{f^n f' - b}\right) \\ &\leq m\left(r, \frac{1}{f^n f'} + \frac{1}{f^n f' - b}\right) + S(r, f) \qquad (2.1) \\ &\leq m\left(r, \frac{(f^n f')'}{f^n f'} + \frac{(f^n f' - b)'}{f^n f' - b}\right) + m\left(r, \frac{1}{(f^n f')'}\right) + S(r, f) \end{aligned}$$

$$\leq T(r,(f^nf')') - N\left(r,\frac{1}{(f^nf')'}\right) + S(r,f)$$
$$\leq T(r,f^nf') + \overline{N}(r,f) - N\left(r,\frac{1}{(f^nf')'}\right) + S(r,f).$$

Then, by adding $N\left(r,\frac{1}{f^{n+1}}\right) + N\left(r,\frac{1}{f^nf'-b}\right)$ to both sides of (2.1), we have

$$m\left(r,\frac{1}{f^{n+1}}\right) + m\left(r,\frac{1}{f^nf'-b}\right) + N\left(r,\frac{1}{f^{n+1}}\right) + N\left(r,\frac{1}{f^nf'-b}\right)$$
$$\leq N\left(r,\frac{1}{f^{n+1}}\right) + N\left(r,\frac{1}{f^nf'-b}\right) + T(r,f^nf') + \overline{N}(r,f) - N\left(r,\frac{1}{(f^nf')'}\right) + S(r,f).$$

It follows

$$T\left(r,\frac{1}{f^{n+1}}\right) + T\left(r,\frac{1}{f^nf'-b}\right) \leq N\left(r,\frac{1}{f^{n+1}}\right) + N\left(r,\frac{1}{f^nf'-b}\right) + T(r,f^nf') + \overline{N}(r,f) - N\left(r,\frac{1}{(f^nf')'}\right) + S(r,f).$$

By Nevanlinna's first fundamental theorem, we have

$$T\left(r,\frac{1}{f^nf'-b}\right) = T(r,f^nf') + O(1).$$

Thus, we obtain

$$T\left(r,\frac{1}{f^{n+1}}\right) \leq N\left(r,\frac{1}{f^{n+1}}\right) + N\left(r,\frac{1}{f^nf'-b}\right) + \overline{N}(r,f) - N\left(r,\frac{1}{(f^nf')'}\right) + S(r,f).$$

From $T\left(r,\frac{1}{f^{n+1}}\right) = (n+1)T(r,f) + S(r,f)$, we obtain

$$(n+1)T(r,f) \leq N\left(r,\frac{1}{f^{n+1}}\right) + N\left(r,\frac{1}{f^nf'-b}\right) + \overline{N}(r,f) - N\left(r,\frac{1}{(f^nf')'}\right) + S(r,f).$$

If z_0 is a zero of f with multiplicity l_1, then z_0 is a zero of f^{n+1} with multiplicity $(n+1)l_1$. Hence, z_0 must be a zero of $(f^nf')'$ with multiplicity $(n+1)l_1 - 2$. Similarly, if z_0 is a zero of $f^nf'-b$ with multiplicity l_2, then z_0 must be a zero of $(f^nf'-b)'$ with multiplicity $l_2 - 1$, which yields z_0 is also a zero of $(f^nf')'$ with multiplicity $l_2 - 1$. It follows

$$N\left(r,\frac{1}{f^{n+1}}\right) + N\left(r,\frac{1}{f^nf'-b}\right) - N\left(r,\frac{1}{(f^nf')'}\right) \leq 2\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^nf'-b}\right) - N_0^*\left(r,\frac{1}{(f^nf')'}\right),$$

where $N_0^*\left(r,\frac{1}{(f^nf')'}\right)$ is the counting function for the zeros of $(f^nf')'$, which are not zeros of $f^{n+1}(f^nf'-b)$.

Hence, we have

$$(n+1)T(r,f) \leq N\left(r,\frac{1}{f^{n+1}}\right) + N\left(r,\frac{1}{f^nf'-b}\right) + \overline{N}(r,f) - N\left(r,\frac{1}{(f^nf')'}\right) + S(r,f)$$
$$\leq \overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^nf'-b}\right) - N_0^*\left(r,\frac{1}{(f^nf')'}\right) + S(r,f) \tag{2.2}$$
$$\leq \overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^nf'-b}\right) + S(r,f).$$

Next we consider two cases.

