Research Article

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The regularity of solutions to the L_p Gauss image problem

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Abstract: The L_p Gauss image problem amounts to solving a class of Monge-Ampère type equations on the sphere. In this article, we discuss the regularity of solutions to the L_p Gauss image problem.

Keywords: L_p Gauss image measure, L_p Gauss image problem, Monge-Ampère equation, regularity of solution

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1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. The symbol o denotes the origin in $\mathbb{R}^n$. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$. A convex body in $\mathbb{R}^n$ is a compact convex set with non-empty interior. Denote by $\mathcal{K}_o^n$ the set of all convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. We write $\mathcal{H}^k$ for k-dimensional Hausdorff measure in $\mathbb{R}^n$.

The Gauss image measure was discovered by Böröczky et al. in their groundbreaking work [1]. Let λ be an absolutely continuous Borel measure on $\mathbb{S}^{n-1}$. For a convex body $K \in \mathcal{K}_o^n$, the Gauss image measure $\lambda(K, \cdot)$ of λ via K is a spherical Borel measure defined by

$$\lambda(K, \omega) = \lambda(\boldsymbol{\alpha}_K(\omega))$$

for each Borel set $\omega \subset \mathbb{S}^{n-1}$. Here, $\boldsymbol{\alpha}_K$ is the radial Gauss image. We write ρ_K to denote the radial function of a convex body $K \in \mathcal{K}_o^n$. For $p \in \mathbb{R}$, the L_p Gauss image measure was introduced in [2] and can be defined by

$$\mathrm{d}\lambda_p(K, \cdot) = \rho_K^p \mathrm{d}\lambda(K, \cdot).$$

A characterisation problem for the L_p Gauss image measure is called the L_p Gauss image problem recently proposed by Wu et al. [2]. Such type of problem is an analogue of the L_p Minkowski problem concerning the L_p surface area measure. The L_p Minkowski problem and its related problems have been extensively studied in last three decades; see [3–52]. The L_p Gauss image problem asks what are the necessary and sufficient conditions for a Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$ to be the L_p Gauss image measure of a unique convex body K. Namely, this problem is to find a convex body $K \subset \mathbb{R}^n$ such that

$$\mu = \lambda_p(K, \cdot)$$

on $\mathbb{S}^{n-1}$, and if such a body exists, to what extent is it unique?

It will be seen that when μ has a density f and λ has a density g, the L_p Gauss image problem is equivalent to solving the following Monge-Ampère equation on $\mathbb{S}^{n-1}$:

$$g\left(\frac{\nabla h + hv}{|\nabla h + hv|}\right)\frac{h^{1-p}}{(|\nabla h|^2 + h^2)^{\frac{n}{2}}}\det(\nabla^2 h + hI) = f(v), \tag{1.1}$$

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where $h:\mathbb{S}^{n-1}\to\mathbb{R}$ is the unknown function on $\mathbb{S}^{n-1}$, ∇ is the covariant derivative with respect to an orthonormal frame on $\mathbb{S}^{n-1}$, and I is the unit matrix of order $n-1$.

The case where $p=0$ is the Gauss image problem. The existence and uniqueness results of its solutions were established in [1], and the existence of smooth solutions was obtained in [16]. In [2], the L_p Gauss image problem was solved for $p>0$, while for $p<0$, it was solved in the even case. The Gauss image problem was very recently developed to the Musielak-Orlicz case in [23]. The corresponding Minkowski problem in this setting may be called the Musielak-Orlicz-Gauss image problem. The existence of solutions to this problem was studied in [23] using a variational argument. Alternative approach based on a parabolic flow was provided in [53].

In this article, we will consider the regularity of the solution for the L_p Gauss image problem, which are inspired by the recent and important works of [3,5,54,55]. The support function and polar body of a convex body $K\in\mathcal{K}_o^n$ are denoted by h_K and K^*, respectively. Then our regularity result can be stated as follows:

Theorem 1.1. *Suppose that* $\mathrm{d}\mu=f\mathrm{d}\mathcal{H}^{n-1}$ *and* $\mathrm{d}\lambda=g\mathrm{d}\mathcal{H}^{n-1}$ *with* $0<c_1\leqslant f,g\leqslant c_2$ *on* $\mathbb{S}^{n-1}$. *For* $p\in\mathbb{R}$, *let* $K\in\mathcal{K}_o^n$ *satisfy* $\mathrm{d}\lambda_p(K^*,\cdot)=f\mathrm{d}\mathcal{H}^{n-1}$ *on* $\mathbb{S}^{n-1}$. *Then*
- (i) ∂K *is* C^1 *and strictly convex, and* h_K *is* C^1 *on* $\mathbb{R}^n\backslash\{o\}$;
- (ii) *if* f,g *are both continuous, then the restriction of* h_K *to* $\mathbb{S}^{n-1}$ *is in* $C^{1,\beta}$ *for any* $\beta\in(0,1)$;
- (iii) *if* $f,g\in C^\beta(\mathbb{S}^{n-1})$ *for* $\beta\in(0,1)$, *then* h_K *is* $C^{2,\beta}$ *on* $\mathbb{S}^{n-1}$.

