DE GRUYTER Open Mathematics 2025; 23: 20240113

Research Article

Carlos M. da Fonseca, M. Lawrence Glasser, and Victor Kowalenko*

Chebyshev polynomials of the first kind and the univariate Lommel function: Integral representations

https://doi.org/10.1515/math-2024-0113
received June 6, 2024; accepted December 3, 2024

Abstract: This study investigates a number of integrals possessing products of different indices of the univariate Lommel function, $s_{\mu,\nu}(a)$, with various elementary and special functions. As a consequence, connections between the function and Chebyshev polynomials of the first kind are established.

Keywords: Bessel function, Chebyshev polynomial of the first kind, Fourier transform, generalized hypergeometric function, index/order integral, Lommel function, parameter, special functions

MSC 2020: 33C47, 44A15, 44A20

Dedicated to the Memory of José Carlos Petronilho.

1 Introduction

The (univariate) Lommel functions, $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$, are the solutions of an inhomogeneous form of the Bessel differential equation:

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = z^{\mu+1}, \tag{1.1}$$

and, as we shall see later, are closely related to the more familiar Struve functions when $\mu = \nu$. In this study, we shall be primarily interested in $s_{\mu,\nu}(z)$, which is defined as

$$s_{\mu,\nu}(z) \doteqdot \frac{z^{\mu+1}}{(\mu+1)^2 - \nu^2}\, {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right). \tag{1.2}$$

where ${}_1F_2(a; b, c; z)$ represents a generalized hypergeometric function. According to No. 8.574(3) in [1], (1.2) can be expressed alternatively as

$$s_{\mu,\nu}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{\mu+1+2m}}{((\mu+1)^2 - \nu^2)((\mu+3)^2 - \nu^2)\ldots((\mu+2m+1)^2 - \nu^2)}. \tag{1.3}$$

*** Corresponding author: Victor Kowalenko,** School of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia, e-mail: vkowa@unimelb.edu.au

Carlos M. da Fonseca: Kuwait College of Science and Technology, Doha District, Safat 13133, Kuwait; Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Kliment Ohridski Blvd. 8, 1000 Sofia, Bulgaria; Chair of Computational Mathematics, University of Deusto, 48007 Bilbao, Spain, e-mail: c.dafonseca@kcst.edu.kw, carlos.fonseca@tu-sofia.bg

M. Lawrence Glasser: Departamento de Fisica Teorica, Atomica y Optica, Universidad de Valladolid, 47011 Valladolid, Spain; Department of Physics, Clarkson University, Potsdam, NY 13699-5820, United States of America, e-mail: laryg@clarkson.edu

Moreover, it should be noted that $s_{\mu,\nu}(z)$ also appears occasionally in the literature with an upper index (n), in which case it is defined as

$$s_{\mu,\nu}^{(n)}(z) \doteq \frac{z^{\mu+1}}{(\mu+1)^2-\nu^2}\,{}_1F_2\left(1;\,\frac{\mu-n\nu+3}{2},\,\frac{\mu+\nu+3}{2};\,-\frac{z^2}{4}\right). \tag{1.4}$$

Hence we observe from (1.4) that $s_{\mu,\nu}^{(1)}(z)$ reduces to $s_{\mu,\nu}(z)$, while from (1.3), we observe that $s_{\mu,\nu}(z)$ is singular whenever $\mu + 2k + 1 = |\nu|$, for k, a nonnegative integer. As we will be primarily interested in the cases when $\mu = 0$ or -1, here, our results would normally not be valid for ν, an odd integer when $\mu = 0$, and ν, an even integer when $\mu = -1$, but we shall see in these cases that the Lommel function will be accompanied by a zero from a trigonometric function, thereby removing any singularity.

The function $s_{\mu,\nu}(z)$ also satisfies several recurrence relations such as [1, 8.575.1–8.575.5] and various integral transforms appearing between [1, 6.862–6.869]. In addition, Mellin transforms in which the function appears in the integrand with other elementary and special functions such as Bessel functions are presented in Sec. 2.9 of [2], while some new results have been presented more recently in [3]. In this study, however, we aim to present new integral results involving $s_{\mu,\nu}(z)$ by studying the connection between this Lommel function and Chebyshev polynomials of the first kind. Consequently, we shall be able to derive new and interesting results in the future.

In general, not much information about $s_{\mu,\nu}$ exists in the literature, especially in regard to its behavior and properties. However, one can implement (1.2) through the HypergeometricPFQ routine in Mathematica as follows:

Lommel[μ _, ν_, −z^2/4] ≔ z^(μ + 1) HypergeometricPFQ[{1}, {(μ − ν + 3)/2, (μ + ν + 3)/2}, −z^2/4]/((μ + 1)^2 − ν^2).

Then the Plot instruction can be invoked as follows:

Plot[{Lommel[0, 1/2, z], Lommel[0, 2, z], Lommel[0, 7/3, z], Lommel[0, 4, z]}, {z, −100, 100}, PlotLabel → "Lommel Functions with μ = 0," PlotLegends → "Expressions"]

By invoking the aforementioned routine, we arrive at Figure 1, which displays $s_{0,\nu}$ for various values of ν and real values of z lying in the domain of $[-100, 100]$. In this figure, we see that $s_{0,\nu}(z)$, though oscillating wildly, which is typical of Bessel functions over this range, is continuous with the largest amplitudes occurring for small values of ν. Nevertheless, the amplitudes all lie in the range between $[-1, 1]$. Therefore, one expects integrals of $s_{0,\nu}(z)$ not to be large or at least not divergent over large ranges of integration variable, much like $\sin z$ and $\cos z$, provided, of course, the second order or index, ν, is not an odd integer.

Figure 2 presents $s_{-1,\nu}(z)$ for various values of the second index or order, ν. These plots have been obtained by setting μ in the Plot routine given above to -1. In addition, the range has also been extended to $[-3, 3]$. Again, we observe wild oscillations with the greatest amplitudes occurring for small values of $|\nu|$, but unlike Figure 2,

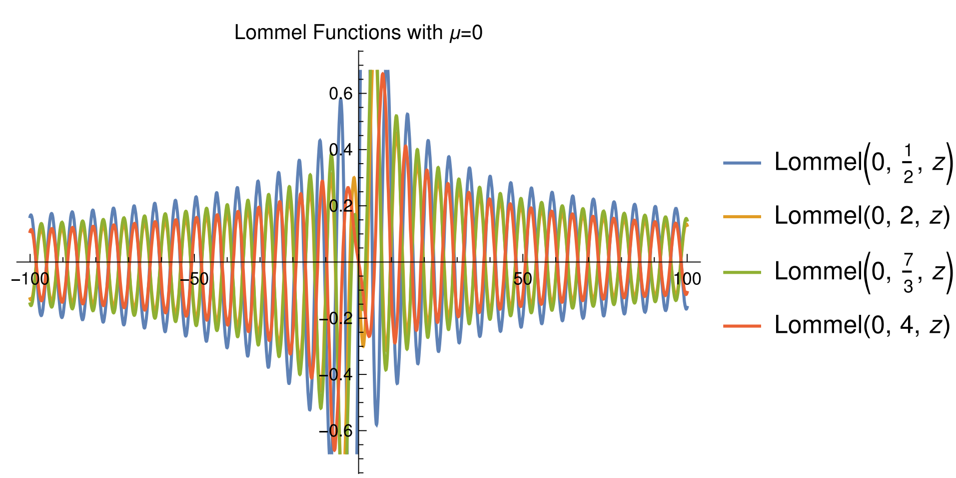

Figure 1: Lommel function $s_{0,\nu}(z)$ for various values of ν.

