Research Article

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Almost periodic functions on time scales and their properties

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Abstract: In this article, we first propose a concept of almost periodic functions on arbitrary time scales, which is defined by trigonometric polynomial approximations with respect to supremum norm, and study some basic properties of these kinds of functions. Then, on almost periodic time scales, we introduce the concepts of the mean value and Fourier series of almost periodic functions and give some related results. Finally, we give the definitions of almost periodic functions in the sense of Bohr and in the sense of Bochner on time scales, respectively, and prove the equivalence of the above three definitions on almost periodic time scales.

Keywords: almost periodic functions, mean-value, Fourier series, time scales

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1 Introduction

As we all know, the time scale calculus theory can well unify the research of continuous and discrete analysis problems, and this theory has huge potential application value in mathematics itself, economics, physics, population dynamics, neural networks, and many other disciplines [1,2]. Therefore, in the past few decades, this theory has attracted more and more attention.

On the other hand, almost periodic phenomenon is a universal phenomenon in nature. Since Bohr [3] introduced the concept of almost periodic functions in 1920s, the study of the existence of almost periodic solutions of differential equations, difference equations, and dynamic systems has become one of the important research objects in these fields.

In order to unify the research on the existence of almost periodic solutions for continuous time systems and discrete time systems, Li et al. [4,5] in 2011 first introduced the definition of almost periodic functions on almost periodic time scales in the sense of Bohr, and they also showed some basic properties of these kinds of functions. Since then, the existence of almost periodic solutions of dynamic equations, ecological models, and neural network models on time scales has been broadly studied [6–23]. However, at present, there is no concept of almost periodic functions defined on arbitrary time scales, and there is no Fourier series theory of almost periodic functions on time scales. These are issues worthy of discussion with theoretical and application value.

Inspired by the above discussion, the main purpose of this study is first to propose a concept of almost periodic functions on arbitrary time scales, that is, to define the collection of almost periodic functions as the completion of the set composed of trigonometric polynomials with respect to the supremum norm, and to study some basic properties of these kinds of functions. Then, on almost periodic time scales, we introduce the

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concepts of mean value and Fourier series of almost periodic functions and give some related results. Finally, we give the definitions of almost periodic functions on time scales in the sense of Bohr and Bochner, respectively, and prove the equivalence of these three definitions on almost periodic time scales.

The rest of the study is organized as follows. In Section 22, we introduce some notations and definitions of time scale calculus. In Section 3, we propose a concept of almost periodic functions on arbitrary time scales defined by the closure of the set of trigonometric polynomials with respect to the supremum norm and investigate some basic properties of these kinds of functions. In Section 4, we propose a concept of Fourier series associated with an almost periodic function on time scales and present some relative results.

2 Preliminaries

In this section, we collect some definitions and lemmas, which will be used later.

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. For the notations $[a, b]_{\mathbb{T}}$, $[a, b)_{\mathbb{T}}$, and so on, we will denote time scale intervals $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$, where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. The collection of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T})$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T})$.

Definition 2.1. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^k$. If for every $\varepsilon > 0$, there exist a $\delta := \delta(\varepsilon, t) > 0$ and a number $f^\Delta(t)$ such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s|$$

for all $s \in \mathcal{U}(t, \delta) =: (t - \delta, t + \delta) \cap \mathbb{T}$, then we call $f^\Delta(t)$ the delta derivative of f at t.

If f is continuous, then f is right-dense continuous, and if f is delta differentiable at t, then f is continuous at t.

Let $f : \mathbb{T} \to \mathbb{R}$ be right-dense continuous. If $F^\Delta(t) = f(t)$, then we define the delta integral by $\int_a^t f(s)\Delta s = F(t) - F(a)$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \ne 0$ for all $t \in \mathbb{T}_k$. The set of all regressive and right-dense continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \ \forall t \in \mathbb{T}\}$.

Lemma 2.1. [2] *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}(\mathbb{T})$, then*

1. $\int_a^b (f + g)(t)\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$,
2. $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$,
3. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
4. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^d f(t)\Delta t$,

5. $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$,
6. $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$.

Lemma 2.2. [2] *If f, g are delta differentiable on $\mathbb{T}^k$, then*
(i) $(v_1 f + v_2 g)^{\Delta} = v_1 f^{\Delta} + v_2 g^{\Delta}$, *for any constants* v_1, v_2;
(ii) $(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$.

Lemma 2.3. [2] *If f is continuous on $[a, b]$ and delta differentiable in $[a, b)$, then there exist $\xi_1, \xi_2 \in [a, b)$ such that*

$$f^{\Delta}(\xi_1)(b-a) \le f(b) - f(a) \le f^{\Delta}(\xi_2)(b-a).$$

Lemma 2.4. [2] *Let f be a function defined on $[a, b)$ and $c \in \mathbb{T}$ with $a < c < b$. If f is Δ-integrable from a to c and c to b, respectively, then f is Δ-integrable on $[a, b)$ and*

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^d f(t)\Delta t.$$

Throughout this study, $\mathbb{X}$ denotes a Banach space with the norm $\|\cdot\|_{\mathbb{X}}$. Let $\mathbb{T}$ be a given time scale, and $\mathbb{T}$ is a complete metric space with the metric d defined by

$$d(t, t_0) = |t - t_0| \quad \text{for } t, t_0 \in \mathbb{T}.$$

For a given $\delta > 0$, the δ-neighborhood $\mathcal{U}(t_0, \delta)$ of a given point $t_0 \in \mathbb{T}$ is the set of all points $t \in \mathbb{T}$ such that $d(t_0, t) < \delta$.

Definition 2.2. [4] Function $f : D \to \mathbb{X}$ is called continuous at $t_0 \in D \subseteq \mathbb{T}$ if and only if (iff) for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $s \in \mathcal{U}(t_0, \delta) := (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$,

$$\|f(s) - f(t_0)\|_{\mathbb{X}} < \varepsilon.$$

The f is called continuous on D, provided that it is continuous for every $t \in D$.

Definition 2.3. [4] Function $f : D \to \mathbb{X}$ is called uniformly continuous on $D \subseteq \mathbb{T}$ iff for any $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that for any $t_1, t_2 \in D$ with $|t_1 - t_2| < \delta(\varepsilon)$, it is implied that

$$\|f(t_1) - f(t_2)\|_{\mathbb{X}} < \varepsilon.$$

3 Almost periodic functions on arbitrary time scales

In this section, we first introduce a new definition of almost periodic functions on arbitrary time scales, then we discuss some properties of these kinds of functions.

Let $C(\mathbb{T}, \mathbb{X})$ denote the space of all continuous functions from $\mathbb{T}$ to $\mathbb{X}$ and $BC(\mathbb{T}, \mathbb{X})$ denote the space of all bounded continuous functions from $\mathbb{T}$ to $\mathbb{X}$. It is easy to see that $BC(\mathbb{T}, \mathbb{X})$ with the norm $\|x\|_\infty = \sup_{t\in\mathbb{T}} \|x(t)\|_{\mathbb{X}}$ is a Banach space.

We give the following definition of trigonometric polynomials defined on $\mathbb{T}$.

Definition 3.1. A function $T : \mathbb{T} \to \mathbb{X}$ defined by

$$T(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}, \quad t \in \mathbb{T}, \tag{3.1}$$

where for $k = 1, 2, \dots, n$, $\lambda_k \in \mathbb{R}$, and $a_k \in \mathbb{X}$ is called a trigonometric polynomial with values in $\mathbb{X}$.

We denote by $\mathcal{T}$ the set of all trigonometric polynomials with values in $\mathbb{X}$. It is obvious that $\mathcal{T} \subset BC(\mathbb{T}, \mathbb{X})$.

Definition 3.2. A function $f \in C(\mathbb{T}, \mathbb{X})$ is said to be almost periodic if for every $\varepsilon > 0$ there exists a trigonometric polynomial $T(t) \in \mathcal{T}$ such that

$$\|f - T\|_\infty < \varepsilon.$$

We denote the space of all such functions by $AP(\mathbb{T}, \mathbb{X})$.

Remark 3.1. Obviously, the space $AP(\mathbb{T}, \mathbb{X})$ is the closure $\overline{\mathcal{T}}$ of $\mathcal{T}$ in the sense of convergence in the norm $\|\cdot\|_\infty$.

Remark 3.2. A function $f \in AP(\mathbb{T}, \mathbb{X})$ iff there exists a sequence of trigonometric polynomials $\{T_n(t)\} \subset \mathcal{T}$ such that $\lim_{n\to\infty} T_n(t) = f(t)$ uniformly on $\mathbb{T}$.