Case 1. $n \geq 3$. From (2.2), we obtain

$$(n-2)T(r,f) \leq \overline{N}\left(r, \frac{1}{f^n f' - b}\right) + S(r,f).$$

It follows

$$T(r,f) \leq \overline{N}\left(r, \frac{1}{f^n f' - b}\right) + S(r,f). \tag{2.3}$$

Case 2. $n = 2$. From (2.2), we have

$$T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f^2 f' - b}\right) + S(r,f). \tag{2.4}$$

□

Lemma 2.6. [6] *Let f be a nonconstant meromorphic function, and let k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.7. [9] *Let f be a transcendental meromorphic function, let ϕ ($\not\equiv 0$) be a small function of f, and let k be a positive integer. If $N\left(r, \frac{1}{f}\right) = S(r,f)$, then $N\left(r, \frac{1}{f^{(k)} - \phi}\right) \neq S(r,f)$.*

Lemma 2.8. [10] *Let f be a nonconstant meromorphic function, and let k (≥ 2) be an integer. If $f \neq 0, f^{(k)} \neq 0$, then either $f = e^{az+b}$ or $f = (az+b)^{-n}$, where a ($\neq 0$), b are constants and n is a positive integer.*

Remark 2.9. $f \neq 0$ refer to the fact that for any $z \in \mathbb{C}$, it holds that $f(z) \neq 0$. $f^{(k)} \neq 0$ refer to the fact that for any $z \in \mathbb{C}$, it holds that $f^{(k)}(z) \neq 0$.

It follows from Theorem 2.2 in [11, p. 423] the following result.

Lemma 2.10. *Let f be a nonconstant meromorphic function, and let k (≥ 2) be an integer. If f and $f^{(k)}$ have finitely many zeros, then $f = Re^P$, where R is a rational function and P is a polynomial.*

3 Proof of Theorem 1.2

Set $L = \begin{vmatrix} f' & \varphi \\ f'' & \varphi' \end{vmatrix}$. By Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f' - \varphi}\right) &\leq m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{1}{f'}\right) + m\left(r, \frac{1}{f' - \varphi}\right) \\ &\leq m\left(r, \frac{1}{f'} + \frac{1}{f' - \varphi}\right) + S(r,f) \\ &\leq m\left(r, \frac{L}{f'} + \frac{L}{f' - \varphi}\right) + m\left(r, \frac{1}{L}\right) + S(r,f) \\ &\leq m\left(r, \frac{L}{f'}\right) + m\left(r, \frac{L}{f' - \varphi}\right) + m\left(r, \frac{1}{L}\right) + S(r,f). \end{aligned} \tag{3.1}$$

From the definition of L, $\varphi(\not\equiv 0)$ is a small function of f, and from Lemma 2.1, we have

$$m\left(r, \frac{L}{f'}\right) = m\left(r, \varphi' - \varphi\frac{f''}{f'}\right) \le m(r, \varphi') + m(r, \varphi) + m\left(r, \frac{f''}{f'}\right) + S(r, f) \le S(r, f) \tag{3.2}$$

and

$$m\left(r, \frac{L}{f' - \varphi}\right) = m\left(r, \varphi' - \varphi\frac{(f' - \varphi)'}{f' - \varphi}\right) \le m(r, \varphi') + m(r, \varphi) + m\left(r, \frac{(f' - \varphi)'}{f' - \varphi}\right) + S(r, f) \le S(r, f). \tag{3.3}$$

By combining (3.1)–(3.3) and Nevanlinna's first fundamental theorem, we obtain

$$\begin{aligned} m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f' - \varphi}\right) &\le m\left(r, \frac{1}{L}\right) + S(r, f) \\ &\le T\left(r, \frac{1}{L}\right) - N\left(r, \frac{1}{L}\right) + S(r, f) \\ &\le T(r, L) - N\left(r, \frac{1}{L}\right) + S(r, f). \end{aligned} \tag{3.4}$$

