Research Article

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The ill-posedness of the (non-)periodic traveling wave solution for the deformed continuous Heisenberg spin equation

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Abstract: Based on an equivalent derivative non-linear Schrödinger equation, we derive some periodic and non-periodic two-parameter solutions of the deformed continuous Heisenberg spin (DCHS) equation. The ill-posedness of these solutions is demonstrated through Fourier integral estimates in the Sobolev space $H^s_{\mathbb{S}^2}$ (for the periodic solution in $H^s_{\mathbb{S}^2}(\mathbb{T})$ and the non-periodic solution in $H^s_{\mathbb{S}^2}(\mathbb{R})$, respectively). When $\alpha \neq 0$, the range of the weak ill-posedness index is $1 < s < \frac{3}{2}$ for both periodic and non-periodic solutions. However, the periodic solution exhibits a strong ill-posedness index in the range of $\frac{3}{2} < s < \frac{7}{2}$, whereas for the non-periodic solution, the range is $1 < s < 2$. These findings extend our previous work on the DCHS model to include the case of periodic solutions and explore a different fractional Sobolev space.

Keywords: Heisenberg spin, soliton, ill-posedness, Fourier integral

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1 Introduction

The deformed continuous Heisenberg spin (DCHS) equation is an important physical model that has received significant attention in the past few decades. Mikhailov and Shabat [1] were the first to construct an integrable SO(3)-invariant DCHS equation, which can be written as

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx} + \alpha \mathbf{S}_x (\mathbf{S}_x)^2, \tag{1}$$

where $\wedge$ denotes the cross-product of the vector $\mathbf{S}(x,t) = (S_1(x,t), S_2(x,t), S_3(x,t))$, with $\mathbf{S} \cdot \mathbf{S} = 1$ and $(\mathbf{S}_x)^2 = \mathbf{S}_x \cdot \mathbf{S}_x$.

The DCHS equations encompass a wide range of equations that can be transformed into various orders of non-linear Schrödinger (NLS) equations. Porsezian et al. [2] were the first to demonstrate that (1) is gauge equivalent to the integrable derivative NLS equation, which has applications in two-photon self-induced transparency and the propagation of ultra-short light pulses in optical fibers. Similarly, Lakshmanan et al. [3–7] explored higher-order integrable DCHS equations and found that they can be transformed into higher-order NLS equations by associating the spin vector with the tangent to a moving curve in Euclidean space. For the higher-dimensional integrable DCHS [8], methods for deriving the corresponding gauge-equivalent NLS equations have also been proposed [9].

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If $\alpha = 0$, then (1) reduces to an isotropic Heisenberg spin (IHS) equation [10], which is an exactly integrable equation and can be considered as the simplest case of the Landau-Lifshitz (LL) equation [11–13]. The following articles illustrate some typical progress on the LL equation. Alouges and Soyeur [14] established some necessary conditions for the existence of a global weak solution. When the spatial dimension is $n = 1$ with periodic boundary conditions, Guo and Huang [15] proved the existence of a unique smooth solution using the technique of spatial differences. In $\mathbb{R}^3$, Carbou and Fabrie [16] proved the local existence and uniqueness of regular solutions, as well as the global existence when the initial data are sufficiently small. Chang et al. [17] established the existence of small-data global solutions in cylindrical coordinates. In normal coordinates, a global solution with small initial values was also shown to exist [18] under certain norms. In dimensions larger than three, the global existence and uniqueness of mild solutions were demonstrated [19] under a smallness condition. Similarly, under a smallness constraint in Morrey spaces, Lin et al. [20] extended this result to establish the existence of a global solution. Moreover, the solution with small initial data in critical Besov space was shown [21] to be globally well-posed in dimensions $n \geq 3$.

Inspired by studies of heat flow in harmonic maps [22] and the Ginzburg-Landau equation [23], estimates of the concentration set of the stationary weak solutions of the LL equation have been made [24–26] to analyze the solution's behavior near singular points. Moreover, the singularity properties, including finite-time blowup, have been demonstrated for a special type of solution. In particular, when the topological degree is one, the equivariant solution exhibits blowup behavior, and its blowup rate was predicted in previous studies [27–29]. It is well known that exact solutions to the LL equation provide a more intuitive way to study its dynamic behavior. For further details, we refer the reader to [30–35].

Since (1) is a quasilinear equation, analyzing the well-posedness of the system directly presents significant challenges. The theory of well-posedness for partial differential equations (PDEs) is not fully developed. As a result, many studies rely on an equivalent system of the original equation (particularly the equivalent NLS equation) to demonstrate the well-posedness of solutions to these PDEs. Examples include the Schrödinger equation with derivative:

$$-\mathrm{i}W_t = \Delta W - \frac{2W^*}{1 + |W|^2}\nabla W \cdot \nabla W$$

and its equivalent covariant derivative form:

$$(iD_t - D_jD_j)\Psi_k = -\mathrm{i}\,\mathrm{Im}(\Psi_j\Psi_j^*)\Psi_j.$$

In a similar manner, we use an equivalent complex equation to study the DCHS equation. Let the curvature κ and torsion τ be defined as

$$\kappa = (\mathbf{S}_x \cdot \mathbf{S}_x)^{\frac{1}{2}} \quad \text{and} \quad \tau = \frac{\mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_{xx})}{\kappa^2},$$

respectively.