The organisation of the article is as follows. In Section 2, we list some notions and basic facts regarding convex bodies and the L_p Gauss image measure. In Section 3, we will establish Theorem 1.1 according to the famous regularity results by Caffarelli [56,57].

2 Preliminaries

2.1 Basics regarding convex bodies

In this section, we introduce some basic facts and notions about convex bodies which will be used later. For general references, see the books of Gardner [58] and Schneider [59], and the references of [1,21,37].

The Euclidean norm and inner product on $\mathbb{R}^n$ are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. For $x\in\mathbb{R}^n\backslash\{o\}$, we will use $\bar{x}$ to abbreviate $\frac{x}{|x|}$. Denote by ∂K and $\mathrm{cl}K$ the boundary and closure of a convex body K, respectively.

Associated to each convex body $K\in\mathcal{K}_o^n$ are the support function $h=h_K:\mathbb{S}^{n-1}\to\mathbb{R}$ and the radial function $\rho=\rho_K:\mathbb{S}^{n-1}\to\mathbb{R}$, which are respectively defined by

$$h(v)=\max\{\langle v,y\rangle: y\in K\},\quad \rho(u)=\max\{\lambda:\lambda u\in K\}.$$

We easily see that $\rho_K(u)u\in\partial K$ for all $u\in\mathbb{S}^{n-1}$.

The polar body of $K\in\mathcal{K}_o^n$ is defined by

$$K^*=\{x\in\mathbb{R}^n:\langle x,y\rangle\le 1 \text{ for all } y\in K\}.$$

It easily follows from this definition that $K^*\in\mathcal{K}_o^n$ and $(K^*)^*=K$. Moreover,

$$\rho_K=1/h_{K^*},\quad h_K=1/\rho_{K^*}. \tag{2.1}$$

For each $v\in\mathbb{S}^{n-1}$, the supporting hyperplane $H_K(v)$ of $K\in\mathcal{K}_o^n$ is defined as follows:

$$H_K(v)=\{x\in\mathbb{R}^n:\langle x,v\rangle=h_K(v)\}.$$

Let $\sigma\subset\partial K$ with $K\in\mathcal{K}_o^n$. The spherical image of σ is given by

$$\boldsymbol{\nu}_K(\sigma)=\{v\in\mathbb{S}^{n-1}:x\in H_K(v) \text{ for some } x\in\sigma\}\subset\mathbb{S}^{n-1}.$$

Suppose that $\sigma_K \subset \partial K$ is the set consisting of all $x \in \partial K$ for which the set $\boldsymbol{\nu}_K(\{x\})$, often abbreviated as $\boldsymbol{\nu}_K(x)$, contains more than a single element. As is well known that $\mathcal{H}^{n-1}(\sigma_K) = 0$ (see Schneider [59, p. 84]). Suppose that for each $x \in \partial K \backslash \sigma_K$, $\nu_K(x)$ is the unique element in $\boldsymbol{\nu}_K(x)$. Therefore, we define the function

$$\nu_K : \partial K \backslash \sigma_K \to \mathbb{S}^{n-1},$$

which is called the spherical image map (also known as the Gauss map) of K. Sometimes it is convenient to write $\partial K \backslash \sigma_K$ by $\partial' K$.

The reverse spherical image $\boldsymbol{\nu}_K^{-1}$, of $K \in \mathcal{K}_o^n$ at $\eta \subset \mathbb{S}^{n-1}$, is defined as follows:

$$\boldsymbol{\nu}_K^{-1}(\eta) = \{x \in \partial K : x \in H_K(v) \text{ for some } v \in \eta\} \subset \partial K.$$

The set $\eta_K \subset \mathbb{S}^{n-1}$ consisting of all $v \in \mathbb{S}^{n-1}$ for which the set $\boldsymbol{\nu}_K^{-1}(v) = \boldsymbol{\nu}_K^{-1}(\{v\})$ contains more than a single element is of $\mathcal{H}^{n-1}$-measure 0 (see Schneider [59, Theorem 2.2.11]). $\boldsymbol{\nu}_K^{-1}(v)$ has the unique element for $v \in \mathbb{S}^{n-1} \backslash \eta_K$, which is denoted by $\nu_K^{-1}(v)$. Thus, we define the reverse spherical image map by

$$\nu_K^{-1} : \mathbb{S}^{n-1} \backslash \eta_K \to \partial K,$$

From Lemma 2.2.12 of Schneider [59], it is continuous.