Lemma 3.1. *If $T(t) \in \mathcal{T}$, then T is uniformly continuous on $\mathbb{T}$.*

Proof. Let $T(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$, where $a_k \in \mathbb{X}$, $\lambda_k \in \mathbb{R}$. For any $\varepsilon > 0$, there exists $\delta(\varepsilon) = \frac{\varepsilon}{\sum_{k=1}^{n} 2 \mid \lambda_k \mid \|a_k\|_{\mathbb{X}}}$ such that for any $t_1, t_2 \in \mathbb{T}$ with $|t_1 - t_2| < \delta(\varepsilon)$, one has

$$\begin{aligned}
\|T(t_1) - T(t_2)\|_{\mathbb{X}} &\le \sum_{k=1}^{n} \|a_k\|_{\mathbb{X}} |e^{i\lambda_k t_1} - e^{i\lambda_k t_2}| \\
&\le \sum_{k=1}^{n} \|a_k\|_{\mathbb{X}} (|\cos\lambda_k t_1 - \cos\lambda_k t_2| + |\sin\lambda_k t_1 - \sin\lambda_k t_2|) \\
&\le \sum_{k=1}^{n} 2|\lambda_k| \|a_k\|_{\mathbb{X}} |t_1 - t_2| < \varepsilon.
\end{aligned}$$

Thus, T is uniformly continuous on $\mathbb{T}$. This completes the proof. □

Theorem 3.1. *If $f \in AP(\mathbb{T}, \mathbb{X})$, then f is bounded and uniformly continuous on $\mathbb{T}$.*

Proof. By Definition 3.2, for any $\varepsilon > 0$, there exists $T(t) \in \mathcal{T}$ such that

$$\|f(t) - T(t)\|_{\mathbb{X}} \le \|f - T\|_\infty < \varepsilon.$$

Let $M = \sup_{t\in\mathbb{T}} \|T(t)\|_{\mathbb{X}}$, we have

$$\|f(t)\|_{\mathbb{X}} \le \|f(t) - T(t)\|_{\mathbb{X}} + \|T(t)\|_{\mathbb{X}} \le \varepsilon + M.$$

Hence, f is bounded on $\mathbb{T}$.

Now, we will prove f is uniformly continuous on $\mathbb{T}$. Since $f \in AP(\mathbb{T}, \mathbb{X})$, there exists a sequence of trigonometric polynomials $\{T_n(t); n \ge 1\} \subset \mathcal{T}$ such that

$$\lim_{n\to\infty} T_n(t) = f(t)$$

uniformly on $\mathbb{T}$. Hence, for any $\varepsilon > 0$, one can choose n large enough such that

$$\|T_n(t) - f(t)\|_{\mathbb{X}} < \frac{\varepsilon}{3}, \quad \forall t \in \mathbb{T}.$$

By Lemma 3.1, T_n is uniformly continuous on $\mathbb{T}$. That is, for the previous $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that for any $t, s \in \mathbb{T}$ with $|t - s| < \delta(\varepsilon)$, it is implied that

$$\|T_n(t) - T_n(s)\|_{\mathbb{X}} < \frac{\varepsilon}{3}.$$

Thus, for $|t - s| < \delta(\varepsilon)$, we have

$$\|f(t) - f(s)\|_{\mathbb{X}} \le \|f(t) - T_n(t)\|_{\mathbb{X}} + \|T_n(t) - T_n(s)\|_{\mathbb{X}} + \|T_n(s) - f(s)\|_{\mathbb{X}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which implies that f is uniformly continuous on $\mathbb{T}$. This completes the proof. □

Theorem 3.2. *If* $\{f_n(t)\} \subset AP(\mathbb{T}, \mathbb{X})$ *and* $f_n(t) \to f(t)$ *as* $n \to \infty$ *uniformly on* $\mathbb{T}$, *then* $f \in AP(\mathbb{T}, \mathbb{X})$.

Proof. For any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that

$$\|f(t) - f_{n_0}(t)\|_{\mathbb{X}} < \frac{\varepsilon}{2}, \quad \forall t \in \mathbb{T},$$

for $n_0 > N$. Because $f_{n_0}(t) \in AP(\mathbb{T}, \mathbb{X})$, there exists a trigonometric polynomial $T(t) \in \mathcal{T}$ such that

$$\|f_{n_0}(t) - T(t)\|_{\mathbb{X}} < \frac{\varepsilon}{2}, \quad \forall t \in \mathbb{T}.$$

As a consequence,

$$\|f(t) - T(t)\|_{\mathbb{X}} \le \|f(t) - f_{n_0}(t)\|_{\mathbb{X}} + \|f_{n_0}(t) - T(t)\|_{\mathbb{X}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall t \in \mathbb{T}.$$

This completes the proof. □

Theorem 3.3. *If* $f, g \in AP(\mathbb{T}, \mathbb{X})$ *and* $\lambda \in \mathbb{R}$, *then* $f + g, \lambda f \in AP(\mathbb{T}, \mathbb{X})$.

Proof. Let $f, g \in AP(\mathbb{T}, \mathbb{X})$, for any $\varepsilon > 0$, there exist two trigonometric polynomials $T(t), S(t) \in \mathcal{T}$ such that

$$\|f - T\|_\infty < \frac{\varepsilon}{2}, \quad \|g - S\|_\infty < \frac{\varepsilon}{2}.$$

Thus, one has

$$\|(f + g) - (T + S)\|_\infty \le \|f - T\|_\infty + \|g - S\|_\infty < \varepsilon.$$

From the above inequality, we derive $f + g \in AP(\mathbb{T}, \mathbb{X})$ due to the fact that $T(t) + S(t) \in \mathcal{T}$.

The proof of $\lambda f \in AP(\mathbb{T}, \mathbb{X})$ follows immediately from the fact that if $T(t) \in \mathcal{T}$, then $\lambda T(t) \in \mathcal{T}$. This completes the proof. □

Theorem 3.4. *If* $\mathbb{X}$ *is a Banach algebra and* $\varphi, \psi \in AP(\mathbb{T}, \mathbb{X})$, *then* $\varphi\psi \in AP(\mathbb{T}, \mathbb{X})$.

Proof. Since $\varphi, \psi \in AP(\mathbb{T}, \mathbb{X})$, there exist two sequences of trigonometric polynomials $\{T_n^1(t)\}$ and $\{T_n^2(t)\} \subset \mathcal{T}$ such that

$$\lim_{n\to\infty} T_n^1(t) = \varphi(t) \tag{3.2}$$

uniformly on $\mathbb{T}$ and

$$\lim_{n\to\infty} T_n^2(t) = \psi(t) \tag{3.3}$$

uniformly on $\mathbb{T}$. Because $\mathbb{X}$ is a Banach algebra, $T_n^1(t)T_n^2(t)$ and $\psi(t)\varphi(t)$ are meaningful. Let $T_n^1(t)T_n^2(t) = S_n(t)$, by (3.2) and (3.3), we obtain

$$\lim_{n\to\infty} S_n(t) = \psi(t)\varphi(t)$$

uniformly on $\mathbb{T}$. Because $S_n(t)$ is also a trigonometric polynomial, so $\varphi\psi \in AP(\mathbb{T}, \mathbb{X})$. This completes the proof. □

In the sequel, we denote $\mathbb{C}^n$ by the n-dimensional complex vector space with the norm $\|\cdot\|_{\mathbb{C}^n}$.

Lemma 3.2. [24] *If f is continuous on a bounded and closed subset* Ω *of* $\mathbb{C}^n$, *then for any* $\varepsilon > 0$, *one can choose a polynomial* P_ε *such that*

$$\|f(x) - P_\varepsilon(x)\|_{\mathbb{C}^n} < \varepsilon, \quad x \in \Omega.$$

Lemma 3.3. [24] *Let* $P : \mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial and* $f_1, f_2, \dots, f_n \in AP(\mathbb{R}, \mathbb{C})$. *Then the function* $F(\cdot) = P(f_1(\cdot), \dots, f_n(\cdot)) \in AP(\mathbb{R}, \mathbb{C}^n)$.

By Lemmas 3.2 and 3.3, the following statement is obvious.

Theorem 3.5. *Let* $F : D \to \mathbb{C}^n$ *be a uniformly continuous function, where* D *is a bounded subset of* $\mathbb{C}^n$. *If* $f_1, f_2, \dots, f_n \in AP(\mathbb{T}, \mathbb{C})$ *and for each* $t \in \mathbb{R}, (f_1(t), f_2(t), \dots, f_n(t)) \in D$, *then* $f(\cdot) = F(f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)) \in AP(\mathbb{T}, \mathbb{C}^n)$.

Theorem 3.6. *If* $f, g \in AP(\mathbb{T}, \mathbb{C})$ *and* $m := \inf_{t\in\mathbb{T}} \|g(t)\|_{\mathbb{C}} > 0$, *then* $f/g \in AP(\mathbb{T}, \mathbb{C})$.

