Research Article

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Spectrum boundary domination of semiregularities in Banach algebras

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Abstract: Let $\mathcal{A}$ be a complex unital Banach algebra and (R_1, R_2) be a pair of semiregularities in $\mathcal{A}$. In order to investigate the boundary of spectra in the axiomatic theory of spectrum, this article defines the concept of spectrum boundary domination (SBD) of semiregularities and gives a criterion for a pair (R_1, R_2) possessing SBD property. Furthermore, the conditions such that $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ are described for $a \in \mathcal{A}$ under the assumption that the pair (R_1, R_2) possesses SBD property. In addition, the transfer of SBD property through a Banach algebra homomorphism is discussed.

Keywords: spectrum, regularity, spectrum boundary domination, Banach algebra, homomorphism

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1 Introduction

All algebras in this article are complex and unital. Denote by $B(X)$ the Banach algebra of all linear bounded operators acting on a Banach space X. The ordinary spectrum of an operator $T \in B(X)$ is widely studied in the existence and stability of solutions in the theory of differential equations and integral equations, also closely related to physics, engineering, quantum mechanics, and other disciplines. The rationale of these studies is the exploration of the invertibility of T. It was Atkinson who gave a necessary and sufficient condition for T to be Fredholm, which is that the coset of T is invertible in Calkin algebra $C(X)$, the quotient algebra of $B(X)$ module, and the ideal of compact operators acting on X. He pioneered the technique of constructing Fredholm-type operators through quotient homomorphism, which became a link between invertibility and classic Fredholm theory. As an extension, Fredholm theory involving quotient Banach algebras with respect to inessential ideals was greatly developed. More abstractly, Fredholm theory with respect to a continuous Banach algebra homomorphism was introduced in [1] and developed by many scholars (see for example, [2,3]). This theory is widely applied, especially in the development of spectral theory in Banach algebra [1,4,5].

On the other hand, by means of Atkinson's theorem, different classes of operators appeared, accompanied by the study of associated spectra and their properties, especially spectral mapping theorem and the boundary of these spectra (see, for example, [6,7]). Then, the concept of various spectra of T gradually became independent of Atkinson's theorem. These spectra of T were defined by $\sigma_S(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } S\}$, where S is the set of all linear bounded operators satisfying some condition, which can be seen as generalized invertibility, such as Saphar condition, consistent in invertibility, and consistent in Fredholm index (see [8–10]). Meanwhile, instead of $B(X)$, many scholars turned to study spectral theory in more general Banach algebras or in Hilbert C*-modules [1,11,12].

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Illuminated by the properties of $\mathcal{A}^{-1}$, the set of invertible elements in a Banach algebra $\mathcal{A}$, Kordula and Müller introduced the axiomatic theory of spectrum with the help of regularity [13]. Namely, they assigned to each $a \in \mathcal{A}$ a subset of $\mathbb{C}$ and call it the spectrum of a associated with R, where R is a regularity in $\mathcal{A}$. Later, Müller extended this theory to lower semiregularity and upper semiregularity in [14]. Since then, the axiomatic theory of spectrum has provided guidance for the research of spectral theory [11,15–20].

Along with the research on spectra, both spectral mapping theorem and topological properties of various spectra have been inevitably discussed. The former is emphatically studied in the axiomatic theory of spectrum. More precisely, the spectrum associated with a semiregularity satisfies one-side spectral mapping theorem at least. As to the topological properties, the boundary of spectrum is often closely related to Mobius spectrum, which is one of the issues of great concern in recent decades (see, for example, [5,21–23]). However, the boundary of spectrum was rarely discussed in the axiomatic theory of spectrum.

For a Banach algebra $\mathcal{A}$ with an identity $1_{\mathcal{A}}$, denote the ordinary spectrum of $a \in \mathcal{A}$ by $\sigma_{\mathcal{A}^{-1}}(a)$, which is associated with the smallest regularity $\mathcal{A}^{-1}$ in $\mathcal{A}$. For a closed subalgebra $\mathcal{B}$ of $\mathcal{A}$ with common identity, it is well known that

$$\partial\sigma_{\mathcal{B}^{-1}}(a) \subseteq \partial\sigma_{\mathcal{A}^{-1}}(a) \subseteq \sigma_{\mathcal{A}^{-1}}(a) \subseteq \sigma_{\mathcal{B}^{-1}}(a), \tag{1}$$

for every $a \in \mathcal{A}$, where ∂K denotes the topological boundary of $K \subseteq \mathbb{C}$. Furthermore, with the help of (1), it is known that $\sigma_{\mathcal{A}^{-1}}(a) = \sigma_{\mathcal{B}^{-1}}(a)$ under specific conditions, such as $\mathcal{B}$ being a maximal commutative subalgebra of $\mathcal{A}$, or $\mathcal{A}$ being a C*-algebra and $\mathcal{B}$ a C*-subalgebra of $\mathcal{A}$. Therefore, some scholars studied this inclusion about various spectra in Banach algebra, such as singular spectrum, exponential spectrum, and boundary spectrum [3,21]. On the other hand, the inclusion of two spectra mentioned earlier has been realized for concrete semiregularities in $B(X)$ (see, for example, [7,24,25]). The purpose of this article is to probe the inclusion

$$\partial\sigma_{R_1}(a) \subseteq \sigma_{R_2}(a) \subseteq \sigma_{R_1}(a), \quad (\forall a \in \mathcal{A}), \tag{2}$$

where (R_1, R_2) is a pair of semiregularities in $\mathcal{A}$, and $\sigma_{R_i}(a)$ $(i = 1, 2)$ means the spectrum of a associated with R_i. From now on, we call a pair (R_1, R_2) of semiregularities possessing spectrum boundary domination (SBD) property, provided that it satisfies the inclusion relation (2) for all $a \in \mathcal{A}$.

The aim of Section 3 is to give a criterion for SBD property and, furthermore, to study conditions such that $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ for $a \in \mathcal{A}$ under the assumption that the pair (R_1, R_2) possesses SBD property. In Section 4, given two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ and an algebra homomorphism $T : \mathcal{A} \to \mathcal{B}$, we consider the SBD property transferred from $\mathcal{B}$ to $\mathcal{A}$ through T. Before that, we obtain some results on the transfer of semiregularity through T and on the preservation of the associated spectra in the axiomatic theory of spectrum.

2 Preliminaries and facts

In this article, assume that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras with their respective identities $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$. Denote the spectrum of $a \in \mathcal{A}$ by $\sigma_{\mathcal{A}^{-1}}(a)$ and the spectrum of $b \in \mathcal{B}$ by $\sigma_{\mathcal{B}^{-1}}(b)$.

For a subset K of $\mathbb{C}$, denote the closure of K by $\overline{K}$ and the set of the interior of K by $\mathrm{int}K$. If K is a bounded subset of $\mathbb{C}$, ηK is indicated to the connected hull of $\overline{K}$, the compact set consisting of $\overline{K}$ together with holes of $\mathbb{C}\backslash\overline{K}$, where a hole is a bounded component of $\mathbb{C}\backslash\overline{K}$. Furthermore, the complement of ηK is connected.

In the following, denote by $\mathbb{N}$ and $\mathbb{D}$ the set of all positive integers and the closed unit disk in $\mathbb{C}$, respectively. Let us recall the definition of regularity in a Banach algebra.

Definition 1. [13] A non-empty subset R of $\mathcal{A}$ is called a regularity if it satisfies the following conditions:
(1) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in R \Leftrightarrow a^n \in R$;
(2) if a, b, c, and d are mutually commuting elements of $\mathcal{A}$ such that $ac + bd = 1_{\mathcal{A}}$, then $ab \in R \Leftrightarrow a \in R$ and $b \in R$.

Following [14], conditions (1) and (2) in Definition 1 can be split into two directions, each of them implying a one-sided regularity and giving rise to a one-sided spectral mapping theorem.

Definition 2. [14]

(1) A non-empty subset R of $\mathcal{A}$ is called a upper semiregularity if it satisfies the following conditions:
 (a) if $a \in R$ and $n \in \mathbb{N}$, then $a^n \in R$;
 (b) if $a, b, c,$ and d are mutually commuting elements of $\mathcal{A}$ such that $ac + bd = 1_{\mathcal{A}}$, then $a \in R$ and $b \in R \Rightarrow ab \in R$;
 (c) R contains some neighborhood of identity $1_{\mathcal{A}}$.

(2) A non-empty subset R of $\mathcal{A}$ is called a lower semiregularity if it satisfies the following conditions:
 (a) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a^n \in R \Rightarrow a \in R$;
 (b) if $a, b, c,$ and d are mutually commuting elements of $\mathcal{A}$ such that $ac + bd = 1_{\mathcal{A}}$, then $ab \in R \Rightarrow a \in R$ and $b \in R$.

An interesting fact is that $\mathcal{A}^{-1}$, the set of invertible elements in $\mathcal{A}$, is both the smallest regularity and the smallest lower semiregularity in $\mathcal{A}$. However, it is not the smallest upper semiregularity in $\mathcal{A}$ [26]. For instance, $\operatorname{Exp}(\mathcal{A}) = \{e^{a_1}e^{a_2} \dots e^{a_n} : a_i \in \mathcal{A}, 1 \le i \le n, n \in \mathbb{N}\}$ is an upper semiregularity in $\mathcal{A}$ as well as the principle component of $\mathcal{A}^{-1}$, i.e., the connected component of $\mathcal{A}^{-1}$ containing $1_{\mathcal{A}}$.

Definition 3. A semiregularity $R \subset \mathcal{A}$ assigns to each $a \in \mathcal{A}$ a subset of $\mathbb{C}$ naturally by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_{\mathcal{A}} \text{ is not in } R\}.$$

We call it the spectrum of a associated with R.