We apply the following Hasimoto transform [2,36,37]:

$$Q = \kappa \exp\left(\mathrm{i}\int_{-\infty}^{x} \tau(t, x')\mathrm{d}x'\right)$$

to convert (1) into the following non-linear derivative Schrödinger equation (see [2]):

$$\mathrm{i}Q_t + Q_{xx} + \frac{1}{2}Q\,|Q|^2 - \mathrm{i}\alpha(|Q|^2Q)_x = 0, \tag{2}$$

where the parameter α specifies the contribution of the non-linear term $(|Q|^2Q)_x$.

Equation (2) is a combination of the cubic Schrödinger equation and the Alfvén equation. If $i\alpha(|Q|^2Q)_x$ is omitted, (2) reduces to the well-known cubic Schrödinger equation, which has been thoroughly studied. However, if the cubic term $\frac{1}{2}Q\,|Q|^2$ is removed, (2) becomes the Alfvén equation, which originates from plasma physics [38]. For the Alfvén equation, various well-posedness results have been established. Hayashi [39] proved its global well-posedness in H^1. Similar results can also be found in [40,41]. Furthermore, Takaoka [42] examined solutions with rougher initial data and demonstrated local well-posedness in H^s with $s > \frac{1}{2}$ using an equivalent equation. When the initial condition Q_0 satisfies $||Q_0||_{L^2} < \sqrt{2\pi}$, Hayashi and Ozawa [43] used mass conservation to show that the solution is global. However, a soliton-type solution is ill-posed in H^s for $0 < s < \frac{1}{2}$ [44].

In comparison with well-posedness results, studies on the ill-posedness of PDEs are relatively scarce. The ill-posedness often depends on the specific solutions. Different solutions may correspond to different ill-posedness spaces and have varying ill-posedness indices. This variability makes it challenging to obtain a general ill-posedness result. Although Bigioni and Linares established the ill-posedness for a class of solutions as early as 2001, it remains unclear whether solutions with general initial boundary data or more general derivative Schrödinger equations also exhibit ill-posedness in certain specific spaces. Notably, the solution discussed in [44] is considered in the entire space $H^s(\mathbb{R})$.

To the best of our knowledge, there are few studies on the ill-posedness of the DCHS equation. In a recent study [45], we provided a proof of the ill-posedness of the solution and identified an exact index range for the first time. As far as we know, no other studies have addressed this topic, and further research is needed. In light of this, we build upon our previous work [45] to further explore the ill-posedness of the DCHS equation, extending the results to various solutions (periodic and non-periodic solutions) and different fractional Sobolev spaces (which will be defined below).

Here, we investigate the ill-posedness of (1) and (2). For $\sigma \geq 0$, let J^σ denote the Fourier multiplier $\xi \to (1 + |\xi|^2)^{\sigma/2}$. The spaces H^σ (including $H^\sigma(\mathbb{R})$) and the periodic space $H^\sigma(\mathbb{T})$ (where the period is $\mathbb{T}$) are defined by the norm $||f||_{H^\sigma} = ||J^\sigma(f)||_{L^2}$, where L^2 is defined over $\mathbb{R}$ or $\mathbb{T}$, characterizing complex fractional Sobolev spaces.

With the initial condition Q_0, the solution of equation (2) is classified as ill-posed in H^s (whether in $H^s(\mathbb{R})$ or $H^s(\mathbb{T})$) and can be characterized as follows:

(I) Weak ill-posedness: Let $\tilde{C}$ be a given constant. For any real $\delta > 0$, the solution is weakly ill-posed in H^s if and only if

$$||Q_{c_1,\omega_1}(x,0) - Q_{c_2,\omega_2}(x,0)||_{H^s} \leq \delta,$$

$$Q_{c_j,\omega_j}(\cdot,T) \in H^s, \quad ||Q_{c_1,\omega_1}(\cdot,T) - Q_{c_2,\omega_2}(\cdot,T)||_{H^s} \geq \tilde{C}.$$

(II) Strong ill-posedness: For any $\varepsilon > 0$ and $\delta > 0$, the solution is strongly ill-posed in H^s if and only if

$$\begin{aligned} &||Q_{c_1,\omega_1}(x,0) - Q_{c_2,\omega_2}(x,0)||_{H^s} \leq \delta,\\ &||Q_{c_1,\omega_1}(\cdot,T) - Q_{c_2,\omega_2}(\cdot,T)||_{H^s} \geq \varepsilon^{-1},\\ &||Q_{c_j,\omega_j}(\cdot,T)||_{H^s} = ||Q_{c_j,\omega_j}(\cdot,0)||_{H^s} > \varepsilon^{-1}.\end{aligned}$$

Given that $||\kappa|| = ||Q||$ and $\kappa = (\mathbf{S}_x \cdot \mathbf{S}_x)^{\frac{1}{2}}$, there exists an equivalence between the norms of Q and $\mathbf{S}$. Thus, we can use the norm of Q to estimate the norm of $\mathbf{S}$. Let $F = (F_1, F_2, F_3)$ and $G = (G_1, G_2, G_3)$. We define the induced distance as

$$d^\sigma(F,G) = \left[\sum_{l=1}^{3} ||F_l - G_l||^2_{H^\sigma}\right]^{\frac{1}{2}},$$

which is used to define the vector norm in H^σ_* as follows:

$$||F||_{H^\sigma_*} = \left(\sum_{l=1}^{3} ||F_l||^2_{H^\sigma}\right)^{\frac{1}{2}}.$$