Proof. Let $M := \sup_{t\in\mathbb{T}} \|g(t)\|_{\mathbb{C}}$, we note that $G(z) = 1/z$ is continuous in the crown $m \le \|z\|_{\mathbb{C}} \le M$. According to Theorem 3.5, if $z \in AP(\mathbb{T}, \mathbb{C})$, then $G \in AP(\mathbb{T}, \mathbb{C})$. Hence, $1/g(t)$ is almost periodic. Consequently, based on Theorem 3.4, $f(t)/g(t)$ is almost periodic. This completes the proof. □

Theorem 3.7. *If* $f \in C(\mathbb{C}^n, \mathbb{C}^n)$ *satisfies the Lipschitz condition and* $x \in AP(\mathbb{T}, \mathbb{C}^n)$, *then* $f(x(\cdot)) \in AP(\mathbb{T}, \mathbb{C}^n)$.

Proof. Let L represent the Lipschitz constant of f. Since $x \in AP(\mathbb{T}, \mathbb{C}^n)$, for any $\varepsilon > 0$, there exists a trigonometric polynomial $S(t)$ such that

$$\|x(t) - S(t)\|_{\mathbb{C}^n} < \frac{\varepsilon}{2L}, \quad \forall t \in \mathbb{T}. \tag{3.4}$$

In addition, by Lemma 3.2, for any $\varepsilon > 0$, there exists a polynomial $P_\varepsilon(S(t))$ such that

$$\|f(S(t)) - P_\varepsilon(S(t))\|_{\mathbb{C}^n} < \frac{\varepsilon}{2}, \quad \forall t \in \mathbb{T}. \tag{3.5}$$

Obviously, $P_\varepsilon(S(t))$ is also a trigonometric polynomial. Thus, it follows from (3.4) and (3.5) that

$$\begin{aligned} \|f(x(t)) - P_\varepsilon(S(t))\|_{\mathbb{C}^n} &\le \|f(x(t)) - f(S(t))\|_{\mathbb{C}^n} + \|f(S(t)) - P_\varepsilon(S(t))\|_{\mathbb{C}^n} \\ &< L\|x(t) - S(t)\|_{\mathbb{C}^n} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall t \in \mathbb{T}. \end{aligned}$$

Hence, $f(x(\cdot)) \in AP(\mathbb{T}, \mathbb{X})$. This completes the proof. □

4 Almost periodic functions on almost periodic time scales

Definition 4.1. [4] A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi = \{\tau \in \mathbb{R} : \tau \pm t \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Lemma 4.1. [12] *Let* $\mathbb{T}$ *be an almost periodic time scale, and let* $K = \inf\{|\alpha| : \alpha \in \Pi \text{ for } \alpha \neq 0\}$. *Then,* $K = 0$ *iff* $\mathbb{T} = \mathbb{R}$, *and* $K > 0$ *iff* $\mathbb{T} \neq \mathbb{R}$. *Moreover,* $\Pi = \mathbb{R}$ *if* $\mathbb{T} = \mathbb{R}$, *and* $\Pi = K\mathbb{Z}$ *if* $\mathbb{T} \neq \mathbb{R}$.

In this section, we always assume that $\mathbb{T}$ be an almost periodic time scale, we will study some basic properties of almost periodic functions on $\mathbb{T}$.

Theorem 4.1. *If* $f \in AP(\mathbb{T}, \mathbb{X})$ *and* $h \in \Pi$, *then* $f(\cdot + h) \in AP(\mathbb{T}, \mathbb{X})$.

Proof. By Definition 3.2, for any $\varepsilon > 0$, there exists a trigonometric polynomial $T(t) \in \mathcal{T}$ such that

$$\|f - T\|_\infty < \varepsilon.$$

It follows immediately from $T(t+h) \in \mathcal{T}$ that $f(\cdot+h) \in AP(\mathbb{T}, \mathbb{X})$.

This completes the proof. □

4.1 Fourier series of almost periodic functions on almost periodic time scales

In order to introduce the Fourier series of an almost periodic function on time scales, we first define the mean value of such a function.

Lemma 4.2. [25] *If* $\lambda \in \mathbb{R}$ *and* $\lambda \neq 0$, *then* $\lim_{l\to\infty}\frac{1}{l}\int_\alpha^{\alpha+l} e^{i\lambda t}\Delta t = 0$, *where* $\alpha \in \mathbb{T}$, $l \in \Pi$.

Definition 4.2. Let $f \in C(\mathbb{T}, \mathbb{X})$, if

$$M\{f\} := \lim_{l\to\infty} \frac{1}{l}\int_\alpha^{\alpha+l} f(t)\Delta t, \quad \text{where } \alpha \in \mathbb{T}, l \in \Pi,$$

exists, then the number $M\{f\}$ is called the mean value of f.

Theorem 4.2. *If* $f \in AP(\mathbb{T}, \mathbb{X})$, *then*

$$M\{f\} = \lim_{l\to\infty} \frac{1}{l}\int_\alpha^{\alpha+l} f(t)\Delta t, \quad \textit{where } \alpha \in \mathbb{T}, l \in \Pi$$

uniformly exists for $\alpha \in \mathbb{T}$. *The number* $M\{f\}$ *is independent of* α *and is called the mean value of* f.

Proof. Let

$$R(t) = c_0 + \sum_{k=1}^m c_k e^{i\lambda_k t}, \quad t \in \mathbb{T},$$

where $\lambda_k \neq 0$, $k = 1, 2, \ldots, n$. It follows from Lemma 4.2 that

$$\lim_{l\to\infty} \frac{1}{l}\int_\alpha^{\alpha+l} R(t)\Delta t = c_0 + \lim_{l\to\infty}\left(\sum_{k=1}^m \frac{c_k}{l}\int_\alpha^{\alpha+l} e^{i\lambda_k t}\Delta t\right) = c_0.$$

That is, for any $R(t) \in \mathcal{T}$, its mean value exists and is independent of α.

Since $f \in AP(\mathbb{T}, \mathbb{X})$, for any $\varepsilon > 0$, there exists a trigonometric polynomial $T(t) \in \mathcal{T}$ such that

$$\|f(t) - T(t)\|_{\mathbb{X}} < \frac{\varepsilon}{3}, \quad \forall t \in \mathbb{T}, \tag{4.1}$$

and T takes on the following form:

$$T(t) = d_0 + \sum_{k=1}^m d_k e^{i\lambda_k t}, \quad t \in \mathbb{T},$$

where $\lambda_k \neq 0$, $k = 1, 2, \ldots, n$. Because the mean value of T exists, we can choose $N(\varepsilon) > 0$ such that

$$\left\|\frac{1}{l_1}\int_\alpha^{\alpha+l_1} T(t)\Delta t - \frac{1}{l_2}\int_\alpha^{\alpha+l_2} T(t)\Delta t\right\|_{\mathbb{X}} < \frac{\varepsilon}{3} \tag{4.2}$$

for $l_1, l_2 \in \Pi$ with $l_1, l_2 \geq N(\varepsilon)$. By (4.1) and (4.2), one has

$$\begin{aligned}\left\|\frac{1}{l_1}\int_{\alpha}^{\alpha+l_1} f(t)\Delta t - \frac{1}{l_2}\int_{\alpha}^{\alpha+l_2} f(t)\Delta t\right\|_{\mathbb{X}} &\leq \frac{1}{l_1}\int_{\alpha}^{\alpha+l_1} \|f(t)-T(t)\|_{\mathbb{X}}\Delta t + \left\|\frac{1}{l_1}\int_{\alpha}^{\alpha+l_1} T(t)\Delta t - \frac{1}{l_2}\int_{\alpha}^{\alpha+l_2} T(t)\Delta t\right\|_{\mathbb{X}} + \frac{1}{l_2}\int_{\alpha}^{\alpha+l_2} \|f(t)-T(t)\|_{\mathbb{X}}\Delta t \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Thus, the mean value of $f(t)$ exists.