As we know, $\sigma_{\mathcal{A}^{-1}}(a)$ is a non-empty compact subset of $\mathbb{C}$. For a semiregularity R in $\mathcal{A}$, $\sigma_R(a)$ is bounded, since $\sigma_R(a) \subseteq \eta\sigma_{\mathcal{A}^{-1}}(a)$ is always valid [14]. The following examples illustrate that the spectrum associated with a semiregularity may be neither non-empty nor closed.

Example 1. (1) An element $a \in \mathcal{A}$ is a left zero divisor of $\mathcal{A}$ if $a \neq 0$ and $ax = 0$ for some non-zero element $x \in \mathcal{A}$ [26]. Let $\tilde{R}_1 = \{a \in \mathcal{A} : a \text{ is not a left zero divisor of } \mathcal{A}\}$. Then, $\tilde{R}_1$ is a regularity in $\mathcal{A}$ [26]. However, $\sigma_{\tilde{R}_1}(a)$ may not be closed and, furthermore, may be empty. Indeed, $\sigma_{\tilde{R}_1}(a)$ is empty if a is zero. In addition, taking $\mathcal{A} = B(\ell^2)$, we define $W \in B(\ell^2)$ by

$$W(x_1, x_2, x_3, \dots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right). \tag{3}$$

By calculation, $\sigma_{\tilde{R}_1}(W) \supseteq \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ and $\sigma_{\mathcal{A}^{-1}}(W) = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$. Besides, for $V \in B(\ell^2)$, $WV = 0$ implies $V = 0$ because W is injective. Furthermore, $0 \notin \sigma_{\tilde{R}_1}(W)$, i.e., $\sigma_{\tilde{R}_1}(W)$ is not closed.

(2) An element $a \in \mathcal{A}$ is called consistent in invertibility (CI) if, for each b in $\mathcal{A}$, ab and ba are invertible or noninvertible together [27]. Also, $a \in \mathcal{A}$ is not a CI element if and only if $a \in (\mathcal{A}_l^{-1} \backslash \mathcal{A}^{-1}) \cup (\mathcal{A}_r^{-1} \backslash \mathcal{A}^{-1})$ [27], where $\mathcal{A}_l^{-1}$ is the set of all left invertible elements in $\mathcal{A}$ and $\mathcal{A}_r^{-1}$ is the set of all right invertible elements in $\mathcal{A}$. Denote by $\tilde{R}_2$ the set of all CI elements in $\mathcal{A}$. Then, $\tilde{R}_2$ is an upper semiregularity $\mathcal{A}$, and $\sigma_{\tilde{R}_2}(a)$ is open for all a in $\mathcal{A}$ because $\mathcal{A}_l^{-1} \backslash \mathcal{A}^{-1}$ and $\mathcal{A}_r^{-1} \backslash \mathcal{A}^{-1}$ are disjoint open sets [26].

More specifically, taking $\mathcal{A} = B(\ell^2 \oplus \ell^2)$, we define $S \in B(\ell^2)$ by

$$S(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots). \tag{4}$$

Furthermore, $T \in \mathcal{A}$ is defined by

$$T = \begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix}. \tag{5}$$

By calculation, $\sigma_{\tilde{R}_2}(T) = \emptyset$.

(3) An element $a \in \mathcal{A}$ is said to be generalized Drazin invertible if there exists $b \in \mathcal{A}$ such that $ab = ba$ and $bab = b$ and that $aba - a$ is a quasinilpotent element [6]. Denote by $\tilde{R}_3$ the set of all generalized Drazin invertible elements in $\mathcal{A}$. Then, $\tilde{R}_3$ is a regularity in $\mathcal{A}$ [28], and $\sigma_{\tilde{R}_3}(c)$ is empty provided that $\sigma_{\mathcal{A}^{-1}}(c)$ is finite [6, Theorem 4.2], i.e., c is polynomially quasinilpotent by [29, Theorem 2.1].

For a non-empty subset R of $\mathcal{A}$, the following properties can provide a useful tool to verify whether R is a regularity or a semiregularity.
(P1) $ab \in R \Leftrightarrow a \in R$ and $b \in R$ for all commuting elements $a, b \in \mathcal{A}$;
(P1′) $ab \in R \Rightarrow a \in R$ and $b \in R$ for all commuting elements $a, b \in \mathcal{A}$;
(P1″) R is a semigroup containing some neighborhood of $1_{\mathcal{A}}$.

Combined with Definitions 1 and 2, we have the following conclusion.

Proposition 1. [26] *Let R be a non-empty subset of $\mathcal{A}$ satisfying* (P1) ((P1′), *and* (P1″) *resp.*). *Then, R is a regularity (a lower semiregularity, an upper semiregularity resp.*).

Different from the definition of regularity (lower semiregularity, upper semiregularity resp.), the property of (P1) ((P1′), and (P1″) resp.) does not involve the identity $1_{\mathcal{A}}$. We can see that $\tilde{R}_1$, $\mathcal{A}_l^{-1}$, $\mathcal{A}_r^{-1}$, and $\mathcal{A}^{-1}$ satisfy (P1), and that both $\tilde{R}_2$ and $\mathrm{Exp}(\mathcal{A})$ satisfy (P1″). Indeed, property (P1) ((P1′), and (P1″) resp.) is only a sufficient condition of regularity (lower semiregularity, upper semiregularity resp.). For instance, the set of operators possessing generalized inverse is a regularity in $B(X)$ without property (P1) [26, Chapter II 13 Lemma 4–5].

3 SBD property

Suppose that (R_1, R_2) is a pair of semiregularities in $\mathcal{A}$. The aim of this section is to characterize the relation $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ with the help of SBD property defined below, where $a \in \mathcal{A}$.

3.1 Definition and properties

For some semiregularities R_1 and R_2 in $B(X)$, we have already known that the inclusion $\partial\sigma_{R_1}(T) \subseteq \sigma_{R_2}(T) \subseteq \sigma_{R_1}(T)$ is valid for every $T \in B(X)$. The pair (R_1, R_2) is such an example, where R_1 is the set of Weyl operators in $B(X)$ and R_2 is the set of Fredholm operators in $B(X)$. As a result, one has $\partial\sigma_{R_1}(T) \subseteq \sigma_{R_2}(T) \subseteq \sigma_{R_1}(T)$ for every $T \in B(X)$ [7]. Now, we consider this inclusion in the axiomatic theory of spectrum.

Definition 4. Let (R_1, R_2) be a pair of semiregularities in $\mathcal{A}$. R_2 is said to be an SBD of R_1 (abbreviated as R_2 is R_1–SBD) if $\partial\sigma_{R_1}(a) \subseteq \sigma_{R_2}(a) \subseteq \sigma_{R_1}(a)$ holds for every $a \in \mathcal{A}$. Meanwhile, we say the pair (R_1, R_2) possesses SBD property.

Remark 1. If the pair of semiregularities (R_1, R_2) possesses SBD property, then $\sigma_{R_1}(a)$ is closed for every $a \in \mathcal{A}$. This is because $\partial\sigma_{R_1}(a) \subseteq \sigma_{R_1}(a)$ for every $a \in \mathcal{A}$ when R_2 is R_1–SBD.

The following examples provide some pairs possessing SBD property in Banach algebras.

Example 2. The concepts of the operator mentioned in (1)–(3) can be found in [25].
(1) Now, we consider the collections of classical Fredholm-type operators. Set

$$\tilde{R}_4 = \{T \in B(X) : T \text{ is a Browder operator}\};$$
$$\tilde{R}_5 = \{T \in B(X) : T \text{ is a Weyl operator}\};$$
$$\tilde{R}_6 = \{T \in B(X) : T \text{ is a Fredholm operator}\}.$$

Then, $\tilde{R}_4$ and $\tilde{R}_6$ are regularities in $B(X)$ and $\tilde{R}_5$ is an upper semiregularity in $B(X)$. Referring to inclusions of spectra associated with $\tilde{R}_i (4 \le i \le 9)$ proved by Miličić and Veselić in [7], we know that the pairs $(\tilde{R}_4, \tilde{R}_5)$ and $(\tilde{R}_5, \tilde{R}_6)$ possess SBD property.

(2) The following collections of semi-Fredholm-type operators, as extensions of Fredholm-type operators, form different semiregularities. Set

$$\begin{aligned}
\tilde{R}_7 &= \{T \in B(X) : T \text{ is an upper semi-Browder operator}\};\\
\tilde{R}_8 &= \{T \in B(X) : T \text{ is a lower semi-Browder operator}\};\\
\tilde{R}_9 &= \{T \in B(X) : T \text{ is an upper semi-Weyl operator}\};\\
\tilde{R}_{10} &= \{T \in B(X) : T \text{ is a lower semi-Weyl operator}\};\\
\tilde{R}_{11} &= \{T \in B(X) : T \text{ is an upper semi-Fredholm operator}\};\\
\tilde{R}_{12} &= \{T \in B(X) : T \text{ is a lower semi-Fredholm operator}\};\\
\tilde{R}_{13} &= \{T \in B(X) : T \text{ is a semi-Fredholm operator}\}.
\end{aligned}$$

Then, $\tilde{R}_i$ $(i = 7, 8, 11, 12)$ are regularities, $\tilde{R}_{13}$ is a lower semiregularity, while $\tilde{R}_9$ and $\tilde{R}_{10}$ are upper semiregularities in $B(X)$. Referring to relationships of spectra associated with $\tilde{R}_i$ $(4 \le i \le 13)$ proved by Rakočević in [24] and by Miličić and Veselić in [7], the following pairs possess SBD property:

$$(\tilde{R}_4, \tilde{R}_7), (\tilde{R}_7, \tilde{R}_9), (\tilde{R}_9, \tilde{R}_{11}), (\tilde{R}_{11}, \tilde{R}_{13}), (\tilde{R}_4, \tilde{R}_8), (\tilde{R}_8, \tilde{R}_{10}), (\tilde{R}_{10}, \tilde{R}_{12}),$$
$$(\tilde{R}_{12}, \tilde{R}_{13}), (\tilde{R}_5, \tilde{R}_9), (\tilde{R}_5, \tilde{R}_{10}), (\tilde{R}_6, \tilde{R}_{11}), (\tilde{R}_6, \tilde{R}_{12}), (\tilde{R}_6, \tilde{R}_{13}).$$

The aforementioned conclusions can be shown in the Figure 1, where the notation $\boxed{\tilde{R}_m} \to \boxed{\tilde{R}_n}$ means that the pair $(\tilde{R}_m, \tilde{R}_n)$ possesses SBD property.