Similarly, the induced norm of $H^\sigma_{\mathbb{S}^2}$ ($H^\sigma_{\mathbb{S}^2}(\mathbb{T})$ and $H^\sigma_{\mathbb{S}^2}(\mathbb{R})$) for the vector $\mathbf{S}$ (where $\mathbf{S}_\infty$ is the value of $\mathbf{S}$ at $x = \infty$) is defined as

$$||\mathbf{S}||_{H^\sigma_{\mathbb{S}^2}(\mathbb{T})} = ||\mathbf{S}||_{H^\sigma_*(\mathbb{T})}$$

and

$$\|\mathbf{S}\|_{H^{\sigma}_{\mathbb{S}^2}(\mathbb{R})} = \|\mathbf{S} - \mathbf{S}_\infty\|_{H^{\sigma}_{*}(\mathbb{R})},$$

respectively.

It is straightforward to observe that if $Q \in C([0, T]; H^{\sigma})$ is the solution of (2), then the solution of (1) satisfies $\mathbf{S} \in C([0, T]; H^{\sigma+1}_{\mathbb{S}^2})$. Similarly, in $H^{s}_{\mathbb{S}^2}(\mathbb{T})$ or $H^{s}_{\mathbb{S}^2}(\mathbb{R})$, two types of ill-posedness are defined as follows:

(I) Weak ill-posedness: The solution is weakly ill-posed in $H^{s}_{\mathbb{S}^2}$ if and only if

$$\|\mathbf{S}_{c_1,\omega_1}(x, 0) - \mathbf{S}_{c_2,\omega_2}(x, 0)\|_{H^{s}_{\mathbb{S}^2}} \le \delta,$$

$$\mathbf{S}_{c_j,\omega_j}(\cdot, T) \in H^{s}_{\mathbb{S}^2}, \quad \|\mathbf{S}_{c_1,\omega_1}(\cdot, T) - \mathbf{S}_{c_2,\omega_2}(\cdot, T)\|_{H^{s}_{\mathbb{S}^2}} \ge \tilde{C}.$$

(II) Strong ill-posedness: The solution is strongly ill-posed in $H^{s}_{\mathbb{S}^2}$ if and only if

$$\|\mathbf{S}_{c_1,\omega_1}(x, 0) - \mathbf{S}_{c_2,\omega_2}(x, 0)\|_{H^{s}_{\mathbb{S}^2}} \le \delta,$$

$$\|\mathbf{S}_{c_1,\omega_1}(\cdot, T) - \mathbf{S}_{c_2,\omega_2}(\cdot, T)\|_{H^{s}_{\mathbb{S}^2}} \ge \varepsilon^{-1},$$

$$\|\mathbf{S}_{c_j,\omega_j}(\cdot, T)\|_{H^{s}_{\mathbb{S}^2}}, \|\mathbf{S}_{c_j,\omega_j}(\cdot, 0)\|_{H^{s}_{\mathbb{S}^2}} > \varepsilon^{-1}.$$

Using Fourier analysis, we obtain the following result:

Theorem 1. *There exists a solution* $\mathbf{S}(x, t)$ *of* (1) *in* $H^{s}_{\mathbb{S}^2}$, *and the mapping* $\mathbf{S}_0 \to \mathbf{S}(t)$ *is ill-posed. Specifically, if* $\alpha \ne 0$, *there exist the following two-parameter solitary wave solutions* $\mathbf{S}_{c,\omega}$:

(I) Weak and strong ill-posedness for periodic solutions: *If* $\mathbf{S}_{c,\omega}$ *satisfies the constrained curvature condition*

$$(\mathbf{S}_x \cdot \mathbf{S}_x)^{\frac{1}{2}} = \sqrt{2}A_1\left(B_1 \cos\left(\frac{1}{2}A_1\xi\right)^2 + C_1\right)^{-\frac{1}{2}},$$

where $\xi = x - ct$, $A_1 = \sqrt{c^2 + 4\omega}$, $B_1 = 4\sqrt{-\alpha^2\omega + \alpha c + 1}$, *and* $C_1 = -\alpha c - 2\sqrt{-\alpha^2\omega + \alpha c + 1} - 2$, *then the solution is weakly ill-posed in* $H^{s}_{\mathbb{S}^2}(\mathbb{T})$ *for* $1 < s < \frac{3}{2}$, *and strongly ill-posed in* $H^{s}_{\mathbb{S}^2}(\mathbb{T})$ *for* $\frac{3}{2} < s < \frac{5}{2}$.