Now, we will prove that $M\{f\}$ is independent of $\alpha \in \mathbb{T}$. By Theorem 3.1, note that $\|f\|_\infty = \sup_{t\in\mathbb{T}}\|f(t)\|_{\mathbb{X}} < \infty$. Let $\alpha^*, \alpha \in \mathbb{T}$, for $\alpha^* > \alpha$, by Lemma 2.4, we have

$$\begin{aligned}\left\|\frac{1}{l}\int_{\alpha}^{\alpha+l} f(t)\Delta t - \frac{1}{l}\int_{\alpha^*}^{\alpha^*+l} f(t)\Delta t\right\|_{\mathbb{X}} &= \frac{1}{l}\left\|\int_{\alpha}^{\alpha^*} f(t)\Delta t + \int_{\alpha^*}^{\alpha+l} f(t)\Delta t - \int_{\alpha^*}^{\alpha+l} f(t)\Delta t - \int_{\alpha+l}^{\alpha^*+l} f(t)\Delta t\right\|_{\mathbb{X}} \\ &= \frac{1}{l}\left\|\int_{\alpha}^{\alpha^*} f(t)\Delta t - \int_{\alpha+l}^{\alpha^*+l} f(t)\Delta t\right\|_{\mathbb{X}} \\ &\leq \frac{2(\alpha^*-\alpha)\|f\|_\infty}{l}.\end{aligned}$$

For $\alpha^* < \alpha$, again by Lemma 2.4, we obtain

$$\begin{aligned}\left\|\frac{1}{l}\int_{\alpha}^{\alpha+l} f(t)\Delta t - \frac{1}{l}\int_{\alpha^*}^{\alpha^*+l} f(t)\Delta t\right\|_{\mathbb{X}} &= \frac{1}{l}\left\|\int_{\alpha}^{\alpha^*+l} f(t)\Delta t + \int_{\alpha^*+l}^{\alpha+l} f(t)\Delta t - \int_{\alpha^*}^{\alpha} f(t)\Delta t - \int_{\alpha}^{\alpha^*+l} f(t)\Delta t\right\|_{\mathbb{X}} \\ &= \frac{1}{l}\left\|\int_{\alpha^*+l}^{\alpha+l} f(t)\Delta t - \int_{\alpha^*}^{\alpha} f(t)\Delta t\right\|_{\mathbb{X}} \\ &\leq \frac{2(\alpha-\alpha^*)\|f\|_\infty}{l}.\end{aligned}$$

Hence, for any $\alpha, \alpha^* \in \mathbb{T}$, we have

$$\lim_{l\to\infty}\frac{1}{l}\int_{\alpha}^{\alpha+l} f(t)\Delta t = \lim_{l\to\infty}\frac{1}{l}\int_{\alpha^*}^{\alpha^*+l} f(t)\Delta t,$$

which means that $M\{f\}$ is independent of $\alpha \in \mathbb{T}$. This completes the proof. □

Theorem 4.3. *Let $f, g \in AP(\mathbb{T}, \mathbb{X})$ and $\lambda \in \mathbb{R}$, then*
(*i*) $M\{f+g\} = M\{f\} + M\{g\}$;
(*ii*) $M\{\lambda f\} = \lambda M\{f\}$;
(*iii*) *If* $f(t) \geq 0$, $M\{f\} \geq 0$;
(*iv*) $\|M\{f\}\|_{\mathbb{X}} \leq M\{\|f\|_{\mathbb{X}}\}$;
(*v*) *If* $\lim_{n\to\infty} f_n(t) = f(t)$ *uniformly on* $\mathbb{T}$, *then* $\lim_{n\to\infty} M\{f_n\} = M\{f\}$.

Proof. By Theorem 4.2, the proofs of statements (*i*)–(*iv*) are obvious. The proof of statement (*v*) follows from the fact that

$$\|M\{f_n\} - M\{f\}\|_{\mathbb{X}} = \|M\{f_n - f\}\|_{\mathbb{X}} \leq M\{\|f_n - f\|_{\mathbb{X}}\}.$$

This completes the proof. □

Let $f \in AP(\mathbb{T}, \mathbb{X})$, $\lambda \in \mathbb{R}$, we note the fact that $f(t)e^{-i\lambda t} \in AP(\mathbb{T}, \mathbb{X})$. Therefore, there exists the mean value of $f(t)e^{-i\lambda t}$. We write

$$a(f, \lambda) = M\{f(t)e^{-i\lambda t}\}.$$

The following result is particularly important for the concept of Fourier series corresponding to almost periodic functions.

Theorem 4.4. *For each $f \in AP(\mathbb{T}, \mathbb{X})$, there exists at most a countable set of values of $\lambda \in \mathbb{R}$ such that*

$$a(f, \lambda) \neq 0.$$

Proof. For $f \in AP(\mathbb{T}, \mathbb{X})$, there exists a sequence of trigonometric polynomials $\{R_m(t);\ m \geq 1\} \subset \mathcal{T}$ such that $\lim_{m\to\infty} R_m(t) = f(t)$ uniformly on $\mathbb{T}$. By Theorem 4.3, one has

$$\lim_{m\to\infty} M\{R_m(t)e^{-i\lambda t}\} = M\{f(t)e^{-i\lambda t}\} = a(f, \lambda).$$

Next, we will prove that $M\{R_m(t)e^{-i\lambda t}\} \neq 0$ for each $m \geq 1$. Let $R_m(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$, $a_k \in \mathbb{X}$, $k = 1, 2, \ldots, n$. Then, one can easily see that $M\{R_m(t)e^{-i\lambda t}\} \neq 0$ only for $\lambda = \lambda_k$, $k = 1, 2, \ldots, n$. As a consequence, there exists only a countable set of values of λ such that $M\{R_m(t)e^{-i\lambda t}\} \neq 0$ for at least one m, which implies that $a(\lambda, t) \neq 0$ for such values. This completes the proof. □

Definition 4.3. Let $f \in AP(\mathbb{T}, \mathbb{X})$, and denote by the numbers $\lambda_k \in \mathbb{R}$, $k = 1, 2, \ldots$, such that

$$a_k = a(f, \lambda_k) \neq 0,$$

then the series

$$\sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$$

is called the Fourier series of the function f. We will denote this fact by

$$f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}.$$

$\lambda_k \in \mathbb{R}$, $k = 1, 2, \ldots$, are called the Fourier exponents of the function f, and a_k, $k = 1, 2, \ldots$, are called Fourier coefficients of f.

Theorem 4.5. *Let $f \in AP(\mathbb{T}, \mathbb{X})$. If $f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$, then the following hold:*

(i) $f(t + c) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k c} e^{i\lambda_k t}$, *where* $c \in \mathbb{T}$.

(ii) $e^{i\lambda t} f(t) \sim \sum_{k=1}^{\infty} a_k e^{i(\lambda+\lambda_k)t}$, *where* $\lambda \in \mathbb{R}$.

(iii) *if* $f^{\Delta} \in AP(\mathbb{T}, \mathbb{X})$, *then* $f^{\Delta}(t) \sim \sum_{k=1}^{\infty} A_k e^{i\lambda_k t}$, *where* $A_k = M\{f(t)(e^{-i\lambda_k t})^{\Delta}\}$; *in particular, if* $\mathbb{T} = \mathbb{R}$, *then*

$$f^{\Delta}(t) \sim \sum_{k=1}^{\infty} i\lambda_k a_k e^{i\lambda_k t},$$

if $\mathbb{T} = h\mathbb{Z}$, *then*

$$f^{\Delta}(t) \sim \sum_{k=1}^{\infty} M\left\{\frac{1 - e^{-i\lambda_k h}}{h} f(t + h)e^{-i\lambda_k t}\right\} e^{i\lambda_k t}.$$

Proof. By Definition 4.3, the proofs of statements (i) and (ii) are obvious. Next, we will prove statement (iii). Note that

$$\frac{1}{l}\int_{\alpha}^{\alpha+l} f^{\Delta}(t)e^{-i\lambda t}\Delta t = \frac{1}{l}(f(\alpha+l)e^{-i\lambda(\alpha+l)} - f(\alpha)e^{-i\lambda\alpha}) - \frac{1}{l}\int_{\alpha}^{\alpha+l} f(t)(e^{-i\lambda t})^{\Delta}\Delta t.$$

Letting $l \to \infty$, one obtains

$$M\{f^{\Delta}(t)e^{-i\lambda t}\} = M\{f(t)(e^{-i\lambda t})^{\Delta}\},$$

which means that the Fourier exponents of f^{Δ} are the same as those of f, except for $\lambda = 0$. We denote by A_k the Fourier coefficients of f^{Δ}, we have $A_k = M\{f(t)(e^{-i\lambda_k t})^{\Delta}\}$. Hence, if $\mathbb{T} = \mathbb{R}$, then

$$f^{\Delta}(t) \sim \sum_{k=1}^{\infty} i\lambda_k a_k e^{i\lambda_k t}.$$

If $\mathbb{T} = h\mathbb{Z}$, since

$$(e^{-i\lambda t})^{\Delta} = \frac{e^{-i\lambda h} - 1}{h}e^{-i\lambda t},$$

$A_k = M\left\{\frac{1-e^{-i\lambda_k h}}{h}f(t+h)e^{-i\lambda_k t}\right\}$. Consequently,

$$f^{\Delta}(t) \sim \sum_{k=1}^{\infty} M\left\{\frac{1-e^{-i\lambda_k h}}{h}f(t+h)e^{-i\lambda_k t}\right\}e^{i\lambda_k t}.$$

This completes the proof. □

One can easily prove that

Theorem 4.6. *Let $f \in AP(\mathbb{T},\mathbb{C})$. If $f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$, then $\bar{f}(t) \sim \sum_{k=1}^{\infty} \bar{a}_k e^{i\lambda_k t}$.*

Let $f \in AP(\mathbb{T},\mathbb{C})$, $b_k \in \mathbb{C}$, and $\beta_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$. Denote

$$\phi(b_1, b_2, \ldots, b_n) = M\left\{\left\| f(t) - \sum_{k=1}^{n} b_k e^{i\beta_k t}\right\|^2\right\}, \quad t \in \mathbb{T}.$$