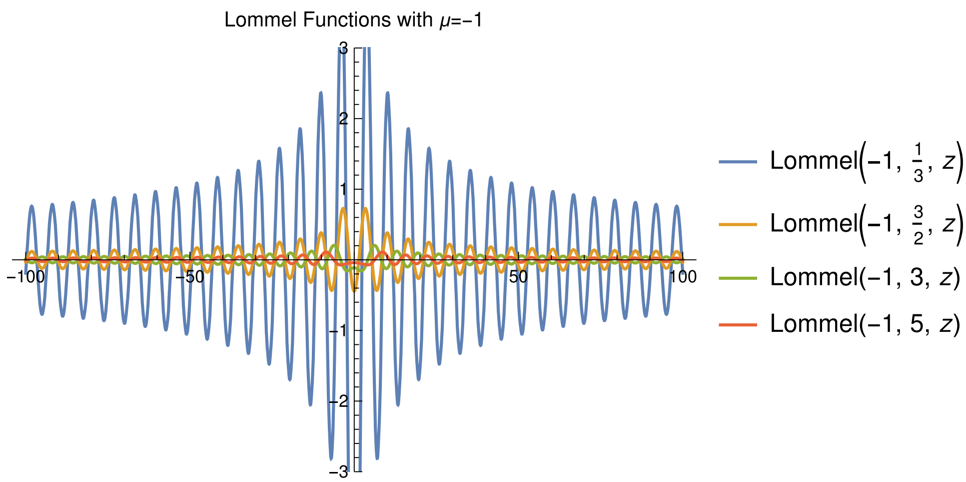

Figure 2: Lommel function $s_{-1,\nu}(z)$ for various values of ν.

we see that the amplitudes can lie well outside of $[-1, 1]$ when z is situated near the origin. Despite this, the plots indicate that $s_{-1,\nu}$ is continuous, provided ν is not an even integer as described earlier. Similarly, one expects integrals of the function not to be divergent over large ranges of z.

As far as the present article is concerned, we shall be mainly interested in the behavior of the Lommel functions where (1) the first order μ is fixed, viz., fixed to zero or unity, (2) the second order, ν, becomes the variable, and (3) z assumes the role of a parameter. In these cases, the behavior of the univariate Lommel functions shown in the figures is very different and we expect the graphs to be singular whenever $\mu + 2k + 1 = |\nu|$.

Figure 3 presents various graphs of $s_{0,\nu}(z)$, where z has been set to the values of ν in Figure 1. These plots were obtained simply by altering $\{z, -100, 100\}$ in the above Mathematica instruction to $\{\nu, -10, 10\}$ and setting z to 1/2, 2, 7/3 and 4 when calling the Lommel functions. On the right side where $\nu > 2$, we observe that graphs begin at $-\infty$, cross the horizontal axis, and continue to ∞, which is reminiscent of the behavior for $\tan z$. This behavior is reflected across the vertical axis for negative values of ν. On the other hand, the central region, $[-2.5, 2.5]$, displays U-shaped curves that eventually diverge to infinity.

Figure 4 presents various graphs of $s_{-1,\nu}(z)$ with z set equal to the various values of ν in Figure 2. The figure displays the same behavior as the previous figure except that the central region is narrower.

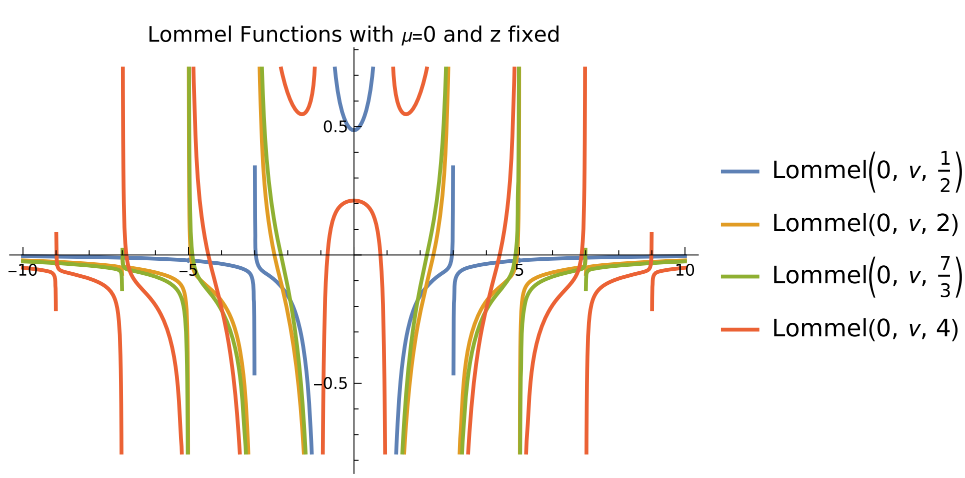

Figure 3: Lommel function $s_{0,\nu}(z)$ for fixed values of z and ν lying between $[-10, 10]$.

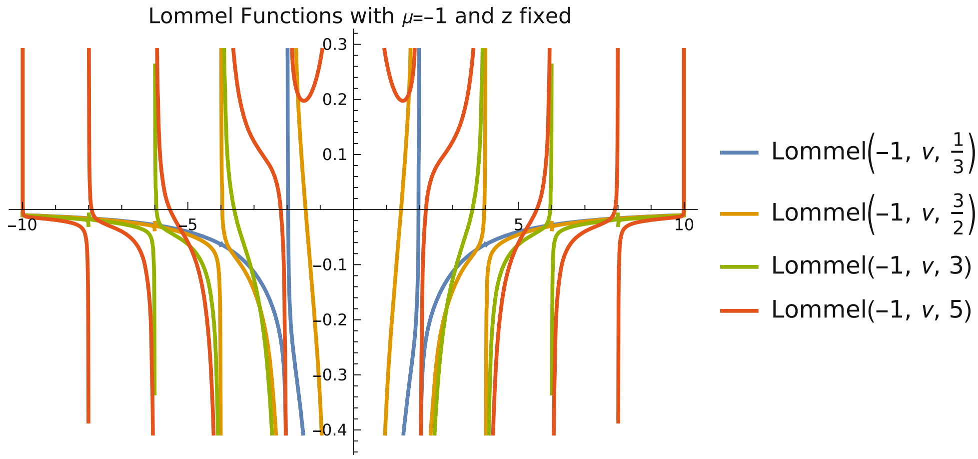

Figure 4: Lommel function $s_{-1,\nu}(z)$ for fixed values of z and ν lying between [−10, 10].

2 Transforms with Lommel functions

In this section, we shall derive several integral identities for particular values of the indices in $s_{\mu,\nu}(z)$. In obtaining these results, it will be necessary to utilize Nos 49 and 50 in Sec. 1.7 of [4], which also appear as Nos 3.715.12 and 3.715.17 in [1]. In actual fact, these results were originally taken from Watson [5]. Since they are prominent in our first theorem, we need to establish their correctness in order to ensure that the proof is not conditional upon them and is, therefore, incomplete. However, before we can prove them, we require the following lemma.