The radial Gauss image of $K \in \mathcal{K}_o^n$ for a Borel set $\omega \subset \mathbb{S}^{n-1}$, denoted by $\boldsymbol{\alpha}_K(\omega)$, is defined as follows:

$$\boldsymbol{\alpha}_K(\omega) = \{v \in \mathbb{S}^{n-1} : \rho_K(u)u \in H_K(v) \text{ for some } u \in \omega\} \subset \mathbb{S}^{n-1}.$$

If $\omega = \{u\}$ is a singleton, we frequently write $\boldsymbol{\alpha}_K(u)$ rather than $\boldsymbol{\alpha}_K(\{u\})$. Set $\omega_K = \{u \in \mathbb{S}^{n-1} : \rho_K(u)u \in \sigma_K\} \subset \mathbb{S}^{n-1}$. Apparently, for each $u \in \omega_K$, $\boldsymbol{\alpha}_K(u)$ contains more than one element. Since $\mathcal{H}^{n-1}(\omega_K) = 0$ from Theorem 2.2.5 of [59], the radial Gauss map of K (denoted by α_K) is the map which is defined on $\mathbb{S}^{n-1} \backslash \omega_K$ that takes each point u in its domain to the unique vector in $\boldsymbol{\alpha}_K(u)$. Therefore, with respect to the spherical Lebesgue measure, α_K is defined almost everywhere on $\mathbb{S}^{n-1}$.

The reverse radial Gauss image $\boldsymbol{\alpha}_K^*(\eta)$, for $K \in \mathcal{K}_o^n$ with a Borel set $\eta \subset \mathbb{S}^{n-1}$, is defined as follows:

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in \mathbb{S}^{n-1} : \rho_K(u)u \in H_K(v) \text{ for some } v \in \eta\} \subset \mathbb{S}^{n-1}.$$

Analogously, we write $\boldsymbol{\alpha}_K^*(v)$ rather than $\boldsymbol{\alpha}_K^*(\{v\})$ for $\eta = \{v\}$. Apparently, $\boldsymbol{\alpha}_K^*(v)$ has the unique element denoted by $\alpha_K^*(v)$ with $v \in \mathbb{S}^{n-1} \backslash \eta_K$. Therefore, we can define the reverse radial Gauss image map

$$\alpha_K^* : \mathbb{S}^{n-1} \backslash \eta_K \to \mathbb{S}^{n-1}.$$

Thus, α_K^* is defined almost everywhere on $\mathbb{S}^{n-1}$ because the set η_K has spherical Lebesgue measure 0.

According to the definitions of $\boldsymbol{\alpha}_K$ and $\boldsymbol{\alpha}_K^*$, it is not difficult to see that for all $\lambda > 0$,

$$\boldsymbol{\alpha}_{\lambda K} = \boldsymbol{\alpha}_K, \quad \boldsymbol{\alpha}_{\lambda K}^* = \boldsymbol{\alpha}_K^*. \tag{2.2}$$

It was proved in [21] that if $K \in \mathcal{K}_o^n$, then for each $\eta \subset \mathbb{S}^{n-1}$,

$$\boldsymbol{\alpha}_K^*(\eta) = \boldsymbol{\alpha}_{K^*}(\eta), \tag{2.3}$$

and if $v \notin \eta_K$, then

$$v \in \boldsymbol{\alpha}_K(\eta) \Leftrightarrow \alpha_K^*(v) \in \eta. \tag{2.4}$$

2.2 Basics for L_p Gauss image measure

For a Borel set $\omega \subset \mathbb{S}^{n-1}$, the surface area measure S_K of a convex body K is a Borel measure on $\mathbb{S}^{n-1}$ which is defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)) = \mathcal{H}^{n-1}(\{x \in \partial K : \nu_K(x) \cap \omega \neq \varnothing\}). \tag{2.5}$$

The following integral representation was given in [1]: If λ is an absolutely continuous Borel measure and $K \in \mathcal{K}_o^n$, then

$$\int_{\mathbb{S}^{n-1}} f(u)\mathrm{d}\lambda(K,u) = \int_{\mathbb{S}^{n-1}} f(\alpha_K^*(v))\mathrm{d}\lambda(v) \tag{2.6}$$

for each bounded Borel $f : \mathbb{S}^{n-1} \to \mathbb{R}$.

It has been proved in [21] that for $K \in \mathcal{K}_o^n$ and each bounded Lebesgue integrable function $f : \mathbb{S}^{n-1} \to \mathbb{R}$,

$$\int_{\mathbb{S}^{n-1}} f(u)\rho_K^n(u)\mathrm{d}\mathcal{H}^{n-1}(u) = \int_{\partial' K} \langle x, \nu_K(x)\rangle f(\bar{x})\mathrm{d}\mathcal{H}^{n-1}(x). \tag{2.7}$$