(II) Weak and strong ill-posedness for non-periodic solutions: *If* $\mathbf{S}_{c,\omega}$ *satisfies the constrained curvature condition*

$$(\mathbf{S}_x \cdot \mathbf{S}_x)^{\frac{1}{2}} = \sqrt{2}\left(\frac{e^{A_2\xi}B_2}{A_2^2} + \frac{e^{-A_2\xi}B_2}{A_2^2} + \frac{C_2}{A_2^2}\right)^{-\frac{1}{2}},$$

where $\xi = x - ct$, $A_2 = \sqrt{-c^2 - 4\omega}$, $B_2 = \sqrt{-\omega\alpha^2 + \alpha c + 1}$, *and* $C_2 = \alpha c + 2$, *then the solution is weakly ill-posed in* $H^{s}_{\mathbb{S}^2}(\mathbb{R})$ *for* $1 < s < \frac{3}{2}$, *and strongly ill-posed in* $H^{s}_{\mathbb{S}^2}(\mathbb{R})$ *for* $1 < s < 2$.

Remark 1. In addition to (1), many more general DCHS models exist. For instance, Lakshmanan and Ganesan [46] proposed a generalized case that includes linear inhomogeneities (as well as higher-order integrable DCHS equations), which is given by

$$\mathbf{S}_t = (\gamma_2 + \mu_2 x)\mathbf{S} \wedge \mathbf{S}_{xx} + \mu_2\mathbf{S} \wedge \mathbf{S}_x - (\gamma_1 + \mu_1 x)\mathbf{S}_x - \gamma\left(\mathbf{S}_{xx} + \frac{3}{2}\mathbf{S}_x^2\mathbf{S}\right)_x, \tag{3}$$

where the parameters γ_1 and γ_2 represent the constant coefficients that modulate the linear terms of the spin field dynamics. The terms with coefficients μ_1 and μ_2 introduce linear inhomogeneities, meaning that the influence of the respective terms varies linearly with the spatial coordinate x. γ is a crucial parameter that regulates the non-linear effects within the spin field.

To the best of our knowledge, the well-posedness and ill-posedness of (3) remain open problems.

The H^{σ} norm of Q is equivalent to the $H^{\sigma+1}_{\mathbb{S}^2}$ norm of $\mathbf{S}$. To prove Theorem 1, it suffices to prove the following equivalent theorem:

Theorem 2. *Let $\alpha \neq 0$ and $\xi = x - ct$; A_i, B_i, and C_i $(i = 1, 2)$ are as defined in* Theorem 1. *Then,* (2) *has a two-parameter solitary wave solution*:

$$Q_{c,\omega}(\xi, t) = e^{-\mathrm{i}\omega t}\phi(\xi)e^{\mathrm{i}\psi(\xi)}, \tag{4}$$

where $\phi(\xi)$ and $\psi(\xi)$ can be given by

$$\phi(\xi) = \sqrt{2}A_1\left(B_1\cos\left(\frac{1}{2}A_1\xi\right)^2 + C_1\right)^{-\frac{1}{2}}, \tag{5}$$

$$\psi(\xi) = \frac{3\alpha A_1}{\sqrt{(B_1 + C_1)C_1}}\arctan\left(\frac{C_1\tan\left(\frac{1}{2}A_1\xi\right)}{\sqrt{(B_1 + C_1)C_1}}\right) + \frac{1}{2}c\xi, \tag{6}$$

or

$$\phi(\xi) = \sqrt{2}\left(\frac{e^{A_2\xi}B_2}{A_2^2} + \frac{e^{-A_2\xi}B_2}{A_2^2} + \frac{C_2}{A_2^2}\right)^{-\frac{1}{2}}, \tag{7}$$

$$\psi(\xi) = -3\arctan\left(\frac{2e^{A_2\xi}B_2 + C_2}{\alpha A_2}\right) + \frac{1}{2}c\xi. \tag{8}$$

These two types of solutions are ill-posed:

(*I*) *Solution* (4), *where $\phi(\xi)$ and $\psi(\xi)$ are given by* (5) *and* (6), *respectively, is weakly ill-posed in $H^s(\mathbb{T})$ for $0 < s < \frac{1}{2}$, and strongly ill-posed in $H^s(\mathbb{T})$ for $\frac{1}{2} < s < \frac{3}{2}$.*

(*II*) *Solution* (4), *where $\phi(\xi)$ and $\psi(\xi)$ are given by* (7) *and* (8), *respectively, is weakly ill-posed in $H^s(\mathbb{R})$ for $0 < s < \frac{1}{2}$, and strongly ill-posed in $H^s(\mathbb{R})$ for $0 < s < 1$.*

Remark 2. The Alfvén equation has a class of ill-posed solutions in $H^s(\mathbb{R})$ for $0 < s < \frac{1}{2}$ [44]. As shown in case (I), when the cubic term $\frac{1}{2}Q\,|Q|^2$ is added to the Alfvén equation, ill-posed solutions still exist. Additionally, previous studies have focused only on the weak ill-posedness of soliton-type solutions over the entire space $H^s(\mathbb{R})$. However, Theorem 2 demonstrates that solutions to the derivative Schrödinger equation with the $\frac{1}{2}Q\,|Q|^2$ term exhibit both weak and strong ill-posedness in $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$.