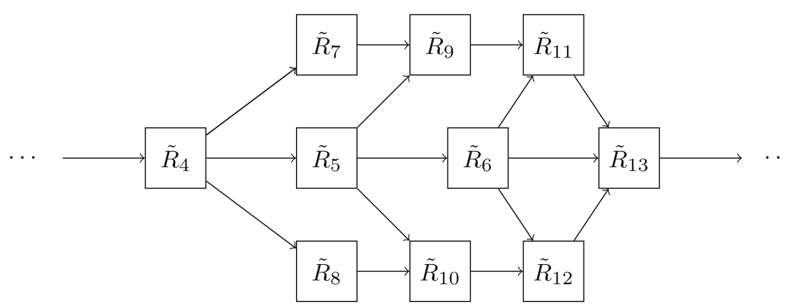

Figure 1: SBD property between $\tilde{R}_m$ and $\tilde{R}_n$ $(4 \le m < n \le 13)$.

Note that in Figure 1, there is no arrow from $\tilde{R}_n$ to $\tilde{R}_m$ even if there exists an arrow from $\tilde{R}_m$ to $\tilde{R}_n$. If not, then $\sigma_{\tilde{R}_m}(T)$ and $\sigma_{\tilde{R}_n}(T)$ are coincident for all $T \in B(X)$, which is impossible. Besides, we also cannot conclude that the pair $(\tilde{R}_m, \tilde{R}_n)$ or $(\tilde{R}_n, \tilde{R}_m)$ possesses SBD property if there is no arrow between them.

For instance, let $A \in B(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, \ldots). \tag{6}$$

Then, $\sigma_{\tilde{R}_5}(A) = \sigma_{\tilde{R}_6}(A) = \mathbb{D}$, and $\sigma_{\tilde{R}_7}(A) = \sigma_{\tilde{R}_9}(A) = \partial\mathbb{D}$, which implies that neither $(\tilde{R}_7, \tilde{R}_5)$ nor $(\tilde{R}_9, \tilde{R}_6)$ possesses SBD property.

In addition, for operators S and T given in equations (4) and (5), we know $\sigma_{\tilde{R}_6}(S) = \partial\mathbb{D}$, $\sigma_{\tilde{R}_9}(S) = \mathbb{D}$, $\sigma_{\tilde{R}_5}(T) = \partial\mathbb{D}$ and $\sigma_{\tilde{R}_7}(T) = \mathbb{D}$. It induces that neither $(\tilde{R}_5, \tilde{R}_7)$ nor $(\tilde{R}_6, \tilde{R}_9)$ possesses SBD property.

On the other hand, from the adjoints of A, S, and T given in equations (4)–(6), respectively, one can see that there is no arrow between $\tilde{R}_5$ and $\tilde{R}_8$, nor between $\tilde{R}_6$ and $\tilde{R}_{10}$.

(3) As a variant of Fredholm-type operators, and semi-Fredholm-type operators, **BR** operators are described as follows. We call $T \in B(X)$ a **BR** operator if there exists some positive integer n such that the restriction of T on the range of T^n is a **R** operator, where **R** denotes any of the following classes: upper (lower) semi-Fredholm (Weyl/Browder) operators, Fredholm (Weyl/Browder) operators, and semi-Fredholm operators.

Let

$$\tilde{R}_{14} = \{T \in B(X) : T \text{ is a B-Browder operator}\};$$
$$\tilde{R}_{15} = \{T \in B(X) : T \text{ is a B-Weyl operator}\};$$
$$\tilde{R}_{16} = \{T \in B(X) : T \text{ is a B-Fredholm operator}\};$$
$$\tilde{R}_{17} = \{T \in B(X) : T \text{ is an upper semi-B-Browder operator}\};$$
$$\tilde{R}_{18} = \{T \in B(X) : T \text{ is a lower semi-B-Browder operator}\};$$
$$\tilde{R}_{19} = \{T \in B(X) : T \text{ is an upper semi-B-Weyl operator}\};$$
$$\tilde{R}_{20} = \{T \in B(X) : T \text{ is a lower semi-B-Weyl operator}\};$$
$$\tilde{R}_{21} = \{T \in B(X) : T \text{ is an upper semi-B-Fredholm operator}\};$$
$$\tilde{R}_{22} = \{T \in B(X) : T \text{ is a lower semi-B-Fredholm operator}\};$$
$$\tilde{R}_{23} = \{T \in B(X) : T \text{ is a semi-B-Fredholm operator}\}.$$

Then, $\tilde{R}_i$ $(i = 14, 16, 17, 18, 21, 22)$ are regularities in $B(X)$, $\tilde{R}_j$ $(j = 15, 19, 20)$ are upper semiregularities in $B(X)$, while $\tilde{R}_{23}$ is a lower semiregularity in $B(X)$. Referring to relationships of spectra associated with $\tilde{R}_i$ $(14 \le i \le 23)$, which can be seen in [25], we obtain the following pairs possessing SBD property:

$$(\tilde{R}_{14}, \tilde{R}_{17}), (\tilde{R}_{17}, \tilde{R}_{19}), (\tilde{R}_{19}, \tilde{R}_{21}), (\tilde{R}_{21}, \tilde{R}_{23}), (\tilde{R}_{14}, \tilde{R}_{18}),$$
$$(\tilde{R}_{18}, \tilde{R}_{20}), (\tilde{R}_{20}, \tilde{R}_{22}), (\tilde{R}_{22}, \tilde{R}_{23}), (\tilde{R}_{15}, \tilde{R}_{19}), (\tilde{R}_{15}, \tilde{R}_{20}),$$
$$(\tilde{R}_{16}, \tilde{R}_{21}), (\tilde{R}_{16}, \tilde{R}_{22}), (\tilde{R}_{14}, \tilde{R}_{15}), (\tilde{R}_{15}, \tilde{R}_{16}), (\tilde{R}_{16}, \tilde{R}_{23}).$$

Continue to use the legend in (2) of Example 2, we can obtain Figure 2.

In Figure 2, arrows in the opposite direction do not exist. Meanwhile, using the operators S, T, and A again, one can see that $\tilde{R}_{15}$ and $\tilde{R}_{17}$(or $\tilde{R}_{18}$), $\tilde{R}_{16}$ and $\tilde{R}_{19}$(or $\tilde{R}_{20}$) cannot be connected by arrows.

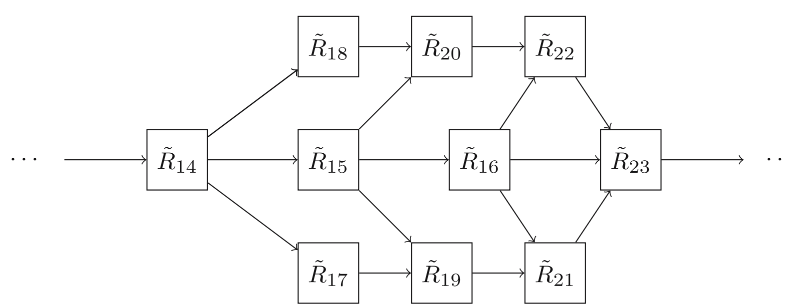

Figure 2: SBD property between $\tilde{R}_m$ and $\tilde{R}_n$ $(14 \le m < n \le 23)$.

(4) The following collections of left(right) Fredholm-type operators, as other extensions of Fredholm-type operators, form semiregularities, which are different from semi-Fredholm-type operators. The concepts of these operators can be found in [30,31]. Set

$$\tilde{R}_{24} = \{T \in B(X) : T \text{ is a left Browder operator}\};$$
$$\tilde{R}_{25} = \{T \in B(X) : T \text{ is a right Browder operator}\};$$
$$\tilde{R}_{26} = \{T \in B(X) : T \text{ is a left Weyl operator}\};$$
$$\tilde{R}_{27} = \{T \in B(X) : T \text{ is a right Weyl operator}\};$$
$$\tilde{R}_{28} = \{T \in B(X) : T \text{ is a left Fredholm operator}\};$$
$$\tilde{R}_{29} = \{T \in B(X) : T \text{ is a right Fredholm operator}\};$$
$$\tilde{R}_{30} = \{T \in B(X) : T \text{ is left Fredholm or right Fredholm}\};$$
$$\tilde{R}_{31} = \{T \in B(X) : T \text{ is an essentially Saphar operator}\}.$$

Then, $\tilde{R}_i$ $(i = 24, 25, 28, 29, 31)$ is a regularity, $\tilde{R}_{26}$ and $\tilde{R}_{27}$ are upper semiregularities, while $\tilde{R}_{30}$ is a lower semiregularity. According to relationships of spectra associated with $\tilde{R}_i$ $(i \in \{4, 5, 6, 24, 25, 26, 27, 28, 29, 30, 31\})$

given by [30, Theorem 10] and [31, Theorem 4.3], the following pairs possess SBD property:

$$\begin{aligned}
&(\tilde{R}_4, \tilde{R}_i), \quad 24 \le i \le 31,\\
&(\tilde{R}_{24}, \tilde{R}_j), \quad j = 26, 28, 30, 31,\\
&(\tilde{R}_{25}, \tilde{R}_k), \quad k = 27, 29, 30, 31,\\
&(\tilde{R}_{26}, \tilde{R}_p), \quad p = 28, 30, 31,\\
&(\tilde{R}_{27}, \tilde{R}_q), \quad q = 29, 30, 31,\\
&(\tilde{R}_{28}, \tilde{R}_{30}), (\tilde{R}_{29}, \tilde{R}_{30}), (\tilde{R}_{28}, \tilde{R}_{31}), (\tilde{R}_{29}, \tilde{R}_{31}), (\tilde{R}_{30}, \tilde{R}_{31}).
\end{aligned}$$