Lemma 2.1. *The cosine transforms of* $\cos(b\cos x)$, $\sin(b\cos x)$, *and* $J_\beta(b\cos x)$ *multiplied by an arbitrary power of* $\cos x$ *are found to be given by*

$$\int_0^{\pi/2} \cos(b\cos x)\cos^{\nu-1} x\cos(ax)dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\nu/2)\Gamma((\nu+1)/2)}{\Gamma((\nu+a+1)/2)\Gamma((\nu-a+1)/2)}\,{}_2F_3\left(\frac{\nu}{2},\frac{\nu}{2}+\frac{1}{2};\frac{1}{2},\frac{\nu+a+1}{2},\frac{\nu-a+1}{2};-\frac{b^2}{4}\right), \tag{2.1}$$

$$\int_0^{\pi/2} \sin(b\cos x)\cos^{\nu} x\cos(ax)dx = \frac{b\sqrt{\pi}}{2}\frac{\Gamma(1+\nu/2)\Gamma((\nu+3)/2)}{\Gamma((\nu+a+3)/2)\Gamma((\nu-a+3)/2)}\,{}_2F_3\left(1+\frac{\nu}{2},\frac{\nu}{2}+\frac{3}{2};\frac{3}{2},\frac{\nu+a+3}{2},\frac{\nu-a+3}{2};-\frac{b^2}{4}\right), \tag{2.2}$$

and more generally,

$$\int_0^{\pi/2} J_\beta(b\cos x)\cos^{\nu-1} x\cos(ax)dx = \frac{2^{\nu-2}b^\beta\sqrt{\pi}}{\Gamma(\beta+1)}\frac{\Gamma((\beta+\nu)/2)}{\Gamma((\beta+\nu+a+1)/2)}\frac{\Gamma((\beta+\nu+1)/2)}{\Gamma((\beta+\nu-a+1)/2)}\,{}_2F_3\left(\frac{\beta+\nu}{2},\frac{\beta+\nu+1}{2};\nu+1,\frac{\beta+\nu+a+1}{2},\frac{\beta+\nu-a+1}{2};-\frac{b^2}{4}\right). \tag{2.3}$$

In these results, $\Re\nu > 0$.

Proof. To establish the lemma, we require [1, 3.631.9], which states that

$$\int_0^{\pi/2} \cos^{\nu-1} x \cos(ax) dx = \frac{\pi\Gamma(\nu)}{2^\nu \Gamma((\nu+a+1)/2)\Gamma((\nu-a+1)/2)} = \frac{\pi}{2^\nu \nu} B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)^{-1}. \tag{2.4}$$

In (2.4), $\Re\nu > 0$, while $B(x,y)$ represents the beta function, which equals $\Gamma(x)\Gamma(y)/\Gamma(x+y)$ for $\Re x, \Re y > 0$. For the sake of completeness, (2.4) is established by evaluating the contour integral given by

$$I = \int_C z^{a-\nu}(1+z^2)^{\nu-1} dz,$$

where one assumes initially that $0 < \Re\nu < \Re a$. This integral is evaluated for two cases, the first where the closed contour C traverses the first quadrant of the unit circle and the second where it traverses the fourth quadrant of the unit circle. In each case, Cauchy's residue theorem is applied, giving zero, since there are no residues due to the initial conditions on a and ν. For example, integrating over the entire first quadrant of the unit circle yields

$$P_+(\nu, a) + 2^{\nu-1} i \int_0^{\pi/2} e^{iat} \cos^{\nu-1} t dt - i e^{i(a-\nu)\pi/2} P_-(\nu, a) = 0,$$

where

$$P_\pm(\nu, a) = \int_0^1 s^{a-\nu}(1 \pm s^2)^{\nu-1} ds.$$

Note that $P_-(\nu, a) = B(\nu, (a-\nu+1)/2)$. A similar result follows for the integration over the fourth quadrant of the unit circle. By taking the difference of the resulting equations, one finds that the left-hand side (LHS) of (2.4) can be expressed purely in terms of the beta function. Finally, by applying the reflection property of the gamma function or [1, 8.335.3], we arrive at the desired result. The result can then be analytically continued to $\Re\, a < \Re\nu$.

As an aside, it should be pointed out that when $a = n$, a nonnegative integer, the right-hand side (RHS) of (2.4) equals the RHS of the integral No. 7.346 in [1]. Therefore, we have

$$\int_0^{\pi/2} \cos^{\nu-1} x \cos(nx) dx = \int_0^1 u^{\nu-1} T_n(u) \frac{du}{\sqrt{1-u^2}}, \tag{2.5}$$

where $T_n(u)$ is the Chebyshev polynomial of the first kind. This result can also be obtained by making the substitution $u = \cos x$ in the integral on the LHS. In the next section, we shall deal with similar integrals to the RHS of the aforementioned result except the algebraic power of u will be replaced by trigonometric functions.

Returning to (2.4), we replace ν by $2n + \nu$ and multiply both sides by $(-1)^n b^{2n}/(2n)!$. Next, we sum over n from 0 to ∞. On the LHS we can interchange the order of the summation and integration since the former is absolutely convergent. Therefore, we obtain

$$\int_0^{\pi/2} \cos(b\cos x) \cos^{\nu-1} x \cos(ax) dx = \frac{\pi}{2^\nu} \sum_{n=0}^{\infty} \frac{(-b^2/4)^n}{(2n)!} \frac{\Gamma(2n+\nu)}{\Gamma(n+(\nu+a+1)/2)} \frac{1}{\Gamma(n+(\nu-a+1)/2)}. \tag{2.6}$$

Next, we apply the duplication formula for the gamma function or No. 8.335.1 in [1] to the ratio of $\Gamma(2n+\nu)/\Gamma(2n+1)$, which yields

$$\frac{\Gamma(2n+\nu)}{\Gamma(2n+1)} = 2^{\nu-1}\frac{\Gamma(n+\nu/2)\Gamma(n+\nu/2+1/2)}{n!\Gamma(n+1/2)}.$$

Hence (2.6) becomes

$$\begin{aligned}&\int_0^{\pi/2}\cos(b\cos x)\cos^{\nu-1}x\cos(ax)dx\\ &= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-b^2/4)^n}{n!\Gamma(n+1/2)}\frac{\Gamma(n+\nu/2)\Gamma(n+\nu/2+1/2)}{\Gamma(n+(\nu+a+1)/2)}\frac{1}{\Gamma(n+(\nu-a+1)/2)},\end{aligned} \tag{2.7}$$

where $\Re\nu > 0$. By multiplying the numerator and denominator of (2.7) by $\Gamma(\nu+1/2)$, $\Gamma(\nu/2)$, $\Gamma((\nu+a+1)/2)$, and $\Gamma((\nu-a+1)/2)$, we arrive at (2.1).

To obtain the second result in the theorem, we put $\nu = 2n+\nu+2$ in (2.4). Then we multiply both sides of the ensuing equation by $(-1)^n b^{2n+1}/(2n+1)!$ and sum from $n=0$ to ∞. Since the sum on the LHS is absolutely convergent, we can, again, interchange the summation and integration. Then we observe that the sum over n can be replaced by $\sin(b\cos x)$. After some algebraic manipulation on the RHS, one eventually arrives at the ${}_2F_3$ hypergeometric function in (2.2). Note that although the parameter, b, is being treated as a real variable, it can be complex or even a function.

The final result in the lemma is obtained by replacing ν by $2n+\beta+\nu$ in (2.4). Next, both sides of the ensuing equation are multiplied by $(-1)^n(b/2)^{2n+\beta}/n!\Gamma(n+\beta+1)$, and n is summed from zero to ∞. Once again, the integration and summation can be interchanged on the LHS. Consequently, the summation over n becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n(b\cos(x)/2)^{2n+\beta}}{n!\Gamma(n+\beta+1)} = J_\beta(b\cos x),$$

where $J_\beta(z)$ represents a Bessel function of order β and variable $b\cos x$. By manipulating the summation over n on the RHS, one eventually arrives at the ${}_2F_3$ hypergeometric function in (2.3). □

If we put $\nu = 1$ in (2.1), then we find that the integral reduces to

$$\int_0^{\pi/2}\cos(b\cos x)\cos(ax)dx = \frac{\pi}{2\Gamma(1-a/2)\Gamma(1+a/2)}\,{}_1F_2\left(1;\,1+\frac{a}{2},1-\frac{a}{2};\,-\frac{b^2}{4}\right). \tag{2.8}$$

Introducing the reflection formula for the gamma function or [1, 8.335.3] and identifying the generalized hypergeometric function as a Lommel function according to (1.2), we find that (2.8) can be expressed as

$$I_1 = \int_0^{\pi/2}\cos(b\cos x)\cos(ax)dx = -a\sin\left(\frac{a\pi}{2}\right)s_{-1,a}(b), \tag{2.9}$$

which is given in [4, 1.7.50] or [1, 3.715.17]. In the second reference, it is also expressed in terms of Anger, $\mathbf{J}_\nu(z)$, and Weber functions, $\mathbf{E}_\nu(z)$, i.e.,

$$I_1 = \frac{\pi}{4}\sec\left(\frac{\pi a}{2}\right)(\mathbf{J}_a(b)+\mathbf{J}_{-a}(b)) = \frac{\pi}{4}\csc\left(\frac{\pi a}{2}\right)(\mathbf{E}_a(b)-\mathbf{E}_{-a}(b)).$$

In the introduction, it was mentioned that $s_{-1,a}(b)$ is singular when $|\nu| = 2k$. However, this does not apply to (2.9) since the zero in the sine factor will remove the singularity in the Lommel function.