Moreover, it has been shown [4] that (3) is geometrically and gauge equivalent to the generalized NLS equation with linear inhomogeneities:

$$\mathrm{i}Q_1 + \mathrm{i}\mu_1 Q + \mathrm{i}(\gamma_1 + \mu_1 x)Q_x + (\gamma_2 + \mu_2 x)(Q_{xx} + 2\,|Q|^2 Q) + 2\mu_2\left(Q_x + Q\int_{-x}^{x}|Q|^2\mathrm{d}x'\right) + \mathrm{i}\gamma(Q_{xxx} + 6\,|Q|^2 Q_x) = 0. \tag{9}$$

Similar to (3), the questions of well-posedness and ill-posedness for (9) remain open.

This article is organized as follows: in Section 2, we construct the (non-)periodic traveling wave solutions for the equivalent non-linear derivative Schrödinger equation associated with the DCHS equation. In Section 3, we prove the ill-posedness of the periodic traveling wave solution and present the range of the ill-posedness index. In Section 4, we establish the weak and strong ill-posedness of the non-periodic solution (soliton solution) and estimate the corresponding ill-posedness indices.

2 (Non-) periodic solutions to the DCHS equation

Under the plane wave variable $\xi = x - ct$, we assume that the soliton solution of (2) is given by

$$Q_{c,\omega}(t,x) = e^{-i\omega t}\phi(\xi)e^{i\psi(\xi)}, \tag{10}$$

where the parameter c represents the wave speed of the traveling wave solution.

By substituting (10) into (2) and separating the real and imaginary parts, we obtain

$$c\phi\frac{d\psi}{d\xi} + \alpha\phi^3\frac{d\psi}{d\xi} - \left(\frac{d\psi}{d\xi}\right)^2\phi + \omega\phi(\xi) + \frac{1}{2}\phi^3 + \frac{d^2\phi}{d\xi^2} = 0 \tag{11}$$

and

$$-c\frac{d\phi}{d\xi} - 3\alpha\phi^2\frac{d\phi}{d\xi} + 2\frac{d\psi}{d\xi}\frac{d\phi}{d\xi} + \left(\frac{d^2\psi}{d\xi^2}\right)\phi = 0, \tag{12}$$

where the parameter α (the same one as in (2)) quantifies the strength of the non-linearity in the system.

Note that (11) and (12) form a system of first-order differential equations with respect to the variable ξ. To solve this system, from (12), we find

$$\psi = c_2 + \frac{c\xi}{2} + \int\frac{3\alpha\phi^4 + 4c_1}{4\phi^2}d\xi. \tag{13}$$

Substituting (13) into (11), we obtain

$$16\left(\frac{d^2\phi}{d\xi^2}\right)\phi^3 + 3\alpha^2\phi^8 + 8\alpha c\phi^6 + 8\phi^6 - 8c_1\alpha\phi^4 + 4c^2\phi^4 + 16\omega\phi^4 - 16c_1^2 = 0. \tag{14}$$

To solve (14), we define an auxiliary function

$$\left(\frac{d\phi}{d\xi}\right)^2 = \sum_{j=0}^{6} h_j\phi^j, \tag{15}$$

where h_j are the undetermined coefficients.

By (15), the second derivative of ϕ must satisfy the following equation:

$$\frac{d^2\phi}{d\xi^2} = \frac{1}{2}\sum_{j=1}^{6} jh_j\phi^{j-1}(\xi). \tag{16}$$

Substituting (15) and (16) into (14) and comparing the powers of ϕ, we obtain

$$\left(\frac{d\phi}{d\xi}\right)^2 = -\frac{\alpha^2}{16}\phi^6 + \left(-\frac{\alpha c}{4} - \frac{1}{4}\right)\phi^4 + \left(\frac{\alpha c_1}{2} - \frac{c^2}{4} - \omega\right)\phi^2 - c_1^2\phi^{-2}. \tag{17}$$

Next, in (17), we consider the case where $c_1 = 0$, which simplifies to

$$\left(\frac{d\phi}{d\xi}\right)^2 = -\left[\frac{\alpha^2}{16}\phi^4 + \left(-\frac{\alpha c}{4} - \frac{1}{4}\right)\phi^2 - \frac{c^2}{4} - \omega\right]\phi^2. \tag{18}$$

For the ordinary differential equation (18), if $h_0 = h_1 = h_3 = h_5 = 0$, $h_6 < 0$, $h_4^2 - 4h_2h_6 > 0$, $h_2 > 0$, and $h_4 < 0$, then (15) has the following bell-shaped solution:

$$\phi(\xi) = \left\{\frac{2h_2\,\mathrm{sech}^2\left(\sqrt{h_2}\,\xi\right)}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right)\mathrm{sech}^2\left(\sqrt{h_2}\,\xi\right)}\right\}^{\frac{1}{2}}$$

and a singular solution:

$$\phi(\xi)=\left\{\frac{2h_2\operatorname{csch}^2\left(\pm\sqrt{h_2}\,\xi\right)}{2\sqrt{h_4^2-4h_2h_6}+\left(\sqrt{h_4^2-4h_2h_6}-h_4\right)\operatorname{csch}^2\left(\pm\sqrt{h_2}\,\xi\right)}\right\}^{\frac{1}{2}}.$$

By (15), we have $h_0=h_1=h_3=h_5=0$, $h_2=-\frac{1}{4}c^2-\omega$, $h_4=-\frac{1}{4}ac-\frac{1}{2}$, and $h_6=-\frac{1}{16}a^2$. Hence, we obtain the following theorem:

Solution 3. Equation (2) has the following solution:

$$Q=e^{-\mathrm{i}\omega t}e^{\mathrm{i}\psi(\xi)}\phi(\xi), \tag{19}$$

where $\xi=x-ct$ and

$$\psi(\xi)=\int\frac{3a\phi^2}{4}\mathrm{d}\xi+\frac{c\xi}{2}. \tag{20}$$

(I) If $a>0$, $c<-2a^{-1}$, and $\omega<\frac{ac+1}{a^2}$, then the equation has the following trigonometric solution:

$$\phi(\xi)=\left\{\frac{2\left(\frac{1}{4}c^2+\omega\right)\sec^2\left(\sqrt{\frac{1}{4}c^2+\omega}\,\xi\right)}{\sqrt{-a^2\omega+ac+1}-\left(\frac{1}{2}\sqrt{-a^2\omega+ac+1}+\frac{ac}{4}+\frac{1}{2}\right)\sec^2\left(\sqrt{\frac{1}{4}c^2+\omega}\,\xi\right)}\right\}^{\frac{1}{2}} \tag{21}$$

and the singular trigonometric solution:

$$\phi(\xi)=\left\{\frac{-2\left(\frac{1}{4}c^2+\omega\right)\csc^2\left(\pm\sqrt{\frac{1}{4}c^2+\omega}\,\xi\right)}{\sqrt{-a^2\omega+ac+1}-\left(\frac{1}{2}\sqrt{-a^2\omega+ac+1}-\frac{ac}{4}-\frac{1}{2}\right)\csc^2\left(\pm\sqrt{\frac{1}{4}c^2+\omega}\,\xi\right)}\right\}^{\frac{1}{2}}. \tag{22}$$

(II) If $a>0$, $\omega<-\frac{1}{4}c^2$, and $-2a^{-1}<c$, then the equation has the following bell-shaped solution:

$$\phi(\xi)=\left\{\frac{-2\left(\frac{1}{4}c^2+\omega\right)\operatorname{sech}^2\left(\sqrt{-\frac{1}{4}c^2-\omega}\,\xi\right)}{\sqrt{-a^2\omega+ac+1}-\left(\frac{1}{2}\sqrt{-a^2\omega+ac+1}-\frac{ac}{4}-\frac{1}{2}\right)\operatorname{sech}^2\left(\sqrt{-\frac{1}{4}c^2-\omega}\,\xi\right)}\right\}^{\frac{1}{2}} \tag{23}$$

and the singular solution:

$$\phi(\xi)=\left\{\frac{-2\left(\frac{1}{4}c^2+\omega\right)\operatorname{csch}^2\left(\pm\sqrt{-\frac{1}{4}c^2-\omega}\,\xi\right)}{\sqrt{-a^2\omega+ac+1}+\left(\frac{1}{2}\sqrt{-a^2\omega+ac+1}+\frac{ac}{4}+\frac{1}{2}\right)\operatorname{csch}^2\left(\pm\sqrt{-\frac{1}{4}c^2-\omega}\,\xi\right)}\right\}^{\frac{1}{2}}. \tag{24}$$

Remark 3. Solutions (21) and (22) can be rewritten in a unified form as follows:

$$\phi(\xi)=\left\{\frac{4(c^2+4\omega)}{4\sqrt{-a^2\omega+ac+1}\cos^2\left(\frac{1}{2}\sqrt{c^2+4\omega}\,\xi\right)-ac-2\sqrt{-a^2\omega+ac+1}-2}\right\}^{\frac{1}{2}}.$$

Similarly, (23) and (24) can also be expressed in a unified form as follows:

$$\phi(\xi)=\left\{\frac{-4(c^2+4\omega)}{4\cosh^2\left(\frac{1}{2}\sqrt{-c^2-4\omega}\,\xi\right)\sqrt{-\omega a^2+ac+1}+ac-2\sqrt{-\omega a^2+ac+1}+2}\right\}^{\frac{1}{2}}.$$