(5) Let $T : \mathcal{A} \to \mathcal{B}$ be a Banach algebra homomorphism with the strong Riesz property defined by [1]. An element $a \in \mathcal{A}$ is T-Fredholm if $T(a)$ is invertible in $\mathcal{B}$, while the element a is T-Weyl if $a = b + c$, where $b \in \mathcal{A}^{-1}$ and $T(c) = 0_{\mathcal{B}}$. In addition, an element $a \in \mathcal{A}$ is T-Browder if $a = b + c$ with $b \in \mathcal{A}^{-1}$, $T(c) = 0_{\mathcal{B}}$ and $bc = cb$ [1]. Set

$$\begin{aligned}
\mathbb{R}_1 &= \{a \in \mathcal{A} : a \text{ is a } T\text{-Browder element}\};\\
\mathbb{R}_2 &= \{a \in \mathcal{A} : a \text{ is a } T\text{-Weyl element}\};\\
\mathbb{R}_3 &= \{a \in \mathcal{A} : a \text{ is a } T\text{-Fredholm element}\}.
\end{aligned}$$

From [29, Theorem 2.3, Corollary 2.1], one can see that the pairs of semiregularities $(\mathbb{R}_1, \mathbb{R}_2)$, $(\mathbb{R}_1, \mathbb{R}_3)$, and $(\mathbb{R}_2, \mathbb{R}_3)$ possess SBD property.

Proposition 2. Figures 1 *and* 2 *cannot be connected by an arrow. Namely, there is no arrow from* $\tilde{R}_j$ $(4 \le j \le 13)$ *to* $\tilde{R}_k$ $(14 \le k \le 23)$ *and no arrow from* $\tilde{R}_k$ *to* $\tilde{R}_j$.

Proof. It is clear that the spectra associated with semiregularities $\tilde{R}_i$ $(4 \le i \le 23)$ are bounded. By virtue of Theorem 1.2 in [32], one has that in Figures 1 and 2, $\eta\sigma_{\tilde{R}_m}(T) = \eta\sigma_{\tilde{R}_n}(T)$ is valid for $T \in B(X)$, provided that the pair $(\tilde{R}_m, \tilde{R}_n)$ possesses SBD property, or equivalently, there is an arrow from $\tilde{R}_m$ to $\tilde{R}_n$. Furthermore, if $\tilde{R}_m$ and $\tilde{R}_n$ are both in Figure 1 or in Figure 2, through the transmission of equality of connected hulls, $\eta\sigma_{\tilde{R}_m}(T) = \eta\sigma_{\tilde{R}_n}(T)$ for $T \in B(X)$.

Now, we prove this proposition by contradiction. Assume that Figures 1 and 2 can be connected by an arrow, which is from some $\tilde{R}_j$ in Figure 1 to some $\tilde{R}_k$ in Figure 2, or from $\tilde{R}_k$ to $\tilde{R}_j$. Then, $\eta\sigma_{\tilde{R}_m}(T) = \eta\sigma_{\tilde{R}_n}(T)$ $(4 \le m, n \le 23)$ for $T \in B(X)$. Taking $X = \ell^2 \oplus \ell^2$, we define $N \in B(\ell^2)$ by

$$N(x_1, x_2, x_3, \ldots) = (0, x_1, 0, 0, \ldots).$$

Then, N is nilpotent, but not Fredholm. Furthermore, $T \in B(X)$ is defined by

$$T = \begin{pmatrix} S + 2I & 0 \\ 0 & N \end{pmatrix},$$

where S is the operator given in equation (4) and I is the identity of $B(X)$. Then,

$$T^2 = \begin{pmatrix} (S + 2I)^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $T^2 |_{R(T^2)} = (S + 2I)^2$ is Fredholm, it follows that $T |_{R(T^2)}$ is Fredholm, which implies that T is B-Fredholm. After a tedious computation, we obtain $\sigma_{\tilde{R}_6}(T) = \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda - 2| = 1\}$ and $\sigma_{\tilde{R}_{16}}(T) = \{\lambda \in \mathbb{C} : |\lambda - 2| = 1\}$. Furthermore, $\eta\sigma_{\tilde{R}_6}(T) = \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda - 2| \le 1\}$, while $\eta\sigma_{\tilde{R}_{16}}(T) = \{\lambda \in \mathbb{C} : |\lambda - 2| \le 1\}$. It contradicts with the equality $\eta\sigma_{\tilde{R}_m}(T) = \eta\sigma_{\tilde{R}_n}(T)$ $(4 \le m, n \le 23)$.

Hence, Figures 1 and 2 cannot be connected by an arrow. □

Proposition 3. *Let* (R_1, R_2) *be a pair of semiregularities in* $\mathcal{A}$ *such that* $\sigma_{R_1}(a)$ *is closed for every* $a \in \mathcal{A}$. *If* R_2 *is open and* R_1 *is the union of some connected components of* R_2, *then the pair* (R_1, R_2) *possesses SBD property.*

Proof. For $a \in \mathcal{A}$, one has that $\sigma_{R_2}(a) \subseteq \sigma_{R_1}(a)$ since $R_1 \subseteq R_2$. Without loss of generality, we assume that $\sigma_{R_2}(a)$ is non-empty. If $a - \mu 1_{\mathcal{A}}$ is in R_2, then either $a - \mu 1_{\mathcal{A}} \in R_2 - R_1$ or $a - \mu 1_{\mathcal{A}} \in R_1$. In the first case, since R_1 is the union of some connected components of R_2, we have μ is an interior point of $\sigma_{R_1}(a)$. The second case induces that μ is not in $\sigma_{R_1}(a)$. Furthermore, $\mu \notin \partial\sigma_{R_1}(a)$ since $\sigma_{R_1}(a)$ is closed. So far, we have proved that the pair (R_1, R_2) possesses SBD property. □

Example 3. Suppose that $\mathcal{A}$ is a Banach algebra.

(1) Let $\tilde{R}_{32} = \mathcal{A}^{-1}$ and $\tilde{R}_{33} = \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$, where $\mathcal{A}_l^{-1}$ $(\mathcal{A}_r^{-1})$ means the set of left (right) invertible elements in $\mathcal{A}$. Then, both $\tilde{R}_{32}$ and $\tilde{R}_{33}$ are open regularities in $\mathcal{A}$ such that $\tilde{R}_{32} \subseteq \tilde{R}_{33}$ [26]. As is known, $\tilde{R}_{33} \backslash \tilde{R}_{32}$ is also open such that $(\tilde{R}_{33} \backslash \tilde{R}_{32}) \cap \tilde{R}_{32} = \emptyset$, which implies that $\tilde{R}_{32}$ is the union of some connected components of $\tilde{R}_{33}$. It follows from [26] that the pair $(\tilde{R}_{32}, \tilde{R}_{33})$ possesses SBD property.

(2) Let $\tilde{R}_{34} = \mathrm{Exp}(\mathcal{A})$. It is well known that $\tilde{R}_{34}$ is the principle component of $\tilde{R}_{32}$. Moreover, the pair $(\tilde{R}_{34}, \tilde{R}_{32})$ possesses SBD property from [3].

(3) Let $\mathcal{A}$ be $B(X)$. Then, $\tilde{R}_5$ is the principle component of $\tilde{R}_6$, while $\tilde{R}_6$ is the union of some connected components of $\tilde{R}_{11}$, $\tilde{R}_{12}$, and $\tilde{R}_{13}$, respectively [33]. Moreover, $\tilde{R}_9$ is the union of some connected components of $\tilde{R}_{11}$, and $\tilde{R}_{10}$ is the union of some connected components of $\tilde{R}_{11}$. SBD property among these semiregularities mentioned above is shown in Example 2.

Set $\mathfrak{R} = \{R : R \subseteq \mathcal{A}$ a semiregularity, $\sigma_R(a)$ is closed for every $a \in \mathcal{A}\}$, and write $R_2 \leq R_1$ if $R_1, R_2 \in \mathfrak{R}$ such that the pair (R_1, R_2) possesses SBD property.

Proposition 4. $(\mathfrak{R}, \leq)$ *is a partial order relation.*

Proof. Reflexivity. Suppose $R \in \mathfrak{R}$ and $a \in \mathcal{A}$, $\partial\sigma_R(a) \subseteq \sigma_R(a) \subseteq \sigma_R(a)$ is valid as $\sigma_R(a)$ is closed.

Antisymmetry. Suppose that $R_2 \leq R_1$ and $R_1 \leq R_2$, which are equivalent to $\partial\sigma_{R_1}(a) \subseteq \sigma_{R_2}(a) \subseteq \sigma_{R_1}(a)$ and $\partial\sigma_{R_2}(a) \subseteq \sigma_{R_1}(a) \subseteq \sigma_{R_2}(a)$ for every $a \in \mathcal{A}$. It follows that $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ for every $a \in \mathcal{A}$.