On the other hand, if we put $\nu = 0$ in (2.2), then we find that

$$\begin{aligned}&\int_0^{\pi/2}\sin(b\cos x)\cos(ax)dx\\ &= \frac{b\sqrt{\pi}}{2}\frac{\Gamma(3/2)}{\Gamma((a+3)/2)\Gamma((3-a)/2)}\,{}_1F_2\left(1;\,\frac{a+3}{2},\frac{3-a}{2};-\frac{b^2}{4}\right).\end{aligned} \tag{2.10}$$

In addition, $b\sqrt{\pi}\Gamma(3/2)/2 = \pi b/4$. Consequently, (2.10) reduces to

$$\int_0^{\pi/2} \sin(b\cos x)\cos(ax)dx = \frac{b\pi}{(1-a^2)}\frac{1}{\Gamma((a+1)/2)\Gamma((1-a)/2)}\ {}_1F_2\left(1;\ \frac{a+3}{2}, \frac{3-a}{2}; -\frac{b^2}{4}\right).$$

The product of the gamma functions in the denominator represents the cosine form of the reflection formula for the gamma function, viz., [1, 8.334.2]. Moreover, using (1.1), we arrive at

$$I_2 = \int_0^{\pi/2} \sin(b\cos x)\cos(ax)dx = \cos\left(\frac{\pi a}{2}\right)s_{0,a}(b). \tag{2.11}$$

This equation, which is the second result required for the lemma appearing shortly, appears as [4, 1.7.49] or as [1, 3.715.12]. Note that the singularity in the Lommel function when b is equal to an odd integer is removed by the zero produced by the cosine factor. It should also be mentioned that in [1] the above integral is expressed in terms of Anger and Weber functions as

$$I_2 = \frac{\pi}{4}\csc\left(\frac{\pi a}{4}\right)(\mathbf{J}_a(b) - \mathbf{J}_{-a}(b)) = -\frac{\pi}{4}\sec\left(\frac{\pi a}{4}\right)(\mathbf{E}_a(b) + \mathbf{E}_{-a}(b)).$$

For the special case of $a = 0$, (2.11) reduces to

$$\int_0^{\pi/2} \sin(b\cos x)dx = \frac{\pi}{2}\mathbf{H}_0(b),$$

where $\mathbf{H}_\nu(z)$ represents the Struve function of order ν. This result can be verified by putting $\nu = 0$ in [1, 8.551.1].

In the introduction, it was mentioned that $s_{\mu,\nu}(z)$ is closely related to the Struve function. In fact, via the power series expansion, viz., [6, 11.2.1], the general Struve function can be expressed as

$$\mathbf{H}_\nu(z) = \frac{z^{\nu+1}}{2^\nu\sqrt{\pi}\Gamma(\nu+3/2)}\ {}_1F_2\left(1;\ \nu+\frac{3}{2}, \frac{3}{2}; -\frac{z^2}{4}\right).$$

Therefore, according to (1.2), we see that

$$\mathbf{H}_\nu(z) = \frac{1}{2^{\nu-1}\sqrt{\pi}\Gamma(\nu+1/2)}s_{\nu,\nu}(z).$$

Moreover, both Anger and Weber functions can be expressed in terms of similar ${}_1F_2$ hypergeometric functions to (1.2). From [1, 8.581] or [6, 11.10.8,11.10.9], the Anger function can be expressed as

$$\mathbf{J}_\nu(z) = \frac{\sin(\nu\pi)}{2\pi}\ {}_1F_2\left(1;\ 1+\frac{\nu}{2}, 1-\frac{\nu}{2}; -\frac{z^2}{4}\right) + \frac{z\sin(\pi z)}{(1-\nu^2)\pi}\ {}_1F_2\left(1;\ \frac{\nu+3}{2}, \frac{3-\nu}{2}; -\frac{z^2}{4}\right),$$

while the Weber function can be expressed as

$$\mathbf{E}_\nu(z) = \frac{\sin^2(\nu\pi/2)}{\pi}\ {}_1F_2\left(1;\ 1+\frac{\nu}{2}, 1-\frac{\nu}{2}; -\frac{z^2}{4}\right) - \frac{2z}{\pi}\frac{\cos^2(\pi z/2)}{(1-\nu^2)}\ {}_1F_2\left(1;\ \frac{\nu+3}{2}, \frac{3-\nu}{2}; -\frac{z^2}{4}\right).$$

Therefore, via (1.2), we arrive at

$$\mathbf{J}_\nu(z) = \frac{\sin(\nu\pi)}{\pi}\left(zs_{0,\nu}(z) - \frac{\nu^2}{2}s_{-1,\nu}(z)\right),$$

and

$$\mathbf{E}_\nu(z) = -\frac{1}{\pi}\left(\nu^2\sin^2\left(\frac{\nu\pi}{2}\right)s_{-1,\nu}(z) + 2\cos^2\left(\frac{\pi\nu}{2}\right)s_{0,\nu}(z)\right).$$

If we put $b = \nu + 2$ in (2.3), then we obtain

$$\int_0^{\pi/2} J_{\nu+2}(b\cos x)\cos^{\nu-1}x\cos(ax)dx = \frac{2^{\nu-2}b^{\nu+2}\sqrt{\pi}}{\nu+1}\frac{\Gamma(\nu+1)}{\Gamma(\nu+(a+3)/2)\Gamma(\nu+(3-a)/2)}\,{}_1F_2\left(\nu+\frac{3}{2};\nu+\frac{a+3}{2},\nu+\frac{3-a}{2};-\frac{b^2}{4}\right). \tag{2.12}$$

It should also be mentioned that $s_{\mu,\nu}(z)$ can be expressed in terms of integrals involving Bessel functions with trigonometric arguments multiplied by powers of trigonometric functions such as (2.12) by applying [6, 11.9.8], which states

$$s_{\mu,\nu}(z) = 2^{(\mu+\nu-1)/2}\Gamma\left(\frac{\mu+\nu+1}{2}\right)z^{(\mu-\nu+1)/2}\sum_{k=0}^{\infty}\frac{(z/2)^k}{k!(2k+\mu-\nu+1)}J_{k+(\mu+\nu+1)/2}(z).$$

Provided that $\Re(\mu-\nu) > -1$, the previous equation can be expressed as

$$s_{\mu,\nu}(z) = 2^{(\mu+\nu-1)/2}\Gamma\left(\frac{\mu+\nu+1}{2}\right)z^{(\mu-\nu+1)/2}\sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}J_{k+(\mu+\nu+1)/2}(z)\int_0^1 t^{2k+\mu-\nu}\mathrm{d}t. \tag{2.13}$$

By interchanging the summation and integration, we can evaluate the resulting summation with the aid of the Bessel multiplication theorem given in Sec. 10.23 of [6] or on p. 377 of [7], which can be expressed as