From the definition of L and (3.3), we have

$$\begin{aligned} m(r, L) &= m\left(r, (f' - \varphi)\left(\varphi' - \varphi\frac{f'' - \varphi'}{f' - \varphi}\right)\right) \\ &\le m(r, f' - \varphi) + m\left(r, \varphi' - \varphi\frac{f'' - \varphi'}{f' - \varphi}\right) + S(r, f) \\ &\le m(r, f' - \varphi) + S(r, f). \end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned} N(r, L) &= N(r, (f' - \varphi)\varphi' - (f'' - \varphi')\varphi) \\ &\le N(r, (f' - \varphi)') + S(r, f) \\ &\le N(r, f' - \varphi) + \overline{N}(r, f' - \varphi) + S(r, f) \\ &\le N(r, f' - \varphi) + \overline{N}(r, f) + S(r, f). \end{aligned} \tag{3.6}$$

By combining (3.4)–(3.6), we obtain

$$\begin{aligned} m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f' - \varphi}\right) &\le T(r, L) - N\left(r, \frac{1}{L}\right) + S(r, f) \\ &\le m(r, L) + N(r, L) - N\left(r, \frac{1}{L}\right) + S(r, f) \\ &\le m(r, f' - \varphi) + N(r, f' - \varphi) + \overline{N}(r, f) - N\left(r, \frac{1}{L}\right) + S(r, f) \\ &\le T(r, f' - \varphi) + \overline{N}(r, f) - N\left(r, \frac{1}{L}\right) + S(r, f). \end{aligned} \tag{3.7}$$

By adding $N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f' - \varphi}\right)$ to both sides of (3.7), we have

$$\begin{aligned} &m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f' - \varphi}\right) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f' - \varphi}\right) \\ &\le N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f' - \varphi}\right) + T(r, f' - \varphi) + \overline{N}(r, f) - N\left(r, \frac{1}{L}\right) + S(r, f). \end{aligned}$$

Hence, we obtain

$$T\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{f' - \varphi}\right) \le N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f' - \varphi}\right) + T(r, f' - \varphi) + \overline{N}(r, f) - N\left(r, \frac{1}{L}\right) + S(r, f).$$

It follows from Nevanlinna's first fundamental theorem that

$$\begin{aligned} T\left(r, \frac{1}{f}\right) &= T(r, f) + O(1), \\ T\left(r, \frac{1}{f' - \varphi}\right) &= T(r, f' - \varphi) + O(1). \end{aligned}$$

Therefore,

$$\begin{aligned} T(r, f) &\le N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f' - \varphi}\right) + \overline{N}(r, f) - N\left(r, \frac{1}{L}\right) + S(r, f) \\ &\le \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f' - \varphi}\right) + S(r, f). \end{aligned} \tag{3.8}$$

It follows from (3.8) and $\overline{N}(r, f) = \overline{N}_{1)}(r, f) + \overline{N}_{(2}(r, f)$ that

$$\begin{aligned} T(r, f) &\le \overline{N}_{1)}(r, f) + \overline{N}_{(2}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f' - \varphi}\right) + S(r, f) \\ &\le N_{1)}(r, f) + \frac{1}{2}N_{(2}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f' - \varphi}\right) + S(r, f) \\ &\le \frac{1}{2}N_{1)}(r, f) + \frac{1}{2}T(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f' - \varphi}\right) + S(r, f). \end{aligned}$$

Hence, we obtain

$$T(r, f) \le N_{1)}(r, f) + 2N\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{f' - \varphi}\right) + S(r, f). \tag{3.9}$$

Obviously, $f''\varphi - \varphi' f' \not\equiv 0$. Set

$$P = \frac{f(f' - \varphi)}{f''\varphi - \varphi' f'}, \quad G = (P\varphi)'' + (P\varphi')' + \varphi.$$

Thus, we have

$$P\varphi f'' - f(f' - \varphi) \equiv P\varphi' f'. \tag{3.10}$$

Let z_0 be a simple pole of f, and satisfy that $\varphi(z_0) \neq 0, \infty, \varphi'(z_0) \neq 0$. Obviously, $P\varphi$ and $P\varphi'$ are holomorphic at z_0, so we obtain

$$f(z) = \frac{c_{-1}}{z - z_0} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots, \tag{3.11}$$

where c_{-1} ($\neq 0$), $c_0, c_1, c_2, \ldots$ are constants.