If $R_1 \neq R_2$, then at least one of the sets $R_1 \backslash R_2$ and $R_2 \backslash R_1$ is non-empty. Without loss of generality, we suppose that $R_2 \backslash R_1$ is non-empty. Then, there is $x \in R_2 \backslash R_1$, which implies $0 \in \sigma_{R_1}(x)$ and $0 \notin \sigma_{R_2}(x)$. It is a contradiction with $\sigma_{R_1}(x) = \sigma_{R_2}(x)$. Therefore, $R_1 = R_2$.

Transitivity. Suppose that $R_2 \leq R_1$ and $R_3 \leq R_2$. It is easy to see that $\mathrm{int}\sigma_{R_2}(a) \subseteq \mathrm{int}\sigma_{R_1}(a)$ for every $a \in \mathcal{A}$. Combined with $\partial\sigma_{R_1}(a) \subseteq \sigma_{R_2}(a)$, one has

$$\partial\sigma_{R_1}(a) \subseteq \partial\sigma_{R_2}(a) \subseteq \sigma_{R_2}(a) \subseteq \sigma_{R_1}(a).$$

Due to $R_3 \leq R_2$,

$$\partial\sigma_{R_1}(a) \subseteq \partial\sigma_{R_2}(a) \subseteq \sigma_{R_3}(a) \subseteq \sigma_{R_2}(a) \subseteq \sigma_{R_1}(a).$$

Hence, $R_3 \leq R_1$. In all, $(\mathfrak{R}, \leq)$ is a partial order relation. □

Figures 1 and 2 partly show the partial order relation for semiregularities in $B(X)$, where the symbol $\tilde{R}_n \leq \tilde{R}_m$ indicates that there exists an arrow from $\tilde{R}_m$ to $\tilde{R}_n$, which is shown in Example 2.

Let R be a semiregularity in $\mathcal{A}$. We write $r_R(a) = \sup_{\lambda \in \sigma_R(a)} |\lambda|$ for $a \in \mathcal{A}$, provided that $\sigma_R(a)$ is non-empty, called spectral radius of a associated with R. The following proposition induces that the equation $r_{R_1}(a) = r_{R_2}(a)$ is valid as long as the pair (R_1, R_2) or (R_2, R_1) possesses SBD property.

Proposition 5. *Let* (R_1, R_2) *be a pair of semiregularities in* $\mathcal{A}$ *possessing SBD property. Then,* $r_{R_1}(a) = r_{R_2}(a)$ *and* $\inf_{\lambda \in \sigma_{R_2}(a)} |\lambda| \geq \inf_{\mu \in \sigma_{R_1}(a)} |\mu|$ *for* $a \in \mathcal{A}$, *provided that* $\sigma_{R_2}(a)$ *is non-empty.*

Proof. Note that $\sup_{\lambda \in \sigma_{R_2}(a)} |\lambda| = \max_{\alpha \in \eta\sigma_{R_2}(a)} |\alpha|$ and $\max_{\lambda \in \sigma_{R_1}(a)} |\lambda| = \max_{\alpha \in \eta\sigma_{R_1}(a)} |\alpha|$ as both $\eta\sigma_{R_2}(a)$ and $\eta\sigma_{R_1}(a)$ are compact and non-empty. It follows from Theorem 1.2 in [32] that $\eta\sigma_{R_2}(a) = \eta\sigma_{R_1}(a)$ since the pair (R_1, R_2) possesses SBD property. Then, we obtain the equality

$$\sup_{\lambda \in \sigma_{R_2}(a)} |\lambda| = \max_{\mu \in \sigma_{R_1}(a)} |\mu|.$$

Moreover, with the help of $\sigma_{R_2}(a) \subseteq \sigma_{R_1}(a)$, we obtain the inequality

$$\inf_{\lambda \in \sigma_{R_2}(a)} |\lambda| \geq \min_{\mu \in \sigma_{R_1}(a)} |\mu|.$$

□

3.2 Equality of spectra related to SBD property

As is well known, for a closed subalgebra C of a Banach algebra $\mathcal{A}$ with common identity $1_{\mathcal{A}}$, $\partial\sigma_{C^{-1}}(a) \subseteq \sigma_{\mathcal{A}^{-1}}(a) \subseteq \sigma_{C^{-1}}(a)$ holds for every $a \in C$ and that $\sigma_{\mathcal{A}^{-1}}(a) = \sigma_{C^{-1}}(a)$ does not hold in general. However, the aforementioned equation is always valid when $\mathcal{A}$ is a C*-algebra and C is C*-subalgebra of $\mathcal{A}$. It is natural to consider when $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ is valid for a in $\mathcal{A}$, where R_1 and R_2 are semiregularities in $\mathcal{A}$.

In view of the axiomatic theory of spectrum, we will give criteria such that $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ for some a in $\mathcal{A}$ with the premise that the pair (R_1, R_2) possesses SBD property.

Theorem 6. *Let (R_1, R_2) be a pair of semiregularities in $\mathcal{A}$ possessing SBD property and $a \in \mathcal{A}$. If the complement of $\sigma_{R_2}(a)$ is open and connected, then $\sigma_{R_1}(a) = \sigma_{R_2}(a)$.*

Proof. As the complement of $\sigma_{R_2}(a)$ is connected, $\mathbb{C}\backslash\sigma_{R_2}(a)$ has only one connected component, i.e., the unbounded connected component. It induces that $\sigma_{R_2}(a) = \overline{\sigma_{R_2}(a)} = \eta\sigma_{R_2}(a)$. It follows from Theorem 7.10.3 in [34] and Theorems 1.2 and 1.3 in [32] that $\sigma_{R_1}(a) \subseteq \eta\sigma_{R_2}(a)$ since (R_1, R_2) possesses SBD property. Hence, $\sigma_{R_1}(a) = \sigma_{R_2}(a)$. □

Example 4. (1) Recall that a Banach algebra $\mathcal{A}$ is semiprime if it has no non-zero nilpotent ideal, and that the socle of a semiprime Banach algebra $\mathcal{A}$ is the algebraic sum of all the minimal left ideals of $\mathcal{A}$ (which coincides with the algebraic sum of all the minimal right ideals of $\mathcal{A}$) [35, Definition 8.2.7]. Assume that $\mathcal{A}$ is a modular annihilator Banach algebra, namely, a semiprime algebra satisfying that the hull of its socle is empty [35, Definition 8.4.6]. By [35, Theorem 8.6.4] and [35, Theorem 4.3.6], $\sigma_{\mathcal{A}^{-1}}(a)$ is at most countable for $a \in \mathcal{A}$, which implies $\partial\sigma_{\mathcal{A}^{-1}}(a) = \sigma_{\mathcal{A}^{-1}}(a) = \eta\sigma_{\mathcal{A}^{-1}}(a)$. For any semiregularity R in $\mathfrak{R}$, the complement of $\sigma_R(a)$ is open and connected. It follows from Theorem 6 that $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ when R_1 and R_2 are semiregularities in $\mathcal{A}$ with $R_2 \leq R_1$.

(2) Recall that an element a in a Banach algebra $\mathcal{A}$ is Riesz if the accumulation of $\sigma_{\mathcal{A}^{-1}}(a)$ is contained in $\{0\}$, which implies that $\sigma_{\mathcal{A}^{-1}}(a)$ is at most countable. Analogous to (1), $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ is valid for semiregularities R_1 and R_2 in $\mathcal{A}$ with $R_2 \leq R_1$.

Remark 2. The condition in Theorem 6 that the complement of $\sigma_{R_2}(a)$ is connected is indispensable. Considering the semiregularities $\tilde{R}_5$ and $\tilde{R}_9$ in $B(X)$ mentioned in Example 2, we have known that $(\tilde{R}_5, \tilde{R}_9)$ possesses SBD property. Taking $X = \ell^2$, we define $A \in B(\ell^2)$ by

$$B(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots). \tag{7}$$

Then, $\sigma_{\tilde{R}_9}(B) = \partial\mathbb{D}$, which implies that the complement of $\sigma_{\tilde{R}_9}(B)$ is not connected. However, $\sigma_{\tilde{R}_5}(B) = \mathbb{D}$ is not coincident with $\sigma_{\tilde{R}_9}(B)$. It shows that if the complement of $\sigma_{R_2}(a)$ is not connected, then $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ may fail.

In the following, we continue to probe the condition such that $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ for some $a \in \mathcal{A}$, where R_1 and R_2 are semiregularities in $\mathcal{A}$. Different from Theorem 6, the conditions such that $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ in the following no longer only depend on the topological properties of spectra. In more detail, we discuss the condition in a hermitian Banach*-algebra $\mathcal{A}$ with a pair (R_1, R_2) possessing SBD property. Let us recall the definition of hermitian Banach *-algebra.

Definition 5. [36] Let $\mathcal{A}$ be a Banach algebra. An involution on $\mathcal{A}$ is a map $* : \mathcal{A} \to \mathcal{A}$ satisfying the following conditions:

(1) $(\lambda a + b)^* = \lambda^* a^* + b^*, \forall \lambda \in \mathbb{C}, a, b \in \mathcal{A}$;

(2) $(ab)^* = b^* a^*, \forall a, b \in \mathcal{A}$;

(3) $(a^*)^* = a, \forall a \in \mathcal{A}$.

An element $h \in \mathcal{A}$ is hermitian if $h = h^*$. Furthermore, $\mathcal{A}$ is called a hermitian Banach*-algebra if each hermitian element h in $\mathcal{A}$ satisfies $\sigma_{\mathcal{A}^{-1}}(h) \subseteq \mathbb{R}$.

For a semiregularity R in Banach *-algebra $\mathcal{A}$, $R^* = R$ is not automatic. For instance, suppose that X is a infinity dimensional Hilbert space. Then, $\tilde{R}_{11}^* = \tilde{R}_{12}$ is not coincident with $\tilde{R}_{11}$.

Definition 6. Let $\mathcal{A}$ be a hermitian Banach *- algebra and $R \subseteq \mathcal{A}$ be a semiregularity. R is called hermitian if $R^* = R$.