Theorem 4.7. *Let $f \in AP(\mathbb{T},\mathbb{C})$. If $b_k = a(f,\beta_k)$, then $\min\phi = M\{\|f(t)\|^2\} - \sum_{k=1}^{n}\|a(f,\beta_k)\|^2$.*

Proof. Note that $\|f(t)\|^2 = f(t)\bar{f}(t)$ is almost periodic, where $\bar{f}(t)$ denotes the conjugate of $f(t)$. So, the mean value of $\|f(t)\|^2$ exists. Thus,

$$\begin{aligned}
M\left\{\left\| f(t) - \sum_{k=1}^{n} b_k e^{i\beta_k t}\right\|^2\right\} &= M\left\{\left(f(t) - \sum_{k=1}^{n} b_k e^{i\beta_k t}\right)\left(\bar{f}(t) - \sum_{k=1}^{n} \bar{b}_k e^{-i\beta_k t}\right)\right\}\\
&= M\{f(t)\bar{f}(t)\} - M\left\{f(t)\sum_{k=1}^{n}\bar{b}_k e^{-i\beta_k t}\right\} - M\left\{\bar{f}(t)\sum_{k=1}^{n} b_k e^{i\beta_k t}\right\} + M\left\{\sum_{k=1}^{n}\sum_{j=1}^{n} b_k\bar{b}_j e^{i(\beta_k-\beta_j)t}\right\} \qquad (4.3)\\
&= M\{\|f(t)\|^2\} - \sum_{k=1}^{n}\bar{b}_k M\{f(t)e^{-i\beta_k t}\} - \sum_{k=1}^{n} b_k M\{\bar{f}(t)e^{i\beta_k t}\} + \sum_{k=1}^{n}\sum_{j=1}^{n} b_k\bar{b}_j M\{e^{i(\beta_k-\beta_j)t}\}.
\end{aligned}$$

If $k \neq j$, by Lemma 4.2, we have

$$M\{e^{i(\beta_k-\beta_j)t}\} = 0.$$

If $k = j$, $M\{e^{i(\beta_k-\beta_j)t}\} = 1$. Therefore, one has

$$\sum_{l=1}^{n}\sum_{j=1}^{n} b_k\bar{b}_j M\{e^{i(\beta_k-\beta_j)t}\} = \sum_{k=1}^{n}\|b_k\|^2. \qquad (4.4)$$

By (4.3) and (4.4), it is easy to obtain that

$$\begin{aligned} &M\left\{\left\| f(t) - \sum_{k=1}^{n} b_k e^{i\beta_k t} \right\|^2\right\} \\ &= M\{\|f(t)\|^2\} - \sum_{k=1}^{n} \bar{b}_k a(f,\beta_k) - \sum_{k=1}^{n} b_k \overline{a(f,\beta_k)} + \sum_{k=1}^{n} \|b_k\|^2 \\ &= M\{\|f(t)\|^2\} + \sum_{k=1}^{n} \|b_k - a(f,\beta_k)\|^2 - \sum_{k=1}^{n} \|a(f,\beta_k)\|^2, \end{aligned}$$

which implies that $\min\phi$ is attained for $b_k = a(f,\beta_k)$ and $\min\phi = M\{\|f(t)\|^2\} - \sum_{k=1}^{n}\|a(f,\beta_k)\|^2$, $k = 1, 2, \ldots, n$. This completes the proof. □

Theorem 4.8. *Let $f \in AP(\mathbb{T},\mathbb{C})$. If $f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$, then the Bessel inequality*

$$\sum_{k=1}^{\infty} \|a_k\|^2 \leq M\{\|f(t)\|^2\}$$

holds.

Proof. Since $\min\phi \geq 0$, by Theorem 4.7,

$$\sum_{k=1}^{n} \|a(f,\beta_k)\|^2 \leq M\{\|f(t)\|^2\}, \tag{4.5}$$

that is,

$$\sum_{k=1}^{n} \|a_k\|^2 \leq M\{\|f(t)\|^2\}.$$

Since $M\{\|f(t)\|^2\}$ is a constant, let $n \to \infty$, we immediately obtain that

$$\sum_{k=1}^{\infty} \|a_k\|^2 \leq M\{\|f(t)\|^2\}.$$

This completes the proof. □

Theorem 4.9. *Let $f \in AP(\mathbb{T},\mathbb{C})$. If $f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$, then the Parseval equality*

$$\sum_{k=1}^{\infty} \|a_k\|^2 = M\{\|f(t)\|^2\}$$

holds.

Proof. By Theorem 4.8, the following inequality holds:

$$\sum_{k=1}^{\infty} \|a_k\|^2 \leq M\{\|f(t)\|^2\}.$$

Next we will prove that

$$M\{\|f(t)\|^2\} \leq \sum_{k=1}^{\infty} \|a_k\|^2.$$

Since $f \in AP(\mathbb{T},\mathbb{C})$, there exists a sequence of trigonometric polynomials $\{R_n;\ n \geq 1\} \subset \mathcal{T}$ such that

$$\|f(t) - R_n(t)\| < \frac{1}{\sqrt{n}}, \quad t \in \mathbb{T}.$$

Thus, it follows from the above inequality that

$$M\{\|f - R_n\|^2\} = \lim_{l\to\infty} \frac{1}{l} \int_{a}^{a+l} \|f(t) - R_n(t)\|^2 \Delta t \le \frac{1}{n}, \quad l \in \Pi, a \in \mathbb{T}. \tag{4.6}$$

Let $F_n(t)$ be the polynomial, $n \ge 1$. Set $F_n(t) \equiv 0$, if none of the Fourier exponents of $f(t)$ occurs in $R_n(t)$, and $F_n(t) = \sum a_k e^{i\lambda_k t}$, the summation is extended to those ks, where λ_k is a common Fourier exponent to $f(t)$ and $R_n(t)$. So, one has

$$M\{\|f - F_n\|^2\} = M\{\|f\|^2\} - \sum \|a_k\|^2,$$

where the summation is extended to those ks, λ_k is a Fourier exponent to $R_n(t)$. Thus, by (4.5) and (4.6), we have

$$M\{\|f\|^2\} - \sum \|a_k\|^2 \le M\{\|f - R_n\|^2\} \le \frac{1}{n}.$$

Hence,

$$M\{\|f\|^2\} \le \sum \|a_k\|^2 + \frac{1}{n},$$

where the summation is extended to those ks, λ_k is a Fourier exponent to $R_n(t)$. Furthermore, we can write the inequality

$$M\{\|f\|^2\} \le \sum_{k=1}^{\infty} \|a_k\|^2 + \frac{1}{n}.$$

Let $n \to \infty$, it follows that

$$M\{\|f\|^2\} \le \sum_{k=1}^{\infty} \|a_k\|^2.$$

This completes the proof. □

Theorem 4.10. *Let $f, g \in AP(\mathbb{T}, \mathbb{C})$. If they have the same Fourier series, then $f(t) \equiv g(t)$ for all $t \in \mathbb{T}$.*

Proof. In view of Theorem 4.9, one has

$$M\{\|f(t) - g(t)\|^2\} = 0,$$

which means that $f(t) \equiv g(t)$. Hence, we obtain that $f(t) \equiv g(t)$, $t \in \mathbb{T}$. This completes the proof. □

4.2 Equivalent definitions of almost periodic functions on time scales

In this subsection, we first give definitions of almost periodic functions in Bochner's sense and almost periodic functions in Bohr's sense on $\mathbb{T}$ and then discuss some of their properties.

Definition 4.4. A function $f \in BC(\mathbb{T}, \mathbb{X})$ is said to be almost periodic in Bochner's sense, if the family of translates $\mathcal{F} = \{f(t + a);\ a \in \Pi\}$ is relatively compact in $BC(\mathbb{T}, \mathbb{X})$. The set of all such functions will be denoted by $AP_N(\mathbb{T}, \mathbb{X})$.

Definition 4.5. A function $f \in BC(\mathbb{T}, \mathbb{X})$ is said to be almost periodic in Bohr's sense, if for each $\varepsilon > 0$, the ε-translation set of

$$E\{\varepsilon, f\} = \{\tau \in \Pi : \|f(t + \tau) - f(t)\|_{\mathbb{X}} < \varepsilon,\ \forall t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{R}$, i.e., for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains at least one $\tau(\varepsilon) \in E\{\varepsilon, f\}$ such that

$$\|f(t+\tau) - f(t)\|_{\mathbb{X}} < \varepsilon, \quad \forall t \in \mathbb{T}.$$

The $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$ and the τ is called ε-translation number of f. The set of all such functions will be denoted by $AP_B(\mathbb{T}, \mathbb{X})$.

To prove the equivalence of Definitions 3.2, 4.4, and 4.5, the introduction of the Fourier series of almost periodic functions in Bohr's sense is very critical. Therefore, we must first derive the existence of the mean value.