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{zt}{2}\right)^k J_{k+\nu}(z) = (1+t)^{-\nu/2}J_\nu(\sqrt{1+t}\,z).$$

Then we find that (2.13) reduces to

$$s_{\mu,\nu}(z) = 2^{(\mu+\nu-1)/2}\Gamma\left(\frac{\mu+\nu+1}{2}\right)z^{(\mu-\nu+1)/2}\int_0^{\pi/2}\sin^{\mu-\nu}\theta\cos^{(\mu+\nu+3)/2}\theta J_{(\mu+\nu+1)/2}(z\cos\theta)d\theta,$$

after making the substitution, $t = \sin\theta$. If we replace $(\mu+\nu+1)/2$ by β, then (2.13) becomes

$$s_{2\beta-\nu-1,\nu}(z) = 2^{\beta-1}\Gamma(\beta)z^{\beta-\nu}\int_0^{\pi/2}\sin^{2\beta-2\nu-1}\theta\cos^{\beta+1}\theta J_\beta(z\cos\theta)d\theta, \tag{2.14}$$

where the condition on the integral becomes $\Re(\beta-\nu) > 0$. For $\beta = (\nu+1)/2$, the condition becomes $\Re\nu < 1$, while (2.14) reduces to

$$s_{0,\nu}(z) = 2^{(\nu-1)/2}\Gamma((\nu+1)/2)z^{(1-\nu)/2}\int_0^{\pi/2}\sin^{-\nu}\theta\cos^{(\nu+3)/2}\theta J_{\nu+1/2}(z\cos\theta)d\theta.$$

For the proof of the upcoming theorem, we shall require the inverted forms of (2.9) and (2.11). In order to derive these results, we need to use the properties of the Dirac delta function [8,9], in particular, the orthogonality result or closure equation for Bessel functions [10,11]. This is given by

$$s\int_0^{\infty} yJ_\nu(sy)J_\nu(ty)dy = \delta(s-t), \tag{2.15}$$

where

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0. \end{cases}$$

Other representations for the function are

$$\delta(x) = \int_{-\infty}^{\infty} e^{-2\pi ikx} dk$$

and

$$\delta(x) = \lim_{p\to\infty} \frac{1}{\pi x} \sin(px).$$

For $\nu = -1/2$, (2.15) reduces to

$$\int_0^{\infty} \cos(sy)\cos(ty)dy = \frac{\pi}{2}\delta(s-t), \tag{2.16}$$

while for $\nu = 1/2$, it reduces to

$$\int_0^{\infty} \sin(sy)\sin(ty)dy = \frac{\pi}{2}\delta(s-t). \tag{2.17}$$

As another aside, the general result or (2.15) is derived in Sec. 6.3 of [11] from studying the Bessel differential equation that arises when the Helmholtz equation is expressed in polar coordinates. The eigenfunctions of this equation are given by Bessel functions, whose orthogonality in the continuum limit results in (2.15).

As a consequence, we can multiply both sides of (2.9) by $\cos(ay)$ and integrate over a from zero to p. Then we find that

$$\int_0^{p}\int_0^{\pi/2} \cos(b\cos x)\cos(ax)\cos(ay)dxda = -\int_0^{p} a\sin\left(\frac{a\pi}{2}\right)s_{-1,a}(b)da.$$

Since both integrals on the LHS are finite/bounded provided $|\nu| \neq 2k$ as outlined in the introduction, we can interchange their order in accordance with Fubini's theorem [12]. By taking the limit as $p \to \infty$, we observe that the inner integral becomes (2.16). From the properties of the Dirac delta function [8,9], in particular,

$$\int_0^{\infty} f(x)\delta(x-a)dx = f(a), \tag{2.18}$$

we arrive at

$$\frac{\pi}{2}\cos(b\cos x)\Theta\left(\frac{\pi}{2}-x\right) = -\int_0^{\infty} y\sin\left(\frac{y\pi}{2}\right)s_{-1,y}(b)\cos(xy)dy. \tag{2.19}$$

The aforementioned method will also be adopted in the proof of Theorem 3.1. In obtaining the above result, a has been replaced by y, while $\Theta(x)$ represents the Heaviside step function, which is defined as

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

This has arisen because the upper limit in (2.18) is finite. Similarly, if we apply the same procedure to (2.11), then we obtain

$$\frac{\pi}{2}\sin(b\cos x)\Theta\left(\frac{\pi}{2}-x\right) = \int_0^{\infty} \cos\left(\frac{\pi y}{2}\right)s_{0,y}(b)\cos(xy)dy. \tag{2.20}$$

Multiplying (2.9) with (2.11) yields

$$\frac{\pi^2}{4}\sin(a\cos x)\cos(b\cos x)\Theta\left(\frac{\pi}{2}-x\right)$$
$$= -\int_0^{\infty}\cos\left(\frac{\pi y}{2}\right)s_{0,y}(a)\cos(xy)dy\int_0^{\infty} z\sin\left(\frac{z\pi}{2}\right)s_{-1,z}(b)\cos(xz)dz.$$

We now integrate both sides over x from 0 to p, where p is to be greater than $\pi/2$. Thus, we arrive at

$$\begin{aligned}&\frac{\pi^2}{4}\int_0^p \sin(a\cos x)\cos(b\cos x)\Theta\left(\frac{\pi}{2}-x\right)dx\\&=\int_0^p\int_0^\infty \cos\left(\frac{\pi y}{2}\right)s_{0,y}(a)\cos(xy)dy\int_0^\infty z\sin\left(\frac{z\pi}{2}\right)s_{-1,z}(b)\cos(xz)dzdx.\end{aligned}\tag{2.21}$$

Since the integral on the LHS of (2.21) is absolutely convergent, namely, bounded by $p\pi^2/4$, and equals the integral on the RHS, the result on the RHS is also absolutely convergent. Therefore, by applying Fubini's theorem [12], we can interchange the order of the integration on the RHS. Next, we take the limit as $p\to\infty$. Consequently, the integral over x reduces to (2.16), which, in turn, means that it can be replaced by $\pi\delta(y-z)/2$. Hence the RHS of (2.21) reduces to a one-dimensional integral, and we find that

$$\int_0^{\pi/2}\sin(a\cos x)\cos(b\cos x)dx=-\frac{1}{\pi}\int_0^\infty x\sin(\pi x)s_{0,x}(a)s_{-1,x}(b)dx,\tag{2.22}$$

where the double angle formula for sine has been applied to obtain $\sin(\pi x)$ in the integrand on the RHS.

Alternatively, we can derive (2.22) via the Parseval-Plancherel identity/theorem [13–15], which states that

$$\int_{-\infty}^\infty F(x)\overline{G}(x)dx=\frac{1}{2\pi}\int_{-\infty}^\infty f(\lambda)\overline{g}(\lambda)d\lambda,$$

where the bar denotes the complex conjugate and $f(\lambda)$ and $g(\lambda)$ represent the Fourier transforms of $F(x)$ and $G(x)$, respectively. By introducing the exponential form for $\cos(xt)$ into I_1 and I_2, we can eventually express them as the Fourier transforms, namely,

$$\frac{1}{2}\int_{-\infty}^\infty e^{ixt}\sin(a\cos x)\Theta\left(\frac{\pi^2}{4}-x^2\right)dx=\cos\left(\frac{\pi t}{2}\right)s_{0,t}(a),$$

and

$$\frac{1}{2}\int_{-\infty}^\infty e^{ixt}\cos(a\cos x)\Theta\left(\frac{\pi^2}{4}-x^2\right)dx=-t\sin\left(\frac{\pi t}{2}\right)s_{-1,t}(a).$$

Note that both integrals are also absolutely convergent.