Theorem 7. *Let $\mathcal{A}$ be a hermitian Banach *-algebra and $a \in \mathcal{A}$ be a normal element. If R_1 and R_2 are nonempty subsets of $\mathcal{A}$ satisfying the following conditions*:
(1) *both R_1 and R_2 satisfy property* $(P1)$;
(2) *the pair* (R_1, R_2) *possesses SBD property*;
(3) *R_2 is hermitian,*
then $\sigma_{R_1}(a) = \sigma_{R_2}(a)$.

Proof. Due to $\partial\sigma_{R_1}(a) \subseteq \sigma_{R_2}(a) \subseteq \sigma_{R_1}(a)$, $\sigma_{R_1}(a)$ is an empty set when $\sigma_{R_2}(a)$ is empty. Without loss of generality, we suppose that $\sigma_{R_2}(a)$ is non-empty.

First, we assume that $a \in \mathcal{A}$ is a hermitian element. It follows from Proposition 1 that R_1 and R_2 are regularities in $\mathcal{A}$. Then,

$$\sigma_{R_2}(a) \subseteq \sigma_{R_1}(a) \subseteq \sigma_{\mathcal{A}^{-1}}(a) \subseteq \mathbb{R},$$

as $\mathcal{A}$ is a hermitian Banach*-algebra and (R_1, R_2) possesses SBD property. It induces that

$$\sigma_{R_1}(a) = \partial\sigma_{R_1}(a) \subseteq \sigma_{R_2}(a),$$

which implies $\sigma_{R_1}(a) = \sigma_{R_2}(a)$.

Now, let a be a normal element of $\mathcal{A}$. For $\lambda \notin \sigma_{R_2}(a)$, both $a - \lambda 1_{\mathcal{A}}$ and $(a - \lambda 1_{\mathcal{A}})^*$ are in R_2 since R_2 is hermitian. Meanwhile, we have

$$(a - \lambda 1_{\mathcal{A}})^*(a - \lambda 1_{\mathcal{A}}) = (a - \lambda 1_{\mathcal{A}})(a - \lambda 1_{\mathcal{A}})^* \in R_2,$$

since R_2 satisfies (P1). Furthermore, the hermitian element $(a - \lambda 1_{\mathcal{A}})^*(a - \lambda 1_{\mathcal{A}}) \in R_1$ since $0 \notin \sigma_{R_2}((a - \lambda 1_{\mathcal{A}})^*(a - \lambda 1_{\mathcal{A}})) = \sigma_{R_1}((a - \lambda 1_{\mathcal{A}})^*(a - \lambda 1_{\mathcal{A}}))$. Then, $a - \lambda 1_{\mathcal{A}}$ is in R_1 because of property (P1). Therefore, $\sigma_{R_1}(a) \subseteq \sigma_{R_2}(a)$.

The converse inclusion is natural. □

In fact, the condition in Theorem 7 that a is a normal element is indispensable. For instance, let $\mathcal{A} = B(\ell^2)$ and the operator S be given in equation (4). Then, $\mathcal{A}$ is a hermitian Banach*-algebra and S is a non-normal operator. Choose $R_1 = \tilde{R}_4$ and $R_2 = \tilde{R}_6$. It has been shown in Example 2(2) that the pair (R_1, R_2) possesses SBD property. Also, one can see from [26] that R_2 is hermitian, and both R_1 and R_2 satisfy property (P1). However, $\sigma_{R_1}(S) = \mathbb{D}$ is not equal to $\sigma_{R_2}(S) = \partial\mathbb{D}$. Therefore, $\sigma_{R_1}(a) = \sigma_{R_2}(a)$ may fail when a is not a normal element.

Example 5. (1) Take $\mathcal{A} = B(L^2(\mathbb{D}))$, the algebra of bounded linear operators on Hilbert space $L^2(\mathbb{D})$, where $\mathbb{D}$ denotes the closed unit disk in $\mathbb{C}$. Then, $\mathcal{A}$ is a unital hermitian Banach*-algebra, where for each $T \in \mathcal{A}$, T^* is the adjoint of T.

Considering the regularities $\tilde{R}_{32}$ and $\tilde{R}_{33}$ mentioned in Example 3, we know that both $\tilde{R}_{32}$ and $\tilde{R}_{33}$ satisfy the property (P1), and that the pair $(\tilde{R}_{32}, \tilde{R}_{33})$ possesses SBD property. Moreover, $\tilde{R}_{33}$ is hermitian since $T \in B(L^2(\mathbb{D}))$ is left (right) invertible if and only if T^* is right(left) invertible. Let $N \in B(L^2(\mathbb{D}))$ be defined by $Nf(z) = zf(z)$ for all f in $L^2(\mathbb{D})$. Referring to Proposition 2.5 in [37], N is a normal operator with $\sigma_{\tilde{R}_{32}}(N) = \mathbb{D}$. By virtue of Theorem 7, one has $\sigma_{\tilde{R}_{32}}(N) = \sigma_{\tilde{R}_{33}}(N)$.

(2) Let $\mathcal{A}$ be the algebra of all continuous functions on Ω, where Ω is a non-empty compact subset of $\mathbb{C}$. For f in $\mathcal{A}$, define $f^*(z) = \overline{f(z)}$. Then, $\mathcal{A}$ is a unital commutative C*-algebra with $\|f\| = \sup_{z\in\Omega}|f(z)|$; furthermore,

a hermitian Banach*-algebra. Let us think of regularities $\tilde{R}_{32}$ and $\tilde{R}_{35}$ in $\mathcal{A}$, where $\tilde{R}_{35}$ is the complement of the set of topological zero divisors in $\mathcal{A}$. From Example 5 and Definition 6 in [26, Chapter I 6], both $\tilde{R}_{32}$ and $\tilde{R}_{35}$ are nonempty subsets of $\mathcal{A}$ satisfying conditions (1), (2) and (3) mentioned in Theorem 7. Then, $\sigma_{\tilde{R}_{35}}(f) = \sigma_{\tilde{R}_{32}}(f) = f(\Omega)$.

4 SBD property relative to a Banach algebra homomorphism

The purpose of this section is to study the transfer of SBD property in view of the axiomatic theory of spectrum through a Banach algebra homomorphism T. For this aim, we consider the transfer of semiregularity through T and the preservation of the associated spectra in the axiomatic theory of spectrum at first.

4.1 Axiomatic theory of spectrum relative to a Banach algebra homomorphism

Fredholm theory relative to a Banach algebra homomorphism was initiated in [1] and has been enriched by many scholars. Along with the direction, many spectra have been defined and studied. More abstractly, spectra relative to a Banach algebra homomorphism deserve being studied in axiomatic theory of spectrum. Now, we consider the preservation of semiregularities by a homomorphism T.

Proposition 8. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. Let $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism satisfying $T(1_{\mathcal{A}}) = T(1_{\mathcal{B}})$.*
(1) *If R is a regularity in $\mathcal{B}$, then $T^{-1}(R)$ is a regularity in $\mathcal{A}$.*
(2) *If R is a lower semiregularity in $\mathcal{B}$, then $T^{-1}(R)$ is a lower semiregularity in $\mathcal{A}$.*
(3) *If R is an upper semiregularity in $\mathcal{B}$ and T is bounded, then $T^{-1}(R)$ is an upper semiregularity in $\mathcal{A}$.*

Proof. (1) It is easy to see that $T^{-1}(R)$ is non-empty. If $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $T(a) \in R \Leftrightarrow T(a)^n = T(a^n) \in R$ as T is a homomorphism. It follows that

$$a \in T^{-1}(R) \Leftrightarrow a^n \in T^{-1}(R).$$

For elements a, b, c, and d mutually commuting in $\mathcal{A}$, if $ac + bd = 1_{\mathcal{A}}$, we have

$$T(a)T(c) + T(b)T(d) = 1_{\mathcal{B}}.$$

Since R is a regularity in $\mathcal{B}$,

$$T(a)T(b) = T(ab) \in R \Leftrightarrow T(a) \in R, T(b) \in R.$$

It shows that $ab \in T^{-1}(R)$ if and only if $a \in T^{-1}(R), b \in T^{-1}(R)$. Hence, $T^{-1}(R)$ is a regularity in $\mathcal{A}$.

(2) It is analogous to (1).

(3) It suffices to verify $T^{-1}(R)$ contains a neighborhood of $1_{\mathcal{A}}$, provided that T is bounded. Assume R contains an open neighborhood V of $1_{\mathcal{B}}$. Then, the non-empty subset $T^{-1}(V) \subseteq T^{-1}(R)$ is open since T is bounded. Furthermore, $T^{-1}(V)$ is an open neighborhood of $1_{\mathcal{A}}$ as $T(1_{\mathcal{A}}) = T(1_{\mathcal{B}})$. □

Proposition 9. *Let $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism and $R \subseteq \mathcal{B}$ be a non-empty set.*
(1) *If R satisfies $(P1)$, then $T^{-1}(R) \subseteq \mathcal{A}$ either is an empty set or satisfies $(P1)$.*
(2) *If R satisfies $(P1')$, then $T^{-1}(R) \subseteq \mathcal{A}$ either is an empty set or satisfies $(P1')$.*
(3) *If R satisfies $(P1'')$ and T is bounded such that $T(1_{\mathcal{A}}) = T(1_{\mathcal{B}})$, then $T^{-1}(R) \subseteq \mathcal{A}$ satisfies $(P1'')$.*

Proof. (1) Without loss of generality, we assume that $T^{-1}(R)$ is non-empty. For arbitrary commuting elements $a, b \in \mathcal{A}$, we have $T(a)T(b) = T(b)T(a)$. Then,

$$a, b \in T^{-1}(R) \Leftrightarrow T(a), T(b) \in R \Leftrightarrow T(a)T(b) = T(ab) \in R \Leftrightarrow ab \in T^{-1}(R).$$

(2) It is analogous to (1).