Theorem 4.11. *If $f \in AP_B(\mathbb{T}, \mathbb{X})$, then*

$$M\{f\} := \lim_{r\to\infty} \frac{1}{r}\int_{\beta}^{\beta+r} f(t)\Delta t, \quad \text{where } \beta \in \mathbb{T},\ r \in \Pi$$

uniformly exists for $\beta \in \mathbb{T}$. $M\{f\}$ is independent of β and is called the mean value of f.

Proof. For any $\varepsilon > 0$, there exists a constant $l = l(\varepsilon) > 0$ such that each interval $(\beta, \beta + l)$, $\beta \in \mathbb{T}$ contains at least one $\tau = \tau(\varepsilon) \in \Pi$ such that

$$\|f(t+\tau) - f(t)\|_{\mathbb{X}} < \frac{\varepsilon}{2}.$$

For a given $a \in \Pi$, one has

$$\frac{1}{r}\int_{\beta+a}^{\beta+a+r} f(t)\Delta t = \frac{1}{r}\int_{\beta+a}^{\beta+\tau} f(t)\Delta t + \frac{1}{r}\int_{\beta+\tau}^{\beta+r+\tau} f(t)\Delta t + \frac{1}{r}\int_{\beta+r+\tau}^{\beta+a+r} f(t)\Delta t. \tag{4.7}$$

Denote $M = \sup_{t\in\mathbb{T}} \|f(t)\|_{\mathbb{X}}$. Then, in view of (4.7), we have

$$\begin{aligned}
&\frac{1}{r}\left\|\int_{\beta}^{\beta+r} f(t)\Delta t - \int_{\beta+a}^{\beta+a+r} f(t)\Delta t\right\|_{\mathbb{X}} \\
&\leq \frac{1}{r}\left\|\int_{\beta}^{\beta+r} f(t)\Delta t - \int_{\beta+\tau}^{\beta+r+\tau} f(t)\Delta t\right\|_{\mathbb{X}} + \frac{1}{r}\left\|\int_{\beta+a}^{\beta+\tau} f(t)\Delta t\right\|_{\mathbb{X}} + \frac{1}{r}\left\|\int_{\beta+r+\tau}^{\beta+a+r} f(t)\Delta t\right\|_{\mathbb{X}} \\
&\leq \frac{1}{r}\int_{\beta}^{\beta+r} \|f(t) - f(t+\tau)\|_{\mathbb{X}}\Delta t + \frac{1}{r}\int_{\beta+a}^{\beta+\tau} \|f(t)\|_{\mathbb{X}}\Delta t + \frac{1}{r}\int_{\beta+r+\tau}^{\beta+a+r} \|f(t)\|_{\mathbb{X}}\Delta t \\
&< \frac{\varepsilon}{2} + \frac{2|\tau - a|M}{r}.
\end{aligned} \tag{4.8}$$

Taking $a = (k-1)r$, $k = 1, 2, \ldots, n$, from (4.8), one can deduce that

$$\frac{1}{r}\left\|\int_{\beta}^{\beta+r} f(t)\Delta t - \int_{\beta+(k-1)r}^{\beta+kr} f(t)\Delta t\right\|_{\mathbb{X}} < \frac{\varepsilon}{2} + \frac{2|\tau - a|M}{r}, \quad k = 1, 2, \ldots, n. \tag{4.9}$$

It follows from inequalities (4.15) that

$$\frac{1}{r}\left\|\int_{\beta}^{\beta+r} f(t)\Delta t - \frac{1}{m}\int_{\beta}^{\beta+mr} f(t)\Delta t\right\|_{\mathbb{X}} = \frac{1}{r}\left\|\int_{\beta}^{\beta+r} f(t)\Delta t - \frac{1}{m}\sum_{k=1}^{m}\int_{\beta+(k-1)r}^{\beta+kr} f(t)\Delta t\right\|_{\mathbb{X}} \tag{4.10}$$

$$\leq \frac{1}{rm}\sum_{k=1}^{m}\left\|\int_{\beta}^{\beta+r} f(t)\Delta t - \int_{\beta+(k-1)r}^{\beta+kr} f(t)\Delta t\right\|_{\mathbb{X}}$$
$$< \frac{1}{m} m\left(\frac{\varepsilon}{2} + \frac{2|\tau - a|M}{r}\right)$$
$$= \frac{\varepsilon}{2} + \frac{2|\tau - a|M}{r}, \quad k = 1, 2, \ldots, n.$$

Take $r_1, r_2 \in \Pi$ with $r_1, r_2 > 0$ such that $m_1 r_1 = m_2 r_2$, for some natural numbers m_1 and m_2, according to (4.10), we can derive that

$$\begin{aligned}
&\left\|\frac{1}{r_1}\int_{\beta}^{\beta+r_1} f(t)\Delta t - \frac{1}{r_2}\int_{\beta}^{\beta+r_2} f(t)\Delta t\right\|_{\mathbb{X}} \\
&\leq \frac{1}{r_1}\left\|\int_{\beta}^{\beta+r_1} f(t)\Delta t - \frac{1}{m_1}\int_{\beta}^{\beta+m_1 r_1} f(t)\Delta t\right\|_{\mathbb{X}} + \frac{1}{r_2}\left\|\frac{1}{m_2}\int_{\beta}^{\beta+m_2 r_2} f(t)\Delta t - \int_{\beta}^{\beta+r_2} f(t)\Delta t\right\|_{\mathbb{X}} \\
&< \varepsilon + 2|\tau - a|M\left(\frac{1}{r_1} + \frac{1}{r_2}\right).
\end{aligned} \tag{4.11}$$

We choose $r_1, r_2 > 4|\tau - a|M$, it follows from (4.11) that

$$\left\|\frac{1}{r_1}\int_{\beta}^{\beta+r_1} f(t)\Delta t - \frac{1}{r_2}\int_{\beta}^{\beta+r_2} f(t)\Delta t\right\|_{\mathbb{X}} < 2\varepsilon,$$

which implies that the mean value of $f(t)$ exists. From inequalities (4.8), choose $r \in \Pi$ with $r > 2|\tau - a|M$, for any $\beta \in \mathbb{T}$ and $a \in \Pi$, one has

$$\frac{1}{r}\left\|\int_{\beta}^{\beta+r} f(t)\Delta t - \int_{\beta+a}^{\beta+a+r} f(t)\Delta t\right\|_{\mathbb{X}} < \varepsilon,$$

which means that $M\{f\}$ is independent of $\beta \in \mathbb{T}$. This completes the proof. □

Remark 4.1. It is useful to point out the fact from Theorem 4.11 that the Fourier series of $f \in AP_B(\mathbb{T}, \mathbb{X})$ exists, and if $f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$, then, similar to the proof on page 28 of [26], one can easily prove that Parseval's equality

$$\sum_{k=1}^{\infty} \|a_k\|_{\mathbb{X}}^2 = M\{\|f(t)\|_{\mathbb{X}}^2\}$$

is also valid.

Next we will prove the equivalence of Definitions 3.2, 4.4, and 4.5.

Theorem 4.12. *If $f \in AP(\mathbb{T}, \mathbb{X})$, then $f \in AP_N(\mathbb{T}, \mathbb{X})$.*

Proof. Obviously, to prove the theorem, it is suffice to show that every sequence $\{f(t + \alpha_n); \alpha_n \in \Pi, n \geq 1\} \subset AP(\mathbb{T}, \mathbb{X})$ contains a subsequence that converges in $AP(\mathbb{T}, \mathbb{X})$.

Let $R(t) = e^{i\lambda t}$, where $\lambda \in \mathbb{R}$, $t \in \mathbb{T}$. If $\{b_k; k \geq 1\} \subset \Pi$ is an arbitrary sequence, then one obtains that $R(t + b_k) = e^{i\lambda t} \cdot e^{i\lambda b_k}$, $k \geq 1$. Since $|e^{i\lambda b_k}| = 1$, in view of the Bolzano-Weierstrass criterion, there exists a subsequence $\{b_{1k}; k \geq 1\} \subset \{b_k; k \geq 1\}$ such that $\{e^{i\lambda b_{1k}}; k \geq 1\}$ is convergent. Since

$$|R(t + b_{1j}) - R(t + b_{1k})| = |e^{i\lambda b_{1j}} - e^{i\lambda b_{1k}}|,$$

it follows from Cauchy's criterion that $\{R(t + b_{1k}); k \geq 1\}$ is uniformly convergent on $\mathbb{T}$.