Next, we multiply the integrals by each other and integrate both sides of the resulting over t from $-\infty$ to ∞. Then we obtain

$$\begin{aligned}&\frac{1}{4}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{ixt}\sin(a\cos x)\Theta\left(\frac{\pi^2}{4}-x^2\right)dx\int_{-\infty}^\infty e^{ixt}\cos(a\cos x)\Theta\left(\frac{\pi^2}{4}-x^2\right)dxdt\\&=-\int_{-\infty}^\infty t\sin\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)s_{0,t}(a)s_{-1,t}(b)dt.\end{aligned}$$

Now, we apply the Parseval-Plancherel identity above by replacing the Fourier transforms on the LHS by 2π multiplied by the moduli of the two integrands. Hence we arrive at

$$\frac{2\pi}{4}\int_{-\infty}^\infty\sin(a\cos t)\cos(a\cos t)\Theta\left(\frac{\pi^2}{4}-t^2\right)dt=-\int_{-\infty}^\infty t\sin\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)s_{0,t}(a)s_{-1,t}(b)dt.$$

The Heaviside step function alters the limits of integration to $-\pi/2$ to $\pi/2$, while the factor outside the integral on the LHS is removed by multiplying both sides by $2/\pi$. The factor of 2 on the RHS allows us to apply the double angle formula for sine. Furthermore, since the integrals are even, we can replace the lower limit on both sides by zero. Hence, we obtain (2.22) again. Since we have seen that the first method has yielded the same result as the second and perhaps, more classical, approach, we can be confident of applying the first method to other examples later in this article.

Theorem 2.2. *As a consequence of the preceding material, the following integrals involving products of Lommel functions integrated with respect to their secondary order apply*

$$\int_0^\infty x\sin(\pi x)s_{-1,x}(a)s_{0,x}(b)dx = \frac{\pi^2}{4}(\mathbf{H}_0(b-a)-\mathbf{H}_0(a+b)), \tag{2.23}$$

$$\int_0^\infty \cos^2(\pi x/2)s_{0,x}(a)s_{0,x}(b)dx = \frac{\pi^2}{8}(J_0(|a-b|)-J_0(a+b)), \tag{2.24}$$

and

$$\int_0^\infty x^2\sin^2(\pi x/2)s_{-1,x}(a)s_{-1,x}(b)dx = \frac{\pi^2}{8}(J_0(|a-b|)+J_0(a+b)). \tag{2.25}$$

In these results, as earlier, $\mathbf{H}_\nu(z)$ *denotes the Struve function of order* ν *and variable* z*.*

Proof. To prove (2.23), we require (2.22) and express the product of trigonometric functions in the integrand on the LHS as the difference of two sine functions, i.e.,

$$\sin(a\cos x)\cos(b\cos x) = \frac{1}{2}(\sin((a+b)\cos x)-\sin((b-a)\cos x)). \tag{2.26}$$

Therefore, (2.22) can be expressed as

$$\int_0^\infty x\sin(\pi x)s_{-1,x}(a)s_{0,x}(a)dx = -\frac{\pi}{2}\int_0^{\pi/2}(\sin((a+b)\cos x)-\sin((b-a)\cos x))dx. \tag{2.27}$$

From [1, 8.551], we have

$$\int_0^{\pi/2}\sin(z\cos x)dx = \frac{\pi}{2}\mathbf{H}_0(z).$$

Hence, the RHS of (2.27) becomes the difference of two Struve functions, both with the order equal to zero, but with the argument of the first one being $a+b$, and the argument of the second one, $b-a$. In addition, both functions are multiplied by $-\pi/2$. Thus, we arrive at the first result in the theorem.

If we multiply (2.20) with the corresponding $\sin(b\cos x)$ version of itself, then we obtain

$$\frac{\pi^2}{4}\sin(a\cos x)\sin(a\cos x)\Theta\left(\frac{\pi}{2}-x\right) = \int_0^\infty\int_0^\infty \cos\left(\frac{\pi y}{2}\right)s_{0,y}(a)\cos(xy)\cos\left(\frac{\pi z}{2}\right)s_{0,z}(a)\cos(xz)dydz.$$

Integrating both sides over x from 0 to ∞ yields

$$\frac{\pi^2}{4}\int_0^{\pi/2}\sin(a\cos x)\sin(a\cos x)dx = \int_0^\infty\int_0^\infty \cos\left(\frac{\pi y}{2}\right)s_{0,y}(a)\cos\left(\frac{\pi z}{2}\right)s_{0,z}(a)\int_0^\infty\cos(xy)\cos(xz)dxdydz.$$

By applying (2.16), we replace the integration over x with $\pi\delta(y-z)/2$. Consequently, we arrive at

$$\frac{\pi}{2}\int_0^{\pi/2}\sin(a\cos x)\sin(a\cos x)dx = \int_0^\infty \cos^2\left(\frac{\pi y}{2}\right)s_{0,y}(a)s_{0,y}(b)dy. \tag{2.28}$$

Since the product of the sine functions on the LHS can be expressed as the difference of two cosines, viz.,

$$\sin(a\cos x)\sin(b\cos x) = \frac{1}{2}(\cos((a-b)\cos x) - \cos((a+b)\cos x)),$$

(2.28) becomes

$$\int_0^\infty \cos^2\left(\frac{\pi y}{2}\right) s_{0,y}(a) s_{0,y}(b) dy = \frac{\pi}{4}\int_0^{\pi/2} (\cos((a-b)\cos x) - \cos((a+b)\cos x))dx. \tag{2.29}$$

From [1, 8.411.4], we immediately recognize that the first integral on the RHS is an integral representation of $J_0(a-b)$, while the second represents $J_0(a+b)$. Introducing these results into (2.29) yields (2.24) given in the theorem.

To prove the final result given by (2.25), we adopt a similar approach to the previous proof except on this occasion, and we multiply (2.19) by the $\cos(a\cos x)$. Integrating both sides over x from 0 to ∞ yields

$$\frac{\pi^2}{4}\int_0^{\pi/2} \cos(a\cos x)\cos(b\cos x)dx$$
$$= \int_0^\infty\int_0^\infty yz\sin\left(\frac{\pi y}{2}\right) s_{-1,y}(a)\sin\left(\frac{\pi z}{2}\right) s_{-1,z}(b)\int_0^\infty \cos(xy)\cos(xz)dxdydz.$$

As we have seen already, the integral over x on the RHS yields $\pi\delta(y-z)/2$. Hence, we find that

$$\frac{\pi}{2}\int_0^{\pi/2} \cos(a\cos x)\cos(b\cos x)dx = \int_0^\infty y^2\sin^2\left(\frac{\pi y}{2}\right) s_{-1,y}(a) s_{-1,y}(b)dy. \tag{2.30}$$

The product of the cosines on the LHS of (2.30) can be expressed as the sum of two cosines, one involving $\cos((a+b)\cos x)$ and the other involving $\cos((a-b)\cos x)$. Thus, the integral on the RHS is equal to the sum of the same two Bessel functions in the proof of (2.24). Hence, we arrive at (2.26). □

It should be mentioned that if one wishes to check the results in Theorem 2.2, then it cannot be done directly since integration routines are unable to handle integrating products of the Lommel functions over infinity. For example, when one applies the Integrate routine to the LHS of (2.23) after expressing the Lommel functions as hypergeometric functions as in (1.2), Mathematica will only print out the entire integral again. However, from (2.22), we can express the LHS of (2.23) in Mathematica as

$f[a\_, b\_]$:= Integrate[Sin[$(a+b)$ Cos[x]] − Sin[$(b-a)$ Cos[x]], {x, 0, $Pi/2$}]/2.