(3) Using Proposition 8(3), it suffices to show that $T^{-1}(R)$ is a semigroup. Indeed, for any $a, b \in T^{-1}(R)$, $T(ab) = T(a)T(b) \in R$ since R is a semigroup in $\mathcal{B}$. And it is evident that $T^{-1}(R)$ satisfies associative law. □

Next, we will consider the relation between $\sigma_R(T(a))$ and $\sigma_{T^{-1}(R)}(a)$ associated with a semiregularity R in $\mathcal{B}$ and $T^{-1}(R)$ in $\mathcal{A}$, respectively.

Theorem 10. *Let $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism satisfying $T(1_{\mathcal{A}}) = T(1_{\mathcal{B}})$. If R is a regularity in $\mathcal{B}$, then for every $a \in \mathcal{A}$ and $c \in \ker T$,*

$$\sigma_R(T(a)) = \sigma_{T^{-1}(R)}(a) = \sigma_{T^{-1}(R)}(a + c).$$

Proof. If $\sigma_R(T(a))$ is empty for some $a \in \mathcal{A}$, then $T(a - \lambda 1_{\mathcal{A}}) = T(a) - \lambda 1_{\mathcal{B}} \in R$ for all $\lambda \in \mathbb{C}$. It follows that $\sigma_{T^{-1}(R)}(a)$ is empty.

Assume that $\sigma_R(T(a))$ is non-empty for $a \in \mathcal{A}$. If $a - \lambda 1_{\mathcal{A}} \in T^{-1}(R)$, which is equivalent to $T(a - \lambda 1_{\mathcal{A}}) \in R$, then $\lambda \notin \sigma_R(T(a))$ by means of $T(1_{\mathcal{A}}) = T(1_{\mathcal{B}})$. Conversely, if $T(a) - \mu 1_{\mathcal{B}} \in R$, then $T(a - \mu 1_{\mathcal{A}}) \in R$, which implies $\mu \notin \sigma_{T^{-1}(R)}(a)$. Hence, $\sigma_R(T(a)) = \sigma_{T^{-1}(R)}(a)$.

We obtain $\sigma_{T^{-1}(R)}(a + c) = \sigma_R(T(a + c)) = \sigma_R(T(a))$ immediately since $T(a) = T(a + c)$ for every $a \in \mathcal{A}$, $c \in \ker T$. □

Moreover, one has the following results for semiregularities.

Remark 3. Let $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism satisfying $T(1_{\mathcal{A}}) = T(1_{\mathcal{B}})$.

(1) If R is a lower semiregularity in $\mathcal{B}$, then for every $a \in \mathcal{A}$ and $c \in \ker T$,

$$\sigma_R(T(a)) = \sigma_{T^{-1}(R)}(a) = \sigma_{T^{-1}(R)}(a + c).$$

(2) If R is an upper semiregularity in $\mathcal{B}$ and T is bounded, then for every $a \in \mathcal{A}$ and $c \in \ker T$,

$$\sigma_R(T(a)) = \sigma_{T^{-1}(R)}(a) = \sigma_{T^{-1}(R)}(a + c).$$

Note that the equation $\sigma_{T^{-1}(R)}(a) = \sigma_{T^{-1}(R)}(a + c)$ in Theorem 10 may fail for $c \in \overline{\ker T}$ as T is not bounded. However, this equation is possible when $T^{-1}(R)$ has some continuity property. In [13], Müller considered several continuity properties of regularity, and we list them as follows. Let $R \subseteq \mathcal{A}$ be a regularity and σ_R be the associated spectrum.

(P2) If $a_n, a \in \mathcal{A}$, $a_n \to a$, $\lambda_n \in \sigma_R(a_n)$, $\lambda_n \to \lambda$, then $\lambda \in \sigma_R(a)$.

(P3) If $a_n, a \in \mathcal{A}$, $a_n \to a$, $a_n a = a a_n$ for every n, $\lambda_n \in \sigma_R(a_n)$, $\lambda_n \to \lambda$, then $\lambda \in \sigma_R(a)$.

(P4) If $a_n, a \in \mathcal{A}$, $a_n \to a$, $a_n a = a a_n$ for every n, then $\lambda \in \sigma_R(a)$ if and only if there exists a sequence $\lambda_n \in \sigma_R(a_n)$ such that $\lambda_n \to \lambda$.

Naturally, properties (P2)–(P4) can be applied to semiregularity. Then, $\sigma_{T^{-1}(R)}(a) = \sigma_{T^{-1}(R)}(a + c)$ for $c \in \overline{\ker T}$ when $T^{-1}(R)$ is a semiregularity satisfying (P2). Next, we will consider the preservation of properties (P2)–(P4) of the spectrum associated with a semiregularity $R \subseteq \mathcal{B}$ by homomorphism T.

Corollary 11. *Let $T : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism satisfying $T(1_{\mathcal{A}}) = T(1_{\mathcal{B}})$. If R is a semiregularity in $\mathcal{B}$ possessing property* (P2) ((P3), (P4) *resp.*), *then $T^{-1}(R) \subseteq \mathcal{A}$ is a semiregularity possessing property* (P2) ((P3), (P4) *resp.*) *as well.*

Proof. If $a_n, a \in \mathcal{A}$, $a_n \to a$, then $T(a_n), T(a) \in \mathcal{A}$, and $T(a_n) \to T(a)$ by the continuity of T. And $T(a_n)T(a) = T(a)T(a_n)$ if $a_n a = a a_n$.

Assume that a semiregularity $R \subseteq \mathcal{B}$ satisfies property (P2) (or (P3)). If $\lambda_n \in \sigma_{T^{-1}(R)}(a_n)$, $\lambda_n \to \lambda$, then $\lambda_n \in \sigma_R(T(a_n))$ by Theorem 10 and Corollary 3; furthermore, $\lambda \in \sigma_R(T(a)) = \sigma_{T^{-1}(R)}(a)$. It induces that the semiregularity $T^{-1}(R) \subseteq \mathcal{A}$ satisfies property (P2) (or (P3)).

Assume that a semiregularity $R \subseteq \mathcal{B}$ satisfies property (P4). It remains to prove that if $\lambda \in \sigma_{T^{-1}(R)}(a)$, then there exists a sequence $\lambda_n \in \sigma_{T^{-1}(R)}(a_n)$ such that $\lambda_n \to \lambda$. According to Theorem 10, we have $\lambda \in \sigma_R(T(a))$. Furthermore, there exists a sequence $\lambda_n \in \sigma_R(T(a_n)) = \sigma_{T^{-1}(R)}(a_n)$ such that $\lambda_n \to \lambda$ since R satisfies (P4). The proof is complete. □

Specially, let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}$ with the same identity and $i : \mathcal{A} \to \mathcal{B}$ be the inclusion mapping. Referring to Theorem 10, we have the following result.

Corollary 12. *If R is a semiregularity in $\mathcal{B}$, then $\sigma_R(a) = \sigma_{R\cap\mathcal{A}}(a)$ for every $a \in \mathcal{A}$.*

It should be noted that $R \cap \mathcal{A}$ in Corollary 12 is a semiregularity in $\mathcal{A}$, but not a semiregularity in $\mathcal{B}$.

4.2 SBD property transferred through a Banach algebra homomorphism

From now on, we always assume that $T : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism satisfying $T(1_\mathcal{A}) = T(1_\mathcal{B})$. This subsection devotes to the transfer of SBD property from $\mathcal{B}$ to $\mathcal{A}$ through T.

Evidently, if (R_1, R_2) is a pair of semiregularities in $\mathcal{B}$ possessing SBD property, then the pair $(T^{-1}(R_1), T^{-1}(R_2))$ possesses SBD property by virtue of Theorem 10. As an application, we will give the conditions such that $T^{-1}(R)$ is $\mathcal{A}^{-1}$–SBD or $\mathrm{Exp}(\mathcal{A})$–SBD.

Theorem 13. *Let R be a semiregularity in $\mathcal{B}$ such that R is $\mathcal{B}^{-1}$-SBD. If T satisfies one of the following conditions*:
(1) *T is bounded below*;
(2) *T is an injective open mapping,*
then $T^{-1}(R)$ is $\mathcal{A}^{-1}$-SBD.

Proof. First, suppose that T is bounded below. We know $\partial\sigma_{\mathcal{A}^{-1}}(a) \subseteq \sigma_{\mathcal{B}^{-1}}(T(a))$ from [1, (3.1)]. Furthermore, $\partial\sigma_{\mathcal{A}^{-1}}(a) \subseteq \partial\sigma_{\mathcal{B}^{-1}}(T(a))$ since $T(\mathcal{A}^{-1})$ is contained in $\mathcal{B}^{-1}$, or equivalently, $\sigma_{\mathcal{B}^{-1}}(T(a)) \subseteq \sigma_{\mathcal{A}^{-1}}(a)$. According to Theorem 10,

$$\partial\sigma_{\mathcal{A}^{-1}}(a) \subseteq \partial\sigma_{\mathcal{B}^{-1}}(T(a)) \subseteq \sigma_R(T(a)) = \sigma_{T^{-1}(R)}(a) \subseteq \sigma_{T^{-1}(\mathcal{B}^{-1})}(a), \quad (\forall a \in \mathcal{A}).$$

It follows from Proposition 8 (1) that $T^{-1}(\mathcal{B}^{-1})$ is a regularity containing $\mathcal{A}^{-1}$ because $\mathcal{B}^{-1}$ is a regularity in $\mathcal{B}$. Then, $\sigma_{T^{-1}(\mathcal{B}^{-1})}(a) \subseteq \sigma_{\mathcal{A}^{-1}}(a)$.