Let $p \in \mathbb{R}$. The L_p Gauss image measure of $K \in \mathcal{K}_o^n$ is given, in [2], by

$$\int_{\mathbb{S}^{n-1}} f(u)\mathrm{d}\lambda_p(K,u) = \int_{\mathbb{S}^{n-1}} f(\alpha_K^*(v))\rho_K^p(\alpha_K^*(v))\mathrm{d}\lambda(v) \tag{2.8}$$

for each continuous $f : \mathbb{S}^{n-1} \to \mathbb{R}$. By combining (2.6) with (2.8), we see

$$\mathrm{d}\lambda_p(K,\cdot) = \rho_K^p \mathrm{d}\lambda(K,\cdot).$$

Together (2.4) with (2.8), it can also be, equivalently, written by

$$\lambda_p(K,\omega) = \int_{\omega} \rho_K^p(u)\mathrm{d}\lambda(K,u) = \int_{\boldsymbol{\alpha}_K(\omega)} \rho_K^p(\alpha_K^*(v))\mathrm{d}\lambda(v)$$

for each Borel $\omega \subset \mathbb{S}^{n-1}$.

Let $u = \bar{x}$ with $x \in \partial K$ for $K \in \mathcal{K}_o^n$, and $\mathrm{d}\lambda = g\mathrm{d}\mathcal{H}^{n-1}$ with $g : \mathbb{S}^{n-1} \to [0,\infty)$. Replacing K by K^* in (2.8), and using (2.3) and the fact that $(K^*)^* = K$, it follows that

$$\int_{\mathbb{S}^{n-1}} f(v)\mathrm{d}\lambda_p(K^*,v) = \int_{\mathbb{S}^{n-1}} f(\alpha_K(u))\rho_{K^*}^p(\alpha_K(u))g(u)\mathrm{d}\mathcal{H}^{n-1}(u). \tag{2.9}$$

By substituting $f\rho_K^{-n}$ for f in (2.7), we obtain

$$\int_{\mathbb{S}^{n-1}} f(u)\mathrm{d}\mathcal{H}^{n-1}(u) = \int_{\partial' K} \langle x, \nu_K(x)\rangle |x|^{-n} f(\bar{x})\mathrm{d}\mathcal{H}^{n-1}(x). \tag{2.10}$$

Thus, it follows from (2.9), (2.10), and (2.1) that

$$\int_{\mathbb{S}^{n-1}} f(v)\mathrm{d}\lambda_p(K^*,v) = \int_{\partial' K} \langle x, \nu_K(x)\rangle \frac{f(\nu_K(x))g(\bar{x})}{|x|^n h_K^p(\nu_K(x))}\mathrm{d}\mathcal{H}^{n-1}(x). \tag{2.11}$$

For $K \in \mathcal{K}_o^n$, we use Dh_K to denote the gradient of h_K in $\mathbb{R}^n$. If h_K is viewed as restricted to the unit sphere $\mathbb{S}^{n-1}$, then the gradient of h_K on $\mathbb{S}^{n-1}$ is written by ∇h_K. Since h_K is differentiable at $\mathcal{H}^n$ almost all points in $\mathbb{R}^n$ and is positively homogeneous of degree 1, h_K is differentiable for $\mathcal{H}^{n-1}$ almost all points of $\mathbb{S}^{n-1}$. We suppose that h_K is differentiable at $v \in \mathbb{S}^{n-1}$ and $v = \nu_K(x)$ is an outer unit normal vector at $x \in \partial K$. Then it follows that

$$x = \nu_K^{-1}(v) = Dh_K(v). \tag{2.12}$$

From this, we easily see

$$h_K(v) = h_K(\nu_K(x)) = \langle x, \nu_K(x)\rangle = \langle Dh_K(v), v\rangle, \tag{2.13}$$

$$x = Dh_K(v) = \nabla h_K(v) + h_K(v)v, \tag{2.14}$$

$$|Dh_K(v)|^2 = h_K^2(v) + |\nabla h_K(v)|^2. \tag{2.15}$$

It follows from (2.5), (2.12), (2.13), (2.14), and (2.15) that for $v \in \mathbb{S}^{n-1}$, the integral representation (2.11) implies

$$\mathrm{d}\lambda_p(K^*, v) = g\left(\frac{\nabla h_K(v) + h_K(v)v}{|\nabla h_K(v) + h_K(v)v|}\right)\frac{h_K^{1-p}(v)}{(h_K^2(v) + |\nabla h_K(v)|^2)^{\frac{n}{2}}}\mathrm{d}S(K, v). \tag{2.16}$$

If $K \in \mathcal{K}_o^n$ has a C^2 boundary with everywhere positive Gauss curvature, then for $v \in \mathbb{S}^{n-1}$,

$$\mathrm{d}S(K, v) = \det(\nabla^2 h_K(v) + h_K(v)I)\mathrm{d}\mathcal{H}^{n-1}(v). \tag{2.17}$$

Therefore, we deduce from (2.16) and (2.17) that if μ has a non-negative function f, i.e. $\mathrm{d}\mu = f\mathrm{d}\mathcal{H}^{n-1}$ on $\mathbb{S}^{n-1}$, then the L_p Gauss image problem can be formulated as finding solutions to the following Monge-Ampère equation on $\mathbb{S}^{n-1}$:

$$g\left(\frac{\nabla h^* + h^* v}{|\nabla h^* + h^* v|}\right)\frac{h^{*1-p}}{(|\nabla h^*|^2 + h^{*2})^{\frac{n}{2}}}\det(\nabla^2 h^* + h^* I) = f(v),$$

where $h^* = h_{K^*}$.