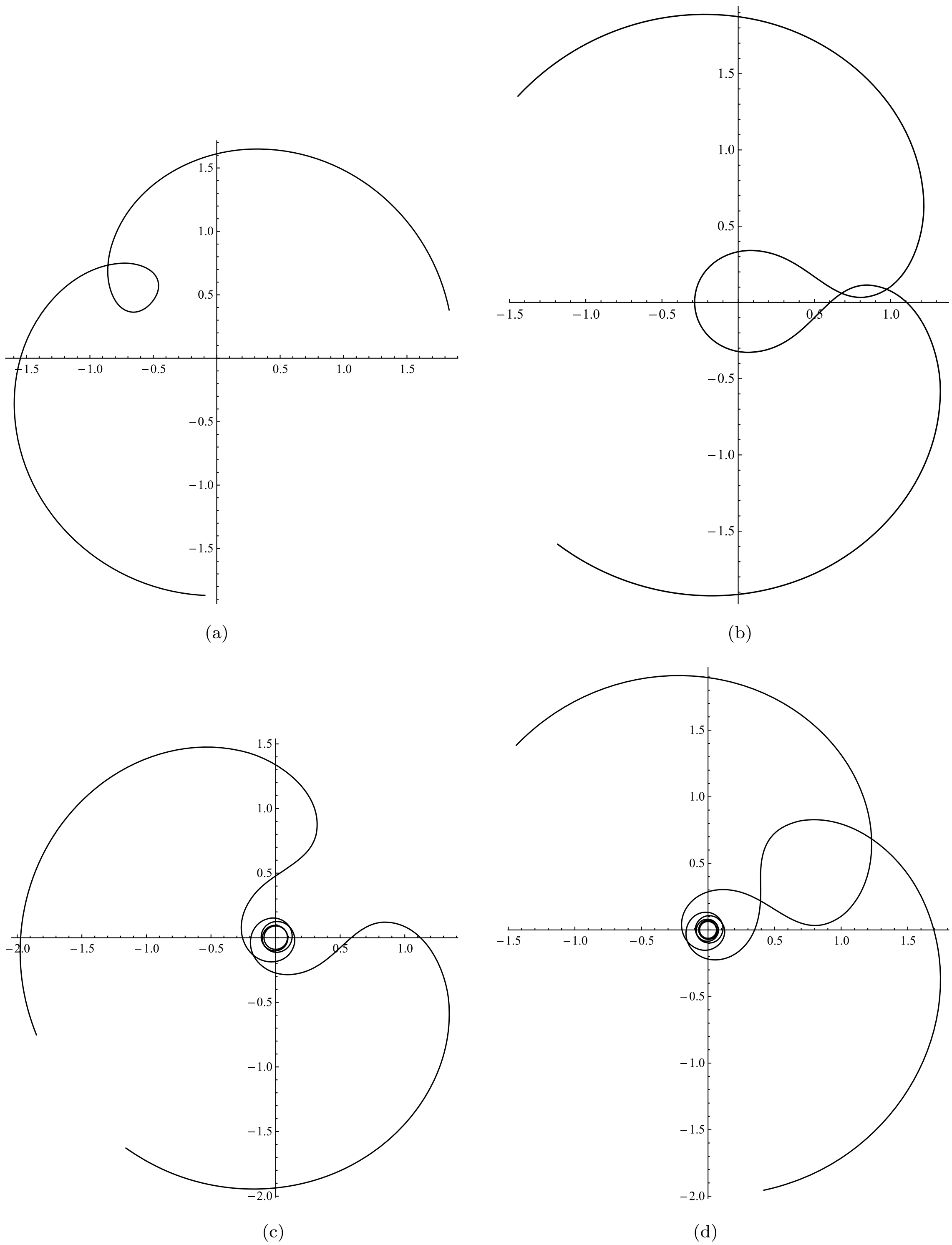

Figure 1: Complex plane image of the solution when $\phi(\xi)$ takes (21), $t = 1$, and $\omega = c - c^2/4$. In order to ensure that the image of periodic solution is drawn continuously, t and x shall meet $-\frac{\pi}{2} \leqslant \frac{1}{2}\sqrt{c^2 + 4\omega}_1(x - ct) \leqslant \frac{\pi}{2}$. It is observed from the figure that the complex plane image of the solution is axisymmetric. Moreover, with the synchronous increase of α and C, the number of times the complex plane images of the solution around the coordinates origin are intertwined with each other will increase: (a) $\alpha = 10$, $c = -10$, $\omega = -15$, $x \in [10.4967, -9.50327]$, (b) $\alpha = 50$, $c = -50$, $\omega = -575$, $x \in [-50.2221, -49.7779]$, (c) $\alpha = 500$, $c = -500$, $\omega = -62{,}000$, $x \in [-500.07, -499.93]$, and (d) $\alpha = 1{,}000$, $c = -1{,}000$, $\omega = -2{,}49{,}000$, $x \in [-1000.05, -999.95]$.

Remark 4. The evolution of solutions (21) and (23) is illustrated in Figures 1–4, which demonstrate the ill-posedness of both (21) and (22). By comparing Figures 3 and 4, it is evident that under different parameter settings, an initial value with a sufficiently small distance at the initial time (Figure 4) can evolve into a solution with a significantly larger distance at a later time (Figure 3).