Second, suppose that T is an injective open mapping. If $\lambda \notin \sigma_{T^{-1}(R)}(a) = \sigma_R(T(a))$, then either $\lambda \in \mathrm{int}\sigma_{\mathcal{B}^{-1}}(T(a))$ or $T(a) - \lambda 1_\mathcal{B} \in \mathcal{B}^{-1}$.

Case 1: $\lambda \in \mathrm{int}\sigma_{\mathcal{B}^{-1}}(T(a))$.

Because of $\sigma_{\mathcal{B}^{-1}}(T(a)) \subseteq \sigma_{\mathcal{A}^{-1}}(a)$, we have $\lambda \in \mathrm{int}\sigma_{\mathcal{B}^{-1}}(T(a)) \subseteq \mathrm{int}\sigma_{\mathcal{A}^{-1}}(a)$, i.e., $\lambda \notin \partial\sigma_{\mathcal{A}^{-1}}(a)$.

Case 2: $T(a) - \lambda 1_\mathcal{B}$ is invertible in $\mathcal{B}$.

We will prove that $T^{-1}(R)$ is $\mathcal{A}^{-1}$–SBD by contradiction. According to Theorem 10, it suffices to prove $\sigma_{T^{-1}(\mathcal{B}^{-1})}(a) \subseteq \sigma_{\mathcal{A}^{-1}}(a)$ for every $a \in \mathcal{A}$.

Assume that $\lambda \in \partial\sigma_{\mathcal{A}^{-1}}(a)$. Then, there exists a sequence $\{\xi_n\}$ in the complement of $\sigma_{\mathcal{A}^{-1}}(a)$ and a sequence $\{\eta_n\}$ in $\sigma_{\mathcal{A}^{-1}}(a)$ such that both $\{\xi_n\}$ and $\{\eta_n\}$ converge to some λ together. In other words, the sequence $\{a - \xi_n 1_\mathcal{A}\}$ in $\mathcal{A}^{-1}$ converges to $a - \lambda 1_\mathcal{A}$, while the sequence $\{a - \eta_n 1_\mathcal{A}\}$ in $\mathcal{A}\backslash\mathcal{A}^{-1}$ also converges to $a - \lambda 1_\mathcal{A}$. It induces that $a - \lambda 1_\mathcal{A}$ is in the boundary of $\mathcal{A}^{-1}$; furthermore, $a - \lambda 1_\mathcal{A}$ is a topological zero divisor in $\mathcal{A}$ [26, Chapter I 1 Theorem 14]. In other words, there exists a sequence $\{s_n\} \subset \mathcal{A}$ with $\|s_n\| = 1$ such that $(a - \lambda 1_\mathcal{A})s_n \to 0$ and $s_n(a - \lambda 1_\mathcal{A}) \to 0$ [26, Chapter I 1 Definition 13].

Since T is an open mapping, there exists some $\delta > 0$ such that $\{y \in \mathcal{B} : \|y\| < \delta\}$ is contained in $\{Tx : x \in \mathcal{A} : \|x\| < 1\}$. Furthermore, $\delta < \|T(s_n)\| < \|T\|$ as T is injective. It follows from the continuity of T that both $\{T(a - \lambda 1_\mathcal{A})T(s_n)\}$ and $\{T(s_n)T(a - \lambda 1_\mathcal{A})\}$ converge to zero.

Let $t_n = \frac{T(s_n)}{\|T(s_n)\|}$. Then, both $\{T(a-\lambda 1_{\mathcal{A}})t_n\}$ and $\{t_n T(a-\lambda 1_{\mathcal{A}})\}$ converge to zero. It induces that $T(a)-\lambda 1_{\mathcal{B}}$ is a topological zero divisor in $\mathcal{B}$, which contradicts with the assumption $T(a)-\lambda 1_{\mathcal{B}} \in \mathcal{B}^{-1}$. □

Remark 4. The condition that T is either a bounded below operator or an injective open mapping in Theorem 13 is necessary. For instance, let X be an infinite dimensional Banach space. We consider the canonical quotient mapping $\pi : B(X) \to C(X)$ and the regularity $R = \{[T] \in C(X) : [T] \text{ is invertible in } C(X)\}$. Then, π is a surjective continuous homomorphism, which implies $\pi(1_{B(X)}) = 1_{C(X)}$. Meanwhile, π is not injective, which implies that π does not satisfy the condition shown in Theorem 13. Note that $\pi^{-1}(R)$ is $\tilde{R}_6$. Then, $\partial\sigma_{B(X)^{-1}}(T) = \sigma_{B(X)^{-1}}(T) \subseteq \sigma_{\tilde{R}_6}(T)$ fails if T is a compact operator with nonzero eigenvalues.

As an application of Theorem 13, we can pull elements back from $\mathrm{QN}(\mathcal{B})$ (or $\mathrm{rad}(\mathcal{B})$) to $\mathrm{QN}(\mathcal{A})$ (or $\mathrm{rad}(\mathcal{A})$) by T. Recall that $\mathrm{QN}(\mathcal{A})$ is the set of all quasinilpotent elements in $\mathcal{A}$ and that the radical $\mathrm{rad}(\mathcal{A})$ is the intersection of all maximal left(or right) ideals of $\mathcal{A}$. In addition, $\mathrm{rad}(\mathcal{B})$ can be written as the set $\{b \in \mathcal{B} : \mathcal{B}b \subseteq \mathrm{QN}(\mathcal{B})\}$.

Corollary 14. *Let T, $\mathcal{A}$, and $\mathcal{B}$ be as mentioned in* Theorem 13.

(1) *If b is in $T(\mathcal{A}) \cap \mathrm{QN}(\mathcal{B})$, then $T^{-1}(b)$ is in* $\mathrm{QN}(\mathcal{A})$.

(2) *If b is in $T(\mathcal{A}) \cap \mathrm{rad}(\mathcal{B})$, then $T^{-1}(b)$ is in* $\mathrm{rad}(\mathcal{A})$.

Proof. (1) Suppose that b is in $T(\mathcal{A}) \cap \mathrm{QN}(\mathcal{B})$. We choose $R = \mathcal{A}^{-1}$ in Theorem 13. Then, $\partial\sigma_{\mathcal{A}^{-1}}(T^{-1}(b)) \subseteq \sigma_{T^{-1}(\mathcal{B}^{-1})}(T^{-1}(b)) = \{0\}$ since $\sigma_{\mathcal{B}^{-1}}(b) = \{0\} = \partial\sigma_{\mathcal{B}^{-1}}(b)$. It induces that $T^{-1}(b)$ is in $\mathrm{QN}(\mathcal{A})$.

(2) Suppose that b is in $T(\mathcal{A}) \cap \mathrm{rad}(\mathcal{B})$. For arbitrary a in $\mathcal{A}$, $T(a)b$ ia a quasinilpotent element of $\mathcal{B}$. Then, $aT^{-1}(b)$ is a quasinilpotent element of $\mathcal{A}$ from (1). It follows that $T^{-1}(b)$ is in $\mathrm{Rad}(\mathcal{A})$. □

Similar to Theorem 13, one can think of the case that $T^{-1}(R)$ is $\mathrm{Exp}(\mathcal{A})$–SBD, where $\mathrm{Exp}(\mathcal{A}) = \{e^{a_1}e^{a_2} \dots e^{a_n} : a_i \in \mathcal{A}, 1 \le i \le n, n \in \mathbb{N}\}$ is the principle component of $\mathcal{A}^{-1}$. Denote by $\varepsilon_{\mathcal{A}}(a)$ the spectrum associated with $\mathrm{Exp}(\mathcal{A})$.

Corollary 15. *Let T, $\mathcal{A}$, and $\mathcal{B}$ be as mentioned in* Theorem 13. *If R is a semiregularity in $\mathcal{B}$ such that R is* $\mathrm{Exp}(\mathcal{B})$*-SBD, then $T^{-1}(R)$ is* $\mathrm{Exp}(\mathcal{A})$*-SBD.*

Proof. It is easy to see that $T(\mathrm{Exp}(\mathcal{A}))$ is connected due to the fact that T is continuous and $\mathrm{Exp}(\mathcal{A})$ is connected. It induces $T(\mathrm{Exp}(\mathcal{A})) \subseteq \mathrm{Exp}(\mathcal{B})$. One can see from Proposition 8 (3) that $T^{-1}(\mathrm{Exp}(\mathcal{B})) \supseteq \mathrm{Exp}(\mathcal{A})$ is an upper semiregularity in $\mathcal{A}$; furthermore,

$$\varepsilon_{\mathcal{B}}(T(a)) = \sigma_{T^{-1}(\mathrm{Exp}(\mathcal{B}))}(a) \subseteq \varepsilon_{\mathcal{A}}(a)$$

due to Theorem 10.

If T is bounded below, then $\partial\varepsilon_{\mathcal{A}}(a) \subseteq \varepsilon_{\mathcal{B}}(T(a))$ by Theorem 2.13 in [21]. Furthermore, $T^{-1}(R)$ is $\mathrm{Exp}(\mathcal{A})$–SBD as $\sigma_R(T(a)) = \sigma_{T^{-1}(R)}(a)$.

Now assume that T is an injective open mapping. We see that the boundary of $\mathrm{Exp}(\mathcal{A})$ is a subset of the boundary of $\mathcal{A}^{-1}$ since $\mathrm{Exp}(\mathcal{A})$ is the principle component of $\mathcal{A}^{-1}$. Furthermore, the boundary of $\mathrm{Exp}(\mathcal{A})$ is a subset of the set of all topological zero divisors in $\mathcal{A}$. Using the same trick as in Theorem 13, we can prove that $\partial\varepsilon_{\mathcal{A}}(a) \subseteq \sigma_{T^{-1}(R)}(a)$ for all $a \in \mathcal{A}$. □

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