3 The regularity of the solution

This section is devoted to the study of the regularity of solutions to the L_p Gauss image problem. Namely, we will prove Theorem 1.1. Let us first recall some basic notions and facts required in this section. We refer to the papers [3] and [5] for more details.

In the following, we assume that K is a convex body. If ∂K contains no segment, then we say that K is strictly convex; if K has a unique tangential hyperplane at $x \in \partial K$, then we say that x is a C^1-smooth point. Apparently, h_K is C^1 on $\mathbb{S}^{n-1}$ if and only if K is strictly convex. In addition, ∂K is C^1 if and only if each $x \in \partial K$ is C^1-smooth.

Let Ω be a convex set in $\mathbb{R}^n$. We say that $z \in \Omega$ is an extremal point if $z = \lambda x_1 + (1-\lambda)x_2$ for $x_1, x_2 \in \Omega$ and $\lambda \in (0,1)$ implies that $x_1 = x_2 = z$.

The normal cone of a convex body K at $z \in K$ is defined by

$$N(K, z) = \{x \in \mathbb{R}^n : \langle x, y\rangle \leqslant \langle x, z\rangle \text{ for all } y \in K\},$$

which is equivalent to

$$N(K, z) = \{x \in \mathbb{R}^n : h_K(x) = \langle x, z\rangle\}.$$

When $z \in \mathrm{int}K$, we have $N(K, z) = \{o\}$, and when $z \in \partial K$ it follows that $\dim N(K, z) \geqslant 1$.

The face of K with outer normal $x \in \mathbb{R}^n$ is given by

$$F(K, x) = \{z \in K : h_K(x) = \langle x, z\rangle\}, \tag{3.1}$$

which lies in ∂K provided $x \neq o$, and

$$F(K, x) = \partial h_K(x). \tag{3.2}$$

Here, $\partial h_K(x)$ is the subgradient of h_K, which is defined by

$$\partial h_K(x) = \{z \in \mathbb{R}^n : h_K(y) \geqslant h_K(x) + \langle z, y - x\rangle \text{ for each } y \in K\},$$

Obviously, it is a non-empty compact convex set. Note that $h_K(x)$ is differentiable at x if and only if $\partial h_K(x)$ consists of exactly one vector which is the gradient of h_K at x.

Let φ be a convex function defined in an open convex set Ω of $\mathbb{R}^n$. We use $D\varphi$ and $D^2\varphi$ to denote its gradient and its Hessian, respectively. Besides, we define

$$N_\varphi(\vartheta) = \bigcup_{x\in\vartheta} \partial\varphi(x),$$

for any Borel subset $\vartheta \subset \Omega$. The Monge-Ampère measure μ_φ is $\mu_\varphi(\vartheta) = \mathcal{H}^n(N_\varphi(\vartheta))$. Let φ be C^2 smooth. Then the subgradient $\partial\varphi$ is equal to the gradient $D\varphi$. Thus, it follows that

$$\mu_\varphi(\vartheta) = \mathcal{H}^n(D\varphi(\vartheta)) = \int_\vartheta \det(D^2\varphi)\mathrm{d}\mathcal{H}^n. \tag{3.3}$$

Note that the surface area measure S_K of a convex body K in $\mathbb{R}^n$ is a Monge-Ampère type measure with h_K restricted to the unit sphere $\mathbb{S}^{n-1}$ since it satisfies

$$S_K(\omega) = \mathcal{H}^{n-1}\left(\bigcup_{v\in\omega} F(K,v)\right) = \mathcal{H}^{n-1}\left(\bigcup_{v\in\omega} \partial h_K(v)\right) = \mu_{h_K}(\omega) \tag{3.4}$$

for any Borel $\omega \subset \mathbb{S}^{n-1}$.

Noting that a convex function φ is the solution of a Monge-Ampère equation in the sense of measure (or in the Aleksandrov sense), this implies that it solves the corresponding integral formula for μ_φ rather than the original formula for $\det(D^2\varphi)$.

In order to obtain the regularity of the solution to the L_p Gauss image problem, we first convert the original Monge-Ampère equation (1.1) on the unit sphere $\mathbb{S}^{n-1}$ into a Euclidean Monge-Ampère equation on $\mathbb{R}^{n-1}$. Thus, we will pay attention to the restriction of a solution h of (1.1) to the hyperplane tangential to $\mathbb{S}^{n-1}$ at $e \in \mathbb{S}^{n-1}$.