$$\varphi(z) = \varphi(z_0) + \varphi'(z_0)(z - z_0) + \frac{1}{2}\varphi''(z_0)(z - z_0)^2 + \ldots, \tag{3.12}$$

$$P\varphi(z) = P\varphi(z_0) + (P\varphi)'(z_0)(z - z_0) + \frac{1}{2}(P\varphi)''(z_0)(z - z_0)^2 + \ldots, \tag{3.13}$$

$$P\varphi'(z) = P\varphi'(z_0) + (P\varphi')'(z_0)(z - z_0) + \frac{1}{2}(P\varphi')''(z_0)(z - z_0)^2 + \ldots. \tag{3.14}$$

By substituting (3.11)–(3.14) into (3.10), and comparing the coefficients of $\frac{1}{z-z_0}$, we obtain

$$(P\varphi)''(z_0) + (P\varphi')'(z_0) + \varphi(z_0) = 0.$$

That is, $G(z_0) = 0$.

Next we consider the following two cases.

Case 1. $G \not\equiv 0$. Since φ is a small function of f, then by Nevanlinna's first fundamental theorem and Lemma 2.1, we have

$$\begin{aligned} N_{1)}(r,f) &\le N\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{\varphi}\right) + \overline{N}\left(r,\frac{1}{\varphi'}\right) + \overline{N}(r,\varphi) \\ &\le N\left(r,\frac{1}{G}\right) + S(r,f) \\ &\le T(r,G) + S(r,f) \\ &\le m(r,P''\varphi + 3P'\varphi' + 2P\varphi'') + m(r,\varphi) + N(r,P''\varphi + 3P'\varphi' + 2P\varphi'') + N(r,\varphi) + S(r,f) \\ &\le m(r,P) + m\left(r,\frac{P''\varphi + 3P'\varphi' + 2P\varphi''}{P}\right) + N(r,P'') + S(r,f) \\ &\le m(r,P) + N(r,P) + 2\overline{N}(r,P) + S(r,f) \\ &\le 3T(r,P) + S(r,f). \end{aligned} \tag{3.15}$$

From Lemma 2.1, we have

$$\begin{aligned} m\left(r,\frac{1}{P}\right) &= m\left(r,\frac{f''\varphi - \varphi'f'}{f(f'-\varphi)}\right) \\ &\le m\left(r,\frac{f''\varphi - \varphi'f'}{f'(f'-\varphi)}\right) + m\left(r,\frac{f'}{f}\right) + S(r,f) \\ &= m\left(r,\frac{\left(\frac{f'-\varphi}{f'}\right)'}{\frac{f'-\varphi}{f'}}\right) + m\left(r,\frac{f'}{f}\right) + S(r,f) \le S(r,f). \end{aligned} \tag{3.16}$$

From the definition of P, we have

$$N\left(r,\frac{1}{P}\right) = N\left(r,\frac{f''\varphi - \varphi'f'}{f(f'-\varphi)}\right).$$

Since φ $(\not\equiv 0)$ is a small function of f, we have $T(r,\varphi) = S(r,f)$, it follows $S(r,\varphi) = S(r,f)$. Hence, we obtain $T(r,\varphi') = S(r,f)$.

Next we consider the following three subcases.

Case 1.1. Let z_0 be a pole of f with multiplicity l_1 (≥ 1), but $\varphi(z_0) \ne \infty$. It follows that z_0 is a pole of $f' - \varphi$ and $f''\varphi - \varphi'f'$ with multiplicity $l_1 + 1$ and $l_1 + 2$, respectively. In this case, the multiplicity of the poles in the numerator is not greater than that in the denominator. Hence, z_0 is not a pole of $\frac{1}{P}$.

Case 1.2. Let z_0 be a pole of f with multiplicity l_1 (≥ 1) and $\varphi(z_0) = \infty$. From $N(r,\varphi) = S(r,f)$, we can deduce that the zeros of this type of P is $S(r,f)$.