Then typing f[a,b] yields

$-(1/4)\pi^2$ (StruveH[0, $a-b$] + StruveH[0, $a+b$]),

which equals the RHS of (2.23) since $\mathbf{H}_0(-x) = -\mathbf{H}_0(x)$. As an aside, it should be mentioned that Mathematica is unable to integrate the LHS of (2.23) directly. Moreover, if one attempts a numerical integration using the NIntegrate instruction, then one obtains spurious or unreliable results due to the software experiencing convergence problems.

For the interesting case of $a = b$, the results in Theorem 2.2 reduce to

$$\int_0^\infty x\sin(\pi x) s_{-1,x}(a) s_{0,x}(a)dx = -\frac{\pi^2}{4}\mathbf{H}_0(2a),$$

$$\int_0^\infty \cos^2(\pi x/2) s_{0,x}^2(a)dx = \frac{\pi^2}{8}(1 - J_0(2a)),$$

and

$$\int_0^\infty x^2\sin^2(\pi x/2) s_{-1,x}^2(a)dx = \frac{\pi^2}{8}(1 + J_0(2a)).$$

To our knowledge, this is the first time that products/squares of Lommel functions have appeared inside an integral over one of its indices/orders.

3 Chebyshev polynomials of the first kind

As a consequence of the results in the previous section, we are now in a position to consider expressing Chebyshev polynomials of the first kind or $T_n(x)$ in terms of integrals involving the Lommel function, and *vice-versa*.

Theorem 3.1. *The Chebyshev polynomials of the first kind and Lommel functions represent the Fourier sine and cosine transforms of one another as demonstrated by the following results*:

$$\int_0^\infty \sin(ut)s_{0,2n}(t)dt = \frac{(-1)^n \pi T_{2n}(u)}{2\sqrt{1-u^2}}\Theta(1-u), \tag{3.1}$$

$$\int_0^\infty \cos(ut)s_{-1,2n+1}(t)dt = \frac{(-1)^{n+1} \pi T_{2n+1}(u)}{2(2n+1)\sqrt{1-u^2}}\Theta(1-u), \tag{3.2}$$

$$\int_0^1 \sin(ut)T_{2n}(u)\frac{du}{\sqrt{1-u^2}} = (-1)^n s_{0,2n}(t), \tag{3.3}$$

and

$$\int_0^1 \cos(ut)T_{2n+1}(u)\frac{du}{\sqrt{1-u^2}} = (-1)^{n+1}(2n+1)s_{-1,2n+1}(t). \tag{3.4}$$

In the first result, u cannot equal zero since the integral vanishes, while the RHS is finite, i.e., the integral is discontinuous at $u = 0$. Furthermore, from the proof, we see that $\Re u > 0$ for (3.1) *and* (3.2). *This, however, does not apply to the other results in the theorem, where $\Re t \geq 0$ can equal zero.*

Remark 3.1. The last two results in the theorem complement the two integrals in No. 2.18.1.17 of [16], which, for $a = 1$ and $b = t$, are

$$\int_0^1 \sin(ut)\frac{T_{2n+1}(u)}{\sqrt{1-u^2}}du = (-1)^n \frac{\pi}{2} J_{2n+1}(t). \tag{3.5}$$

and

$$\int_0^1 \cos(ut)\frac{T_{2n}(u)}{\sqrt{1-u^2}}du = (-1)^n \frac{\pi}{2} J_{2n}(t). \tag{3.6}$$

Remark 3.2. If we multiply (3.5) by $\sin(yt)$ and integrate over t from 0 to p, then we can interchange the order of the integrations in accordance with Fubini's theorem since both integrals are finite. Hence, we obtain

$$\int_0^p \sin(yt)J_{2n+1}(t)dt = \frac{2(-1)^n}{\pi}\int_0^1 \frac{T_{2n+1}(u)}{\sqrt{1-u^2}}\int_0^p \sin(ut)\sin(yt)dtdu.$$

By taking the limit as $p \to \infty$, we can apply (2.17) and introduce the Dirac delta function [8,9]. Employing the integral property of the delta function, viz., (2.18), we find that

$$\int_0^\infty \sin(yt)J_{2n+1}(t)dt = \frac{2(-1)^n}{\pi}\frac{T_{2n+1}(y)}{\sqrt{1-y^2}}\Theta(1-y).$$

On the other hand, if we multiply (3.6) by $\cos(yt)$ and adopt the same procedure except employ (2.16), then we eventually obtain

$$\int_0^\infty \cos(yt)J_{2n}(t)dt = \frac{2(-1)^n}{\pi}\frac{T_{2n}(y)}{\sqrt{1-y^2}}\Theta(1-y).$$

It should also be mentioned that these results can be extracted from Sec. 2.12.15 in [16].

Proof. We shall derive the aforementioned results by employing the orthogonality as in the previous section. To obtain the first result in Theorem 3.1, we put $y = a$ and make the substitution, $x = \arccos u$ in (2.11). This yields

$$\int_0^1 \sin(au)\cos(y\arccos u)\frac{du}{\sqrt{1-u^2}} = \cos\left(\frac{\pi y}{2}\right)s_{0,y}(a). \tag{3.7}$$

Note that (3.7) is now valid for y, an odd integer, because the singularity in $s_{0,y}(a)$ is removed by the zero in the cosine factor preceding it. As an aside, if one were to introduce the power series expansion for sine on the LHS and introduce (1.2) on the RHS, then one would obtain a different version of (2.4) when equating like powers of a.

Returning to (3.7), we now multiply it by $\sin(at)$ and integrate over a from 0 to p. Therefore, we arrive at

$$\int_0^p \sin(at)\int_0^1 \sin(au)\cos(y\arccos u)\frac{du}{\sqrt{1-u^2}}da = \cos\left(\frac{\pi y}{2}\right)\int_0^p \sin(at)s_{0,y}(at)da.$$

Since both integrals on the LHS are both finite, we can interchange their order in accordance with Fubini's theorem [12]. Then the above equation becomes

$$\int_0^1 \frac{\cos(y\arccos u)}{\sqrt{1-u^2}}\int_0^p \sin(at)\sin(au)dadu = \cos\left(\frac{\pi y}{2}\right)\int_0^p \sin(at)s_{0,y}(at)da.$$

In the limit as $p \to \infty$, the inner integral on the LHS becomes (2.17). Then we replace the integral by introducing the RHS of (2.17). With the aid of (2.18), we arrive at

$$\frac{\cos(y\arccos u)}{\sqrt{1-u^2}}\Theta(1-u) = \frac{2}{\pi}\cos\left(\frac{\pi y}{2}\right)\int_0^\infty \sin(ut)s_{0,y}(t)dt, \quad u \neq 0, \tag{3.8}$$

where the Heaviside step-function arises because the range of integration over u is between 0 and unity rather than zero and infinity. If we let $y = 2n$ in (3.8), where n is an integer, then according to [17], the cosine on the LHS becomes $T_{2n}(y)$. After a little algebraic manipulation, we end up with the first result in the theorem. Moreover, (3.8) is not valid for $u = 0$ because the LHS is finite, whereas the integral on the RHS vanishes.

Similarly, to obtain the second result in the theorem, we put $y = a$, $b = a$ and make the same change of variable as before in (2.9). Then we obtain

$$\int_0^1 \cos(bu)\cos(y\arccos u)\frac{du}{\sqrt{1-u^2}} = -y\sin\left(\frac{\pi y}{2}\right)s_{-1,y}(b). \tag{3.9}$$

As in the case of (3.7), by introducing power series expansion for $\cos(bu)$ and $s_{-1,y}(b)$, one would once again arrive at (2.4) after equating like powers of b.