Lemma 3.1. *For $K \in \mathcal{K}_o^n$ and $e \in \mathbb{S}^{n-1}$, we define that $\varphi : e^\perp \to \mathbb{R}$ with $\varphi(y) = h_K(y+e)$. If $h = h_K$ is a solution of* (1.1) *for non-negative functions f and g, then in the sense of measure φ satisfies*

$$g\left(\frac{D\varphi(y) + (\varphi(y) - \langle D\varphi(y), y\rangle)\cdot e}{|D\varphi(y) + (\varphi(y) - \langle D\varphi(y), y\rangle)\cdot e|}\right)\det D^2\varphi(y) = \varphi(y)^{p-1}|D\varphi(y) + (\varphi(y) - \langle D\varphi(y), y\rangle)\cdot e|^n \Theta(y) \quad \text{on } e^\perp. \tag{3.5}$$

Here,

$$\Theta(y) = \frac{f\left(\frac{e+y}{\sqrt{1+|y|^2}}\right)}{(1+|y|^2)^{\frac{n+p}{2}}}.$$

Proof. Let $h = h_K$ be a solution of equation (1.1) for $K \in \mathcal{K}_o^n$. Then from (2.17), (2.14), and (2.15), we have that for $v \in \mathbb{S}^{n-1}$,

$$g\left(\frac{Dh_K(v)}{|Dh_K(v)|}\right)\mathrm{d}S(K,v) = h_K^{p-1}(v)|Dh_K(v)|^n f(v)\mathrm{d}\mathcal{H}^{n-1}(v). \tag{3.6}$$

For $e \in \mathbb{S}^{n-1}$, assume that P_e denotes the hyperplane in $\mathbb{R}^n$ which is tangential to $\mathbb{S}^{n-1}$ at e and $e^\perp$ denotes the orthogonal complement of $\{se : s \in \mathbb{R}\}$ in $\mathbb{R}^n$. For $y \in e^\perp$, we have $y = \sum_{i=1}^{n-1} y_i e_i$, where $\{e_1, \ldots, e_{n-1}\}$ is a basis of $e^\perp$. We define the radial projection $\Pi : e^\perp \to \mathbb{S}^{n-1}$ from $P_e = e + e^\perp$ to $\mathbb{S}^{n-1}$, where $\Pi(y) = (y+e)/\sqrt{1+|y|^2}$. Since

$$\langle \Pi(y), e\rangle = (1+|y|^2)^{-\frac{1}{2}}, \tag{3.7}$$

it follows that for the mapping $y \mapsto v = \Pi(y)$ its Jacobian determinant is

$$|\mathrm{Jac}\,\Pi| = (1+|y|^2)^{-\frac{n}{2}}. \tag{3.8}$$

Suppose that $\varphi : e^\perp \to \mathbb{R}$ is the restriction of h_K on P_e. Thus,

$$h_K(\Pi(y)) = \frac{\varphi(y)}{\sqrt{1+|y|^2}}. \tag{3.9}$$

Then by (3.1) and (3.2), we have that

$$\partial\varphi(y) = F(K, \Pi(y))|e^\perp. \tag{3.10}$$

It follows from the homogeneity of degree 1 and the differentiability of h_K that

$$Dh_K(y+e) = Dh_K(v),$$

where $\nu = \Pi(y)$. Thus, we can deduce $D\varphi(y) = Dh_K(y + e)|e^\perp = Dh_K(\nu)|e^\perp$.

Let

$$Dh_K(\nu) = D\varphi(y) - \varrho e \tag{3.11}$$

for some undetermined constant $\varrho \in \mathbb{R}$. By (2.13), we see

$$h_K(\nu) = \langle Dh_K(\nu), \nu\rangle. \tag{3.12}$$

In addition, we have

$$\nu = \Pi(y) = (y + e)/\sqrt{1 + |y|^2}, \quad h_K(\nu) = \varphi(y)/\sqrt{1 + |y|^2}. \tag{3.13}$$

By substituting (3.11) and (3.13) into (3.12), we have

$$\varrho + \varphi(y) = \langle D\varphi(y), y\rangle. \tag{3.14}$$

Together (3.11) with (3.14), it follows that

$$Dh_K(\nu) - D\varphi(y) = (\varphi(y) - \langle D\varphi(y), y\rangle) \cdot e. \tag{3.15}$$

According to (3.4), we easily see that for a Borel set $\vartheta \subset e^\perp$,

$$\mathcal{H}^{n-1}\left(\bigcup_{\nu\in\pi(\vartheta)} (F(K, \nu)|e^\perp)\right) = \int_{\pi(\vartheta)} \langle \nu, e\rangle \mathrm{d}S_K(\nu). \tag{3.16}$$