Case 1.3. Let z_0 be a zero of $f' - \varphi$ with multiplicity l_2 (≥ 1), but not the zero of f. It follows that z_0 is zero of $f'' - \varphi'$ with multiplicity $l_2 - 1$. From

$$f''\varphi - \varphi'f' = f''\varphi - \varphi\varphi' + \varphi\varphi' - \varphi'f' = \varphi(f'' - \varphi') - \varphi'(f' - \varphi),$$

we have

$$\frac{1}{P} = \frac{f''\varphi - \varphi'f'}{f(f'-\varphi)} = \frac{\varphi(f'' - \varphi') - \varphi'(f'-\varphi)}{f(f'-\varphi)}.$$

It follows that z_0 is a simple pole of $\frac{1}{P}$.

Thus, we have

$$N\left(r,\frac{1}{P}\right)=N\left(r,\frac{f''\varphi-\varphi'f'}{f(f'-\varphi)}\right)+S(r,f)$$
$$\leq N\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f'-\varphi}\right)+S(r,f). \tag{3.17}$$

By (3.15)–(3.17), and Nevanlinna's first fundamental theorem, we obtain

$$N_{1)}(r,f)\leq 3N\left(r,\frac{1}{f}\right)+3\overline{N}\left(r,\frac{1}{f'-\varphi}\right)+S(r,f). \tag{3.18}$$

Case 2. $G\equiv 0$. That is,

$$P''\varphi+3P'\varphi'+2P\varphi''+\varphi\equiv 0. \tag{3.19}$$

Let z_0 be a pole of P with multiplicity l. Then z_0 must be either a zero or a pole of φ.

In fact, if $\varphi(z_0)=a$, where a $(\neq 0,\infty)$ is a constant, then z_0 is a pole of $P''\varphi$ with multiplicity $l+2$, a pole of $P'\varphi'$ with multiplicity at most $l+1$, and a pole of $P\varphi''$ with multiplicity at most l. Thus, by (3.19), we know that z_0 is a pole of $P''\varphi+3P'\varphi'+2P\varphi''+\varphi$, a contradiction.

Hence,

$$\overline{N}(r,P)\leq\overline{N}(r,\varphi)+\overline{N}\left(r,\frac{1}{\varphi}\right)\leq S(r,f).$$

We define $N_0^*\left(r,\frac{1}{f''\varphi-\varphi'f'}\right)$ is the counting function for the zeros of $f''\varphi-\varphi'f'$, which are not zeros of $f(f'-\varphi)$, and $\overline{N}_0^*\left(r,\frac{1}{f''\varphi-\varphi'f'}\right)$ as the corresponding function where the multiplicity is not counted.

Let z_0 be a zero of $f''\varphi-\varphi'f'$, but not the zero of $f(f'-\varphi)$. Then z_0 is a pole of P. It follows from $\overline{N}(r,P)\leq S(r,f)$ that

$$\overline{N}_0^*\left(r,\frac{1}{f''\varphi-\varphi'f'}\right)\leq\overline{N}(r,P)\leq S(r,f). \tag{3.20}$$

Let z_1 be a pole of f with multiplicity l_1 (≥ 2), but $\varphi(z_1)\neq 0,\infty$. Obviously, z_1 is a pole of $f(f'-\varphi)$ and $f''\varphi-\varphi'f'$ with multiplicity $2l_1+1$ and l_1+2, respectively. From the definition of P, we deduce that z_1 must be a pole of P with multiplicity l_1-1. Hence, we have

$$\overline{N}_{(2}(r,f)\leq\overline{N}(r,P)+\overline{N}(r,\varphi)+\overline{N}\left(r,\frac{1}{\varphi}\right)\leq S(r,f). \tag{3.21}$$

Set

$$g=\frac{\varphi^3(f'-\varphi)^3}{(f''\varphi-f'\varphi')^2}=\frac{\varphi(f'-\varphi)}{\left(\frac{f''-\varphi'}{f'-\varphi}-\frac{\varphi'}{\varphi}\right)^2}.$$

Let z_0 be a simple pole of f with $\varphi(z_0)\neq 0,\infty$, $\varphi'(z_0)\neq 0$. Then by (3.11) and (3.12), we obtain

$$g=\frac{-\varphi(z_0)c_{-1}}{4}+\lambda_2(z-z_0)^2+\lambda_3(z-z_0)^3+\ldots,$$

where $\frac{-\varphi(z_0)c_{-1}}{4}\neq 0,\lambda_2,\lambda_3,\ldots$ are constants. Hence, $g(z_0)\neq 0,\infty$, and $g'(z_0)=0$.