Now, consider a trigonometric polynomial $T(t) = \sum_{k=1}^{h} a_k e^{i\lambda_k t}$, where $a_k \in \mathbb{X}$ and $\lambda_k \in \mathbb{R}$. Let $\{\beta_k;\ k \geq 1\} \subset \Pi$ be an arbitrary sequence, then from the previous discussion, we know that there exists a subsequence $\{\beta_k^1;\ k \geq 1\} \subset \{\beta_k;\ k \geq 1\}$ such that $\{a_1 e^{i\lambda_1 \beta_k^1};\ k \geq 1\}$ is uniformly convergent on $\mathbb{T}$. Proceeding in the same way, there exists a subsequence $\{\beta_k^h;\ k \geq 1\} \subset \{\beta_k^m;\ k \geq 1\}$ such that $\{a_n e^{i\lambda_h \beta_k^h};\ k \geq 1\}$ is uniformly convergent on $\mathbb{T}$, where $1 \leq m < h$. Consequently, we have proved that $\{T(t+\beta_k);\ k \geq 1\}$ contains a subsequence $\{T(t+b_k^h);\ k \geq 1\}$ that is uniformly convergent on $\mathbb{T}$.

Finally, since $f \in AP(\mathbb{T}, \mathbb{X})$, there exists a sequence of trigonometric polynomials $\{T_n(t);\ n \geq 1\} \subset \mathcal{T}$ such that

$$\lim_{n\to\infty} T_n(t) = f(t) \tag{4.12}$$

uniformly on $\mathbb{T}$. Let $\{\alpha_n;\ n \geq 1\} \subset \Pi$ be a sequence. Based on the above proof, from the sequence $\{T_1(t+\alpha_n); n \geq 1\}$, one can extract a subsequence $\{\alpha_n^1;\ n \geq 1\} \subset \Pi$ such that $\{T_1(t+\alpha_n^1);\ n \geq 1\}$ is uniformly convergent on $\mathbb{T}$. From the sequence $\{\alpha_n^1;\ n \geq 1\} \subset \Pi$, one can extract a subsequence $\{\alpha_n^2;\ n \geq 1\} \subset \Pi$ such that $\{T_2(t+\alpha_n^2);\ n \geq 1\}$ is uniformly convergent on $\mathbb{T}$. Going on like this, for any integer p, for a sequence of $\{\alpha_n^p;\ n \geq 1\} \subset \Pi$, we can obtain that $\{T_q(t+\beta_n^p);\ n \geq 1\}$ is uniformly convergent on $\mathbb{T}$, $q = 1, 2, \ldots, p$. This means that $\{T_n(t+\beta_m^m);\ m \geq 1\}$ with fixed n is uniformly convergent on $\mathbb{T}$. In the view of (4.12), for any $\varepsilon > 0$, we can choose n large enough such that

$$\|f(t) - T_n(t)\|_{\mathbb{X}} < \frac{\varepsilon}{3}, \quad \text{for all } t \in \mathbb{T}.$$

Hence, there exists $N = N(\varepsilon) > 0$ such that

$$\|T_n(t+b_{mm}) - T_n(t+\alpha_p^p)\|_{\mathbb{X}} < \frac{\varepsilon}{3}, \quad \text{for all } t \in \mathbb{T},$$

for $m, p \geq N(\varepsilon)$. Thus, for $m, p \geq N(\varepsilon)$, we have

$$\begin{aligned}
&\|f(t+\alpha_{mm}^m) - f(t+\alpha_p^p)\|_{\mathbb{X}} \\
&\quad \leq \|f(t+\alpha_m^m) - T(t+\alpha_m^m)\|_{\mathbb{X}} + \|T_n(t+\alpha_m^m) - T_n(t+\alpha_p^p)\|_{\mathbb{X}} + \|f(t+\alpha_p^p) - T_n(t+\alpha_p^p)\|_{\mathbb{X}} \\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \text{for all } t \in \mathbb{T},
\end{aligned}$$

which means that the sequence $\{f(t+\alpha_{mm});\ m \geq 1\}$ is uniformly convergent on $\mathbb{T}$. Thus, $f \in AP_N(\mathbb{T}, \mathbb{X})$. This completes the proof. □

Theorem 4.13. *If $f \in AP_N(\mathbb{T}, \mathbb{X})$, then $f \in AP_B(\mathbb{T}, \mathbb{X})$.*

Proof. On the contrary, if this is not true, then there exists $\varepsilon > 0$ such that for any sufficiently large $l(\varepsilon) > 0$, one can find an interval with length of $l(\varepsilon)$ that contains no ε-translation numbers of f. That is, every point in this interval is not in $E\{\varepsilon, f\}$.

According to Lemma 4.1, we can take a number $\beta_1 \in \Pi$ and an interval (c_1, d_1) with

$$d_1 - c_1 > 2|\beta_1| \tag{4.13}$$

such that (c_1, d_1) contains no ε-translation numbers of f, where $c_1, d_1 \in \mathbb{T}$ with $\frac{d_1+c_1}{2} \in \Pi$. Next, denote

$$\beta_2 = \frac{d_1 + c_1}{2}. \tag{4.14}$$

By (4.13) and (4.14), we obtain $\beta_2 - \beta_1 \in (c_1, d_1)$. Hence, $\beta_2 - \beta_1 \notin E\{\varepsilon, f\}$. Again, by Lemma 4.1, take an interval (c_2, d_2) with $d_2 - c_2 > 2(|\beta_1| + |\beta_1|)$ such that (c_2, d_2) contains no ε-translation numbers of f, where $c_2, d_2 \in \mathbb{T}$ with $\frac{d_2+c_2}{2} \in \Pi$. Next, denote $\beta_3 = \frac{d_2+c_2}{2}$, obviously, $\beta_3 - \beta_2 \notin E\{\varepsilon, f\}$ and $\beta_3 - \beta_1 \notin E\{\varepsilon, f\}$. Proceeding similarly, one can find $\beta_4, \beta_5, \ldots$ such that $\beta_i - \beta_j \notin E\{\varepsilon, f\}$, $i > j$. Thus, for $i > j$, $i, j = 1, 2, 3, \ldots$, we have

$$\sup_{t\in\mathbb{T}} \|f(t+\beta_i) - f(t+\beta_j)\|_{\mathbb{X}} = \sup_{t\in\mathbb{T}} \|f(t+\beta_i-\beta_j) - f(t)\|_{\mathbb{X}} \geq \varepsilon, \quad t \in \mathbb{T},$$

which contradicts relative compactness of the family $\mathcal{F} = \{f(t+a);\ a \in \Pi\}$. By Theorem 4.12, the conclusion is valid. Thus, $f \in AP_B(\mathbb{T}, \mathbb{X})$. This completes the proof. □

Theorem 4.14. *If* $f \in AP_B(\mathbb{T}, \mathbb{X})$, *then* $f \in AP(\mathbb{T}, \mathbb{X})$.

Proof. Let $f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$ and

$$F_n(t) = f(t) - \sum_{k=1}^{n} a_k e^{i\lambda_k t}. \tag{4.15}$$

According to Remark 4.1,

$$M\{\|F_n(t)\|_{\mathbb{X}}^2\} = \sum_{k=n+1}^{+\infty} \|a_k\|_{\mathbb{X}}^2.$$

Because $\sum_{k=1}^{\infty} |a_k|^2$ is convergent, so for any $\xi > 0$, we can find n such that

$$M\{\|F_n(t)\|_{\mathbb{X}}^2\} < \xi. \tag{4.16}$$

Moreover, by Theorem 4.11, there exists $T_0 \in \Pi$ such that

$$\left| \frac{1}{T} \int_{\alpha}^{\alpha+T} \|F_n(t+s)\|_{\mathbb{X}}^2 \Delta t - M\{\|F_n(t)\|_{\mathbb{X}}^2\} \right| < \xi, \quad \alpha \in \mathbb{T}, \tag{4.17}$$

for all $T \in \Pi$ with $T > T_0$ and for all $s \in \Pi$. Hence, it follows from (4.16) and (4.17) that

$$\frac{1}{T} \int_{\alpha}^{\alpha+T} \|F_n(t+s)\|_{\mathbb{X}}^2 \Delta t < 2\xi, \quad \alpha \in \mathbb{T}. \tag{4.18}$$

Let $l(\frac{\varepsilon}{3})$ be the inclusion length of $E\{\frac{\varepsilon}{3}, f\}$. Take $T = Z(l(\frac{\varepsilon}{3}) + 1)$, Z is a nature number. Since $f \in AP_B(\mathbb{T}, \mathbb{X})$, every interval $(\alpha + m(l(\frac{\varepsilon}{3}) + 1), \alpha + m(l(\frac{\varepsilon}{3}) + 1) + l(\frac{\varepsilon}{3}))$ contains a $\tau_m \in E\{\frac{\varepsilon}{3}, f\}$, $m = 0, 2, \dots, Z-1$ such that

$$\|f(\tau_m + s) - f(s)\|_{\mathbb{X}} < \frac{\varepsilon}{3}. \tag{4.19}$$

Then, by Theorem 3.1, we can take a $\delta \in \Pi$ such that for $t_1, t_2 \in \mathbb{T}$ with $|t_1 - t_2| < \delta$,

$$\|f(t_1) - f(t_2)\|_{\mathbb{X}} < \frac{\varepsilon}{3}.$$

Define a function $h(t)$ in the interval $(\alpha, \alpha + T)$ by

$$h(t) = \begin{cases} 1, & t \in (\alpha + \tau_m, \alpha + \tau_m + \delta), \quad m = 0, 2, \dots, Z-1; \\ 0, & t \in (\alpha, \alpha + T) \backslash (\alpha + \tau_m, \alpha + \tau_m + \delta), \quad m = 0, 2, \dots, Z-1. \end{cases}$$