Now, we multiply (3.9) by $\cos(at)$ and integrate over a from 0 to p. We carry out the same procedure as in the first result except we use (2.16). Thus, we arrive at

$$\frac{\cos(y\arccos u)}{\sqrt{1-u^2}}\Theta(1-u) = -\frac{2y}{\pi}\sin\left(\frac{\pi y}{2}\right)\int_0^\infty \cos(ut)s_{-1,y}(t)dt.$$

Putting $y = 2n + 1$ and carrying out a little algebraic manipulation, we arrive at the second result in the theorem.

The third result in the theorem is derived by multiplying both sides of the first result by $\sin(ux)$. By integrating over u from 0 to p, interchanging the double integrals, taking the limit as $p \to \infty$, and utilizing (2.17), one arrives at (3.3) after some algebra.

The last result in the theorem is derived by carrying out the same procedure as the preceding result except that (2.16) is used. □

If we put $u = 0$ in (3.2), then we find that $\int_0^\infty s_{-1,2n+1}(t)dt = 0$, since $T_{2n+1}(0) = 0$. Similarly, both sides of (3.3) vanish for $t = 0$. However, for $t = 0$ in (3.4), we obtain

$$\int_0^1 T_{2n+1}(u)\frac{du}{\sqrt{1-u^2}} = (-1)^{n+1}(2n+1)s_{-1,2n+1}(0).$$

From (1.2), we have $s_{-1,2n+1}(0) = -1/(2n+1)^2$. Hence, we find that

$$\int_0^1 T_{2n+1}(u)\frac{du}{\sqrt{1-u^2}} = \frac{(-1)^n}{2n+1}.$$

This result can be verified by noting that integral is merely the $\nu = 1$ result of (2.5), which is, in turn, the $a = n$ result of (2.4). Therefore, by putting $a = n$ and $\nu = 1$ in (2.4), we obtain the same result after introducing the reflection formula for the gamma function given by either [1, 8.334.3] or [6, 5.5.3].

We can also derive other interesting integrals with the aid of (3.5) and (3.6). In the case of (3.5), we multiply both sides by $\sin(yt)$ and integrate over t from 0 to p. Interchanging the order of the integrations on the LHS in accordance with Fubini's theorem [12,18], and taking the limit as $p \to \infty$, we can use (2.17) to replace the inner integral on the LHS by $\pi\delta(y-u)/2$. This yields

$$\int_0^\infty \sin(yt)J_{2n+1}(t)dt = \frac{(-1)^n T_{2n+1}(y)}{\sqrt{1-y^2}}\Theta(1-y),$$

where we note that both sides vanish for $y = 0$ since $T_{2n+1}(0) = 0$. Now, we consider the second result in Theorem 3.1. Then we find that

$$\int_0^\infty \cos(yt)s_{-1,2n+1}(t)dt = -\frac{\pi}{4n+2}\int_0^\infty \sin(yt)J_{2n+1}(t)dt. \tag{3.10}$$

In a similar manner to the previous example, we multiply (3.6) by $\cos(yt)$ and carry out the same procedure leading to (3.10). In this instance, however, we apply the first result in Theorem 3.1 or (3.1). Thus, we arrive at

$$\frac{2}{\pi}\int_0^\infty \sin(yt)s_{0,2n}(t)dt = \int_0^\infty \cos(yt)J_{2n}(t)dt = (-1)^n\frac{T_{2n}(y)}{\sqrt{1-y^2}}\Theta(1-y),$$

where the first result follows from (3.1). For $y = 0$, the second and third members of the aforementioned result are equal to one another since $T_{2n}(0) = (-1)^n$ and the Bessel integral equals unity, but this does not apply to the first expression since we have already observed that it is not valid when $y = 0$. A more general version of this result can be obtained by: (1) multiplying the upper version of [16, 2.18.1.17] (mentioned in Theorem 3.1) by $\sin(by)$ and the lower entry by $\cos(by)$, (2) integrating over b from zero to ∞, (3) interchanging the order of integrals on the LHS and (4) applying (2.17) and (2.16) as has been done throughout this study. This gives

$$\int_0^\infty \left\{\begin{matrix}\sin(by)J_{2n+1}(ab)\\ \cos(by)J_{2n}(ab)\end{matrix}\right\}db = \frac{(-1)^n}{\sqrt{a^2-y^2}}\left\{\begin{matrix}T_{2n+1}(y/a)\\ T_{2n}(y/a)\end{matrix}\right\}.$$

4 Conclusion

In this study, we have presented new results for integrals (including index/order integrals) of the univariate Lommel function, $s_{\mu,\nu}(z)$, with specific indices and shown their connection to the Chebyshev polynomials of the first kind. Since there are only a few identities involving Lommel functions, these can be greatly increased by re-expressing identities for Chebyshev polynomials of the first kind in terms of $s_{\mu,\nu}(z)$, which was the main motivation behind this study. Finally, it is hoped that the methods and results presented here will enable the derivation of more results involving these esoteric special functions in the future.

Acknowledgement: The authors thank the reviewers for their careful reading and comments, which improved the manuscript.

Funding information: MLG acknowledges partial support from the Spanish Ministerio de Ciencia e Innovación with funding from the European Union NextGenerationEU (PRTRC17.I1) and the Consejeria de Educación of the Junta de Castilla y León through the QCAYLE Project, in addition to MCIN Project PID2020-113406GB-I00. CMdaF and VK received no funding support.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results, and approved the final version of the manuscript. All authors were involved throughout the preparation of the manuscript and contributed to the final version.

Conflict of interest: MLG and VK declare no conflict of interest. CMdaF is an Editor of Open Mathematics, but was not involved in the review and decision-making process of this article.

References

[1] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., Elsevier Academic Press, Oxford, 2007.
[2] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series*, vol. 3: More Special Functions, Gordon and Breach, New York, 1989.
[3] S. B. Yakubovich, *Integral transformations by the index of Lommel's function*, Period. Math. Hungar. **46** (2003), 223–233.
[4] H. Bateman, A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions*, vol. II, McGraw-Hill Book Company, New York, 1953.
[5] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd edn., Cambridge University Press, Cambridge, 1995.
[6] F. W. J. Olver, et al., *NIST Digital Library of Mathematical Functions*, https://dlmf.nist.gov/, release 1.1.10 of 2023-06-15.
[7] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.
[8] Wikipedia Contributors, The Free Encyclopedia, Dirac delta function, https://en.wikipedia.org/wiki/Dirac_delta_function, accessed Feb. 22, 2023.
[9] M. J. Lighthill, *Fourier Analysis and Generalised Functions*, Cambridge University Press, Cambridge, 1975, Ch. 1.
[10] V. Kowalenko, *Employing transform orthogonality in the determination of integral identities involving arbitrary functions*, in: D. Bainov (Ed.), Proceedings of the Eighth International Colloquium on Differential Equations, VSP, Utrecht, 1997, pp. 261–273.
[11] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953, p. 943.
[12] Wikipedia, The Free Encyclopedia, Fubini's Theorem, https://en.wikipedia.org/wiki/Fubinis_theorem, accessed Feb. 20, 2023.
[13] M. R. Spiegel, *Theory and Problems of Laplace Transforms*, McGraw-Hill, New York, 1976, Ch. 6.
[14] E. W. Weisstein, *Plancherel's Theorem, MathWorld - A Wolfram Web Resource*, https://mathworld.wolfram.com/PlancherelsTheorem.html.
[15] E. W. Weisstein, *Parseval's Theorem, MathWorld - A Wolfram Web Resource*, https://mathworld.wolfram.com/ParsevalsTheorem.html.
[16] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series*, vol. 2: Special Functions, Gordon and Breach, 1986.
[17] E. W. Weisstein, *Chebyshev Polynomials of the First Kind*, MathWorld-A Wolfram Web Resource, https://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.
[18] E. DiBenedetto, *Real Analysis*, 2nd edn., Birkhäuser, Basel, 2016, Ch. 4.