Next we consider two subcases.

Case 2.1. $g' \not\equiv 0$. It follows from that φ is a small function of f that

$$N_{1)}(r,f) \le N_0\left(r,\frac{1}{g'}\right) + \overline{N}\left(r,\frac{1}{\varphi}\right) + \overline{N}\left(r,\frac{1}{\varphi'}\right) + \overline{N}(r,\varphi) \le N_0\left(r,\frac{1}{g'}\right) + S(r,f), \tag{3.22}$$

where $N_0\left(r,\frac{1}{g'}\right)$ is the counting function for the zeros of g', which are not the zeros of g.

By Nevanlinna's first fundamental theorem and Lemma 2.1, we have

$$\begin{aligned} N\left(r,\frac{g}{g'}\right) - N\left(r,\frac{g'}{g}\right) &= T\left(r,\frac{g}{g'}\right) - m\left(r,\frac{g}{g'}\right) - T\left(r,\frac{g'}{g}\right) + m\left(r,\frac{g'}{g}\right) \\ &= -m\left(r,\frac{g}{g'}\right) + S(r,g) + S(r,f) \\ &= -m\left(r,\frac{g}{g'}\right) + S(r,f). \end{aligned} \tag{3.23}$$

Referring to line -8 and line -6 in [1, p. 57], we obtain

$$N(r,g') - N(r,g) = \overline{N}(r,g) \tag{3.24}$$

and

$$N\left(r,\frac{1}{g}\right) - N\left(r,\frac{1}{g'}\right) = \overline{N}\left(r,\frac{1}{g}\right) - N_0\left(r,\frac{1}{g'}\right). \tag{3.25}$$

From (3.24), (3.25), and Lemma 2.4, we have

$$\begin{aligned} N\left(r,\frac{g}{g'}\right) - N\left(r,\frac{g'}{g}\right) &= N(r,g) + N\left(r,\frac{1}{g'}\right) - N(r,g') - N\left(r,\frac{1}{g}\right) \\ &= N_0\left(r,\frac{1}{g'}\right) - \overline{N}\left(r,\frac{1}{g}\right) - \overline{N}(r,g). \end{aligned} \tag{3.26}$$

By (3.20), (3.21), and the definition of g, we have

$$\begin{aligned} \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g}\right) &\le \overline{N}_{(2}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f'-\varphi}\right) + \overline{N}_0^*\left(r,\frac{1}{f''\varphi-\varphi'f'}\right) + S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f'-\varphi}\right) + S(r,f). \end{aligned} \tag{3.27}$$

Combining (3.22), (3.23), (3.26) with (3.27), we obtain

$$\begin{aligned} N_{1)}(r,f) &\le N_0\left(r,\frac{1}{g'}\right) + S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g) - m\left(r,\frac{g}{g'}\right) + S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g) + S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f'-\varphi}\right) + S(r,f). \end{aligned} \tag{3.28}$$

Case 2.2. $g' \equiv 0$. Then $g \equiv A$, where A is a constant.

If $A = 0$, then $\frac{\varphi(f'-\varphi)}{\left(\frac{f''-\varphi'}{f'-\varphi} - \frac{\varphi'}{\varphi}\right)^2} \equiv 0$. Hence, we obtain that either $\varphi \equiv 0$ or $f' - \varphi \equiv 0$, a contradiction. Thus, $A \neq 0$. Hence, we have

$$\varphi(f'-\varphi) \equiv A\left(\frac{f''-\varphi'}{f'-\varphi} - \frac{\varphi'}{\varphi}\right)^2. \tag{3.29}$$

Set $F = \frac{f'-\varphi}{\varphi}$. Then by (3.29), we obtain

$$F^3 = \frac{A}{\varphi^2}(F')^2. \tag{3.30}$$

Obviously, $F \not\equiv 0$. Thus, we have $\frac{1}{F} = \frac{A}{\varphi^2}\left[\left(\frac{1}{F}\right)'\right]^2$.