Thus,

$$\int_{\alpha}^{\alpha+T} \|h(t)\|_{\mathbb{X}}^2 \Delta t = \sum_{m=1}^{Z-1} \int_{\alpha+\tau_m}^{\alpha+\tau_m+\delta} \|h(t)\|_{\mathbb{X}}^2 \Delta t = Z\delta. \tag{4.20}$$

Based on (4.18), (4.20), and the Schwarz inequality, we obtain that

$$\left\| \int_{\alpha}^{\alpha+T} F_n(t+s)h(t)\Delta t \right\|_{\mathbb{X}}^2 \leq \int_{\alpha}^{\alpha+T} \|F_n(t+s)\|_{\mathbb{X}}^2 \Delta t \int_{\alpha}^{\alpha+T} \|h(t)\|_{\mathbb{X}}^2 \Delta t \leq 2T\xi Z\delta. \tag{4.21}$$

Observing that

$$\int_{\alpha}^{\alpha+T} F_n(t+s)h(t)\Delta t = \sum_{m=1}^{Z-1} \int_{\alpha+\tau_m}^{\alpha+\tau_m+\delta} F_n(t+s)\Delta t = \sum_{m=1}^{Z-1} \int_{\alpha}^{\alpha+\delta} F_n(t+\tau_m+s)\Delta t,$$

we conclude from (4.21) that

$$\left\| \sum_{m=1}^{Z-1} \int_{\alpha}^{\alpha+\delta} F_n(t+\tau_m+s)\Delta t \right\|_{\mathbb{X}} < \sqrt{2T\xi Z\delta}.$$

Thus, taking $\xi < \frac{\varepsilon^2\delta}{18(l(\frac{\varepsilon}{3})+1)}$, from $T = Z(l(\frac{\varepsilon}{3})+1)$, we have

$$\left\| \frac{1}{Z\delta} \sum_{m=1}^{Z-1} \int_{\alpha}^{\alpha+\delta} F_n(t+\tau_m+s)\Delta t \right\|_{\mathbb{X}} < \sqrt{\frac{2\xi T}{Z\delta}} < \sqrt{\frac{2\xi(l(\frac{\varepsilon}{3})+1)}{\delta}} < \frac{\varepsilon}{3}. \tag{4.22}$$

It follows from (4.15) that

$$\left\| \frac{1}{Z\delta} \int_{\alpha}^{\alpha+\delta} F_n(t+\tau_m+s)\Delta t \right\|_{\mathbb{X}} = \left\| \frac{1}{Z\delta} \int_{\alpha}^{\alpha+\delta} f(t+\tau_m+s)\Delta t - P_m(s) \right\|_{\mathbb{X}}, \tag{4.23}$$

where

$$P_m(s) = \frac{1}{Z\delta} \int_{\alpha}^{\alpha+\delta} \sum_{k=1}^{n} a_k e^{i\lambda_k(t+\tau_m+s)}\Delta t = \frac{1}{Z\delta} \sum_{k=1}^{n} e^{i\lambda_k s} \int_{\alpha}^{\alpha+\delta} a_k e^{i\lambda_k(t+\tau_m)}\Delta t.$$

Thus, P_m is a trigonometric polynomial. Based on (4.22) and (4.23), one has

$$\left\| \frac{1}{Z\delta} \sum_{m=1}^{Z-1} \int_{\alpha}^{\alpha+\delta} f(t+\tau_m+s)\Delta t - P(s) \right\|_{\mathbb{X}} < \frac{\varepsilon}{3}, \tag{4.24}$$

where $P(s) = \sum_{m=1}^{Z-1} P_m(s)$. We can choose value of $\delta \in \Pi$ such that

$$\|f(t+\tau_m+s) - f(\tau_m+s)\|_{\mathbb{X}} < \frac{\varepsilon}{3},$$

for $t \in \mathbb{T}$ with $0 \le t \le \delta$. Consequently,

$$\left\| \frac{1}{\delta} \int_{\alpha}^{\alpha+\delta} f(t+\tau_m+s)\Delta t - f(\tau_m+s) \right\|_{\mathbb{X}} < \frac{\varepsilon}{3}.$$

It follows from the above inequality and (4.19) that

$$\left\| \frac{1}{\delta} \int_{\alpha}^{\alpha+\delta} f(t+\tau_m+s)\Delta t - f(s) \right\|_{\mathbb{X}} < \frac{2\varepsilon}{3}. \tag{4.25}$$

Hence, based on (4.24) and (4.25), one has

$$\begin{aligned} \|f(s) - P(s)\|_{\mathbb{X}} &= \left\| \frac{1}{\delta} \int_{\alpha}^{\alpha+\delta} f(t+\tau_m+s)\Delta t - f(s) \right\|_{\mathbb{X}} + \left\| \frac{1}{\delta} \int_{\alpha}^{\alpha+\delta} f(t+\tau_m+s)\Delta t - P(s) \right\|_{\mathbb{X}} \\ &< \frac{2\varepsilon}{3} + \left\| \frac{1}{Z\delta} \sum_{m=1}^{Z-1} \int_{\alpha}^{\alpha+\delta} f(t+\tau_m+s)\Delta t - P(s) \right\|_{\mathbb{X}} \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This completes the proof. □

Remark 4.2. According to Theorems 4.12, 4.13, and 4.14, it is obvious that Definitions 3.2, 4.4, and 4.5 are equivalent.

Theorem 4.15. *Let $f \in AP(\mathbb{T}, \mathbb{X})$ be a non-negative function. If $f(t) \not\equiv 0$, then $M\{f\} > 0$.*

Proof. At some point $t_0 \in \mathbb{T}$, let $f(t_0) = \beta > 0$. Then, take a number $\delta \in \Pi$ with $\delta > 0$ such that

$$|f(t) - f(t_0)| < \frac{\beta}{3},$$

for $|t - t_0| < \delta$. Since $f \in AP(\mathbb{T}, \mathbb{X})$, there exist a positive number $l_1 > 0$ such that any interval of length l_1 contains $\frac{\beta}{3}$-translation numbers of f. Let $c, l_1 \in \Pi$, then $\tau \in (c - t_0 + \delta, c - t_0 + \delta + l_1)$ is a $\frac{\beta}{3}$-translation number. Thus, $t_0 + \tau \in (c + \delta, c + \delta + l_1)$. Since $|t - t_0| < \delta$, it is easy to see that $t + \tau \in (c, c + l_1 + 2\delta)$. Therefore, one has

$$f(t + \tau) \geq |f(t_0)| - |f(t) - f(t_0)| - |f(t + \tau) - f(t)| > \beta - \frac{\beta}{3} - \frac{\beta}{3} = \frac{\beta}{3}.$$

One can take $l = l_1 + 2\delta$ and conclude that any interval of length l on $\mathbb{T}$ contains a subinterval of length 2δ at all points of which $f(t) > \frac{\beta}{3}$. Thus,

$$\frac{1}{nl} \int_{\alpha}^{\alpha+nl} f(t)\Delta t = \frac{1}{nl} \sum_{k=1}^{n} \int_{\alpha+(k-1)l}^{\alpha+kl} f(t)\Delta t > \frac{\delta\beta}{3l}.$$

Letting $n \to \infty$, one can obtain $M\{f\} \geq \frac{\delta\beta}{3l} > 0$. This completes the proof. □

Theorem 4.16. *Let $f, g \in AP(\mathbb{T}, \mathbb{X})$. If they have the same Fourier series, then $f(t) \equiv g(t)$ for $t \in \mathbb{T}$.*

Proof. If $f(t) \neq g(t)$ and they have the same Fourier series, then by Theorem 4.9, one has

$$M\{|f(t) - g(t)|^2\} = 0,$$

which is a contradiction by Theorem 4.15. Hence, we obtain that $f(t) \equiv g(t)$ for $t \in \mathbb{T}$. This completes the proof. □

5 Conclusion

In this study, we have proposed a concept of almost periodic functions on arbitrary time scales defined by the closure of the set of trigonometric polynomials in the supremum norm and investigated some their basic properties. Moreover, we have introduced the concepts of mean-value and Fourier series associated with an almost periodic function on time scales and present some relative results. Finally, we have given the definitions of almost periodic functions on time scales in Bohr's sense and in Bochner's sense, and discussed the relationship among them. The research in this work lays a foundation for further study of almost periodic function theory on time scales and almost periodic solutions of dynamic equations.

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