Thus, it follows from (3.10), (3.3), (3.6), (3.15), (3.8), and (3.9) that

$$\begin{aligned}
\int_{\vartheta} \det D^2\varphi(y) \mathrm{d}\mathcal{H}^{n-1}(y) &= \int_{\pi(\vartheta)} \langle \nu, e\rangle \mathrm{d}S_K(\nu) \\
&= \int_{\pi(\vartheta)} \langle \nu, e\rangle h_K^{p-1}(\nu) \frac{|Dh_K(\nu)|^n}{g\left(\frac{Dh_K(\nu)}{|Dh_K(\nu)|}\right)} f(\nu) \mathrm{d}\mathcal{H}^{n-1}(\nu) \\
&= \int_{\vartheta} \varphi(y)^{p-1} \frac{|D\varphi(y) + (\varphi(y) - \langle D\varphi(y), y\rangle) \cdot e|^n}{g\left(\frac{D\varphi(y) + (\varphi(y) - \langle D\varphi(y), y\rangle)\cdot e}{|D\varphi(y) + (\varphi(y) - \langle D\varphi(y), y\rangle) \cdot e|}\right)} \frac{f(\pi(y))}{(1 + |y|^2)^{\frac{n+p}{2}}} \mathrm{d}\mathcal{H}^{n-1}(y).
\end{aligned}$$

From this, we have that φ satisfies (3.5) on $e^\perp$.

In the following, two important lemmas by Caffarelli [56,57], see also [5,54,55], are crucial for the proof of Theorem 1.1. □

Lemma 3.2. (Caffarelli [56]). *Let $\lambda_2 > \lambda_1 > 0$, and let φ be a convex function on an open bounded convex set $\Omega \subset \mathbb{R}^n$ such that*

$$\lambda_1 \leqslant \det D^2\varphi \leqslant \lambda_2$$

in the sense of measure.

(*i*) *If φ is non-negative and $W = \{y \in \Omega : \varphi(y) = 0\}$ is not a point, then W has no extremal point in Ω.*

(*ii*) *If φ is strictly convex, then φ is C^1.*

Lemma 3.3. (Caffarelli [57]). *For real functions φ and f on an open bounded convex set $\Omega \subset \mathbb{R}^n$, let φ be strictly convex, and let f be positive and continuous such that*

$$\det D^2\varphi = f$$

in the sense of measure.

(*i*) *Each $z \in \Omega$ has an open ball $B \subset \Omega$ around z such that the restriction of φ to B is in $C^{1,\beta}(B)$ for any $\beta \in (0, 1)$.*

(*ii*) *If f is in $C^\beta(\Omega)$ for some $\beta \in (0, 1)$, then each $z \in \Omega$ has an open ball $B \subset \Omega$ around z such that the restriction of φ to B is in $C^{2,\beta}(B)$.*

By virtue of the above lemmas, we are able to prove Theorem 1.1.

Proof of Theorem 1.1. We first define

$$\Upsilon(e,\iota) = \{v \in \mathbb{S}^{n-1} : \langle v, e\rangle > \iota\},$$

for $e \in \mathbb{S}^{n-1}$ and $0 < \iota < 1$. Since h_K is continuous on $\mathbb{S}^{n-1}$ for $K \in \mathcal{K}_o^n$, we have $0 < \iota_1 < 1$ and $\delta > 0$ such that $h_K(v) \geqslant \delta$, for $v \in \mathrm{cl}\Upsilon(e,\iota_1)$, where ι_1 and δ depend on e and K. Moreover, there exists $0 < \varepsilon < 1$ depending on e and K such that if some $v \in \mathrm{cl}\Upsilon(e,\iota_1)$ is the outer normal at $x \in \partial K$, then

$$\varepsilon < |x| < 1/\varepsilon. \tag{3.17}$$

Noting that for $y \in e^{\perp}$,

$$\Pi(y) = (y+e)/\sqrt{1+|y|^2}\,.$$

we can define

$$\Xi_e = \Pi^{-1}(\Upsilon(e,\iota_1)).$$

Suppose that $\varphi : e^{\perp} \to \mathbb{R}$ satisfies the conditions of Lemma 3.1. Let $y \in \Xi_e$, from (2.12), (3.15), and (3.17), we obtain

$$\varepsilon \leqslant |D\varphi(y) + (\varphi(y) - \langle D\varphi(y), y\rangle)\cdot e| \leqslant \frac{1}{\varepsilon}. \tag{3.18}$$

Since

$$\varphi(y) = \sqrt{1+|y|^2}\, h_K\left(\frac{e+y}{\sqrt{1+|y|^2}}\right) \geqslant \delta$$