Set $H = \frac{1}{F}$. Then we have $H = \left(\frac{BH'}{\varphi}\right)^2$, where $B^2 = A \neq 0$. Hence, $H = h^2$, $H' = 2hh'$, where h is a meromorphic function. It follows $(h')^2 = \left(\frac{\varphi}{2B}\right)^2$. Hence, $T(r, h') = S(r, f)$.

Since $h^2 = \frac{\varphi}{f'-\varphi}$, we have

$$2T(r, h) = T\left(r, \frac{\varphi}{f'-\varphi}\right) = T(r, f') + S(r, f). \tag{3.31}$$

From (3.31), Lemma 2.3, and $T(r, h') = S(r, f)$, we deduce that $T(r, f') \leq S(r, f)$, a contradiction.

By (3.18), (3.28), and (3.9), we obtain

$$T(r, f) \leq 5N\left(r, \frac{1}{f}\right) + 5\overline{N}\left(r, \frac{1}{f'-\varphi}\right) + S(r, f).$$

This completes the proof of Theorem 1.2.

4 Proof of Theorem 1.4

Obviously,

$$m\left(r, \frac{1}{f^{\gamma_Q}}\right) \leq m\left(r, \frac{Q}{f^{\gamma_Q}}\right) + m\left(r, \frac{1}{Q}\right) + S(r, f), \tag{4.1}$$

where $\gamma_Q = \gamma_{Q(f)} = m + \gamma_H$.

By (4.1) and Nevanlinna's first fundamental theorem, we have

$$T(r, f^{\gamma_Q}) \leq T(r, Q) + N\left(r, \frac{1}{f^{\gamma_Q}}\right) - N\left(r, \frac{1}{Q}\right) + m\left(r, \frac{Q}{f^{\gamma_Q}}\right) + S(r, f). \tag{4.2}$$

Since $(m + \gamma_H)T(r, f) = T(r, f^{\gamma_Q}) + S(r, f)$, then by (4.2) and Lemma 2.1, we obtain

$$\begin{aligned}(m + \gamma_H)T(r, f) &\leq T(r, Q) + N\left(r, \frac{1}{f^{\gamma_Q}}\right) - N\left(r, \frac{1}{Q}\right) + m\left(r, \frac{Q}{f^{\gamma_Q}}\right) + S(r, f)\\ &\leq T(r, Q) + (m + \gamma_H)N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{Q}\right) + (\gamma_H - \underline{\gamma}_H)m\left(r, \frac{1}{f}\right) + S(r, f)\end{aligned}$$

$$\le T(r,Q) + (m+\gamma_H)N\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{Q}\right) + (\gamma_H - \underline{\gamma}_H)\left[T\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{f}\right)\right] + S(r,f)$$
$$\le T(r,Q) + (m+\underline{\gamma}_H)N\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{Q}\right) + (\gamma_H - \underline{\gamma}_H)T(r,f) + S(r,f).$$

Thus, we have

$$\begin{aligned}(m+\underline{\gamma}_H)T(r,f) &\le T(r,Q) + (m+\underline{\gamma}_H)N\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{Q}\right) + S(r,f)\\ &\le T(r,Q) + (m+\underline{\gamma}_H)N\left(r,\frac{1}{f}\right) - mN\left(r,\frac{1}{f}\right) + S(r,f)\\ &\le T(r,Q) + \underline{\gamma}_H N\left(r,\frac{1}{f}\right) + S(r,f).\end{aligned}$$

It follows

$$T(r,f) \le \frac{1}{m}T(r,Q) + S(r,f). \tag{4.3}$$

By (4.3) and Lemma 2.5, we prove Theorem 1.4.

5 Proof of Theorem 1.8

Suppose $\phi \not\equiv 0$. By Lemma 2.7, we have $N\left(r,\frac{1}{f^{(k)}-\phi}\right) \ne S(r,f)$, a contradiction. Hence, $\phi \equiv 0$.

Next we consider two cases.

Case 1. $k \ge 2$.

Since f is a transcendental meromorphic function, then by Lemma 2.8, we deduce that $f = e^{az+b}$, where $a\ (\ne 0)$, b are constants.

Case 2. $k = 1$.

Suppose that f'' has finitely many zeros. Then by Lemma 2.10 and $f \ne 0$, we deduce that $f = \frac{1}{Q}e^P$, where P, Q are polynomials. It follows from $f' \ne 0$ that

$$f' = \frac{e^P(P'Q - Q')}{Q^2} \ne 0. \tag{5.1}$$

Obviously, $P'Q - Q' \not\equiv 0$ and $P' \not\equiv 0$.

Next we consider two subcases.

Case 2.1. There exists z such that $P'(z)Q(z) - Q'(z) = 0$.

Let z_1 be a zero of $P'(z)Q(z) - Q'(z)$. By (5.1), we have $Q(z_1) = 0$. Hence, $Q'(z_1) = 0$.

Thus, we obtain

$$P'(z)Q(z) - Q'(z) = (z-z_1)^{l_1}\varphi_1(z), \tag{5.2}$$

and

$$Q(z) = (z-z_1)^{l_2}\varphi_2(z), \tag{5.3}$$

where $l_1, l_2\ (\ge 2)$ are positive integers and $\varphi_1(z), \varphi_2(z)$ are two polynomials with $\varphi_1(z_1) \ne 0$, $\varphi_2(z_1) \ne 0$. It follows from (5.3) that

$$Q'(z) = (z-z_1)^{l_2-1}\varphi_3(z), \tag{5.4}$$

where $\varphi_3(z) = l_2\varphi_2(z) + (z-z_1)\varphi'(z)$. Obviously, $\varphi_3(z_1) = l_2\varphi_2(z_1) \ne 0$.

By (5.3) and (5.4), we have

$$\begin{aligned} P'(z)Q(z) - Q'(z) &= P'(z)(z - z_1)^{l_2}\varphi_2(z) - (z - z_1)^{l_2-1}\varphi_3(z) \\ &= (z - z_1)^{l_2-1}[P'(z)(z - z_1)\varphi_2(z) - \varphi_3(z)] \\ &= (z - z_1)^{l_2-1}\varphi_4(z), \end{aligned} \tag{5.5}$$

where $\varphi_4(z) = P'(z)(z - z_1)\varphi_2(z) - \varphi_3(z)$. Obviously, $\varphi_4(z_1) = -\varphi_3(z_1) \neq 0$.

It follows from (5.2) and (5.5) that $l_1 = l_2 - 1$. Then by (5.2) and (5.4), we know that z_1 is a zero of both $P'(z)Q(z) - Q'(z)$ and $Q'(z)$ with the same multiplicities.

Suppose that the distinct zeros of $P'(z)Q(z) - Q'(z)$ are $z_1, z_2, \ldots, z_s$ with multiplicities are $m_1, m_2, \ldots, m_s$, where $s, m_1, m_2, \ldots, m_s$ are positive integers. It follows

$$P'(z)Q(z) - Q'(z) = A(z - z_1)^{m_1}(z - z_2)^{m_2}\ldots(z - z_s)^{m_s},$$

where A is a nonzero constant and $m_1 + m_2 + \ldots + m_s = \deg(P'Q - Q') = \deg P' + \deg Q \geq \deg Q$. Furthermore, we have

$$Q'(z) = (z - z_1)^{m_1}(z - z_2)^{m_2}\ldots(z - z_s)^{m_s}\varphi_5(z),$$

where $\varphi_5(z)$ is a polynomial. Thus, we obtain $\deg Q' \geq m_1 + m_2 + \ldots + m_s \geq \deg Q$. Then we deduce that Q is a nonzero constant.

Since $f = \frac{1}{Q}e^P$ and $f' \neq 0$, we obtain $f' = \frac{1}{Q}e^P P' \neq 0$. Thus, we obtain $P(z) = az + b_1$, where $a\ (\neq 0), b_1$ are constants. It follows that $f = e^{az+b}$, where b is a constant.

Case 2.2. $P'Q - Q' \neq 0$.

From $P'Q - Q' \neq 0$, $Q \not\equiv 0$ and $P' \not\equiv 0$, we know that P' and Q are nonzero constants. Thus, $P = az + b_2$ and $Q = b_3$, where $a\ (\neq 0)$, b_2 and $b_3\ (\neq 0)$ are constants. It follows that $f = e^{az+b}$, where b is a constant.

This completes the proof of Theorem 1.8.

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