$y \in \mathrm{cl}\Xi_e$, we easily obtain that φ also has an upper bound depending on e and K for $y \in \mathrm{cl}\Xi_e$. Since it is assumed that for positive constants c_1 and c_2, $0 < c_1 \leqslant f, g \leqslant c_2$. According to Lemma 3.1 and (3.18), we obtain that there exists $\Lambda \in (0,1)$ depending on e and K such that for $y \in \Xi_e$,

$$\Lambda \leqslant \det D^2\varphi(y) \leqslant \frac{1}{\Lambda}. \tag{3.19}$$

For $K \in \mathcal{K}_o^n$, we first prove that ∂K is C^1. That is, for any $z \in \partial K$, $\dim N(K,z) = 1$. Assume the contrary and let $z_0 \in \partial K$ be such that $\dim N(K,z_0) \geqslant 2$. Let $e \in N(K,z_0) \cap \mathbb{S}^{n-1}$. According to the definition of support function, and together with $z_0 \in \partial K$, we obtain that for $y \in \Xi_e$,

$$\varphi(y) \geqslant \langle y+e, z_0\rangle.$$

We easily see that

$$y \in \Sigma \coloneqq \Pi^{-1}(N(K,z_0) \cap \Upsilon(e,\iota_1)) \Leftrightarrow \varphi(y) = \langle y+e, z_0\rangle \quad \text{for } y \in \Xi_e.$$

Let $\psi(y) = \langle y+e, z_0\rangle$. Then

$$\varphi(y) - \psi(y)\begin{cases} = 0 & \text{for } y \in \Sigma \\ > 0 & \text{for } y \in \Xi_e \backslash \Sigma. \end{cases}$$

From (3.19), and together with the fact that ψ is the first degree polynomial, it follows that for $y \in \Xi_e$,

$$\Lambda \leqslant \det D^2(\varphi(y) - \psi(y)) \leqslant \frac{1}{\Lambda}.$$

Since $\dim\Sigma \geqslant 1$, we have

$$\Sigma = \Pi^{-1}(N(K,z_0) \cap \Upsilon(e,\iota_1)) = \{y \in \Xi_e : \varphi(y) - \psi(y) = 0\}$$

is not a point. In addition, by the choice of e, the origin o is an extremal point of Σ. From (i) of Lemma 3.2, we obtain a contradiction.

Next, we further show that φ is strictly convex on $\mathrm{cl}\,\Xi_e$ for $e \in \mathbb{S}^{n-1}$. Obviously, Ξ_e is a convex set in $\mathbb{R}^{n-1}$. For $0 < w < 1$ and $y_1, y_2 \in \Xi_e$ with $y_1 \neq y_2$, we let that $e + (wy_1 + (1-w)y_2)$ is an outer normal at $z \in \partial K$. That is,

$$e + (wy_1 + (1-w)y_2) \in N(K, z).$$

Since $z \in \partial K$ is a smooth point, it follows that

$$e + y_1 \notin N(K, z) \quad \text{and} \quad e + y_2 \notin N(K, z).$$

This implies

$$\varphi(y_i) > \langle z, e + y_i \rangle.$$

Thus, we can see

$$w\varphi(y_1) + (1-w)\varphi(y_2) > \langle z, e + (wy_1 + (1-w)y_2) \rangle = \varphi(wy_1 + (1-w)y_2).$$

Therefore,

$$w\varphi(y_1) + (1-w)\varphi(y_2) > \varphi(wy_1 + (1-w)y_2),$$

i.e., φ is strictly convex on an open bounded convex subset in $\mathbb{R}^n$. According to (ii) of Lemma 3.2, it follows from (3.19) and the strict convexity of φ that for any $e \in \mathbb{S}^{n-1}$, φ is C^1 on Ξ_e. From this, it follows that h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$, and the boundary ∂K contains no segment. This implies that the proof of (i) in Theorem 1.1 is completed. □

Let's start with the proof of (ii) in Theorem 1.1. According to the conditions of Theorem 1.1 that f and g are continuous, and recalling that φ is C^1 on $\mathrm{cl}\,\Xi_e$ for any $e \in \mathbb{S}^{n-1}$, we obtain that the right-hand side of (3.5) is continuous. From (i) in Lemma 3.3 together with the strict convexity of φ on Ξ_e, it follows that there is an open ball $B \subset \Xi_e$ centred at o such that φ is $C^{1,\beta}$ on B for any $\beta \in (0, 1)$. From this, we see that h_K is locally $C^{1,\beta}$ on $\mathbb{S}^{n-1}$. Therefore, from the compactness of $\mathbb{S}^{n-1}$, we obtain that h_K is globally $C^{1,\beta}$ on $\mathbb{S}^{n-1}$. The proof of (ii) in Theorem 1.1 is completed.

Finally, we prove (iii) of Theorem 1.1. Note that φ is $C^{1,\beta}$ on B. Since f and g are C^β on $\mathbb{S}^{n-1}$, it follows that the right-hand side of (3.5) is C^β. On the basis of (ii) of Lemma 3.3, we obtain that φ is $C^{2,\beta}$ on an open ball $\widetilde{B} \subset B$ of $e^\perp$ centred at o. This gives that h_K is locally $C^{2,\beta}$ on $\mathbb{S}^{n-1}$. Therefore, h_K is globally $C^{2,\beta}$ on $\mathbb{S}^{n-1}$ from the compactness of $\mathbb{S}^{n-1}$. In view of this, we finish the proof of (iii) in Theorem 1.1.

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