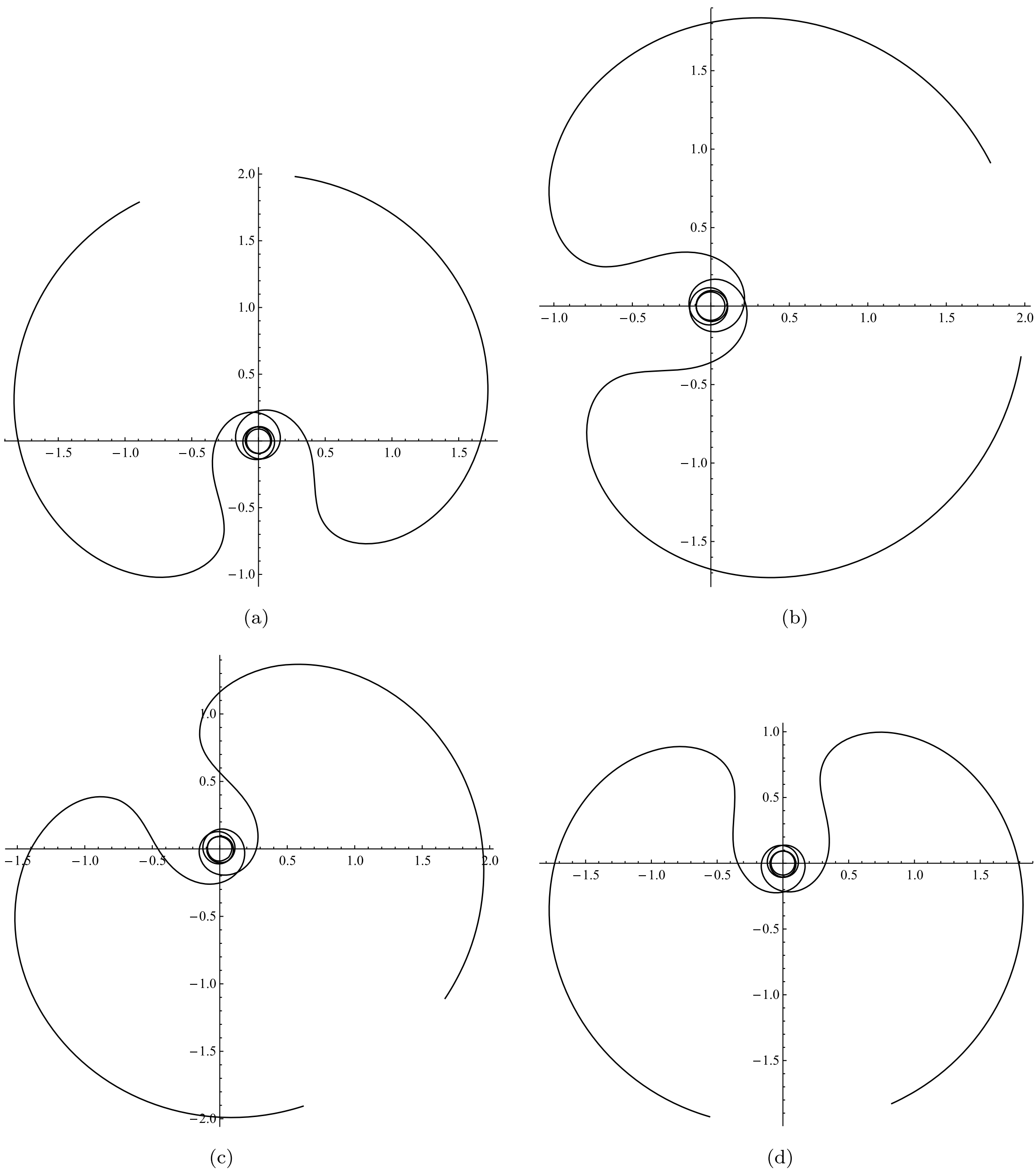

Figure 2: Complex plane image of the solution for different but similar c when $\phi(\xi)$ takes (21), $t = 1$, and $\omega = c - c^2/4$. It can be seen that when α is fixed and $\omega = c - c^2/4$, if c changes slightly, the solution will rotate around the coordinates origin: (a) $\alpha = 500$, $c = -501$, $\omega = 24{,}89{,}974$, $x \in [-501.07, -500.93]$, (b) $\alpha = 500$, $c = -502$, $\omega = -62{,}499$, $x \in [-502.07, -501.93]$, (c) $\alpha = 500$, $c = -503$, $\omega = 25{,}09{,}974$, $x \in [-503.07, -502.93]$, and (d) $\alpha = 500$, $c = -504$, $\omega = -63{,}000$, $x \in [-504.07, -503.93]$.

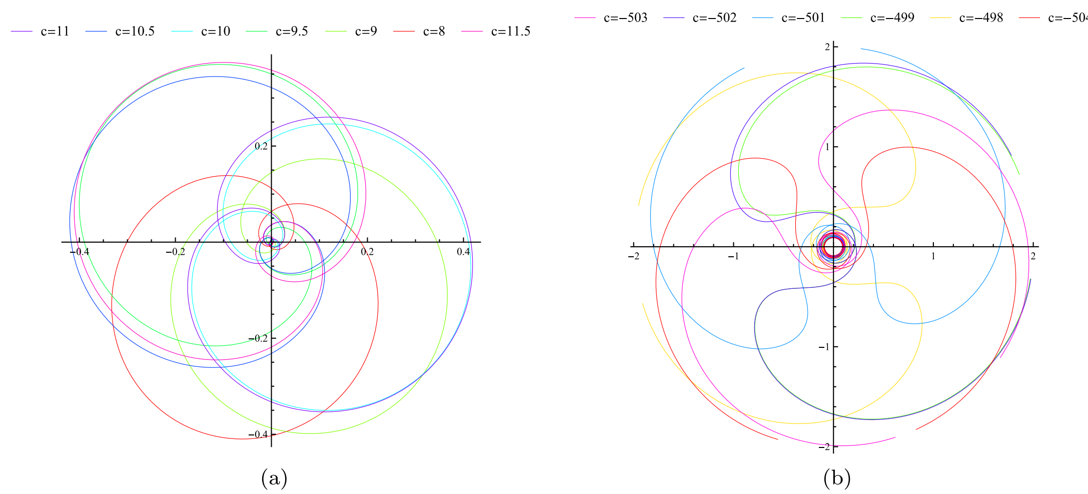

Figure 3: Comparison of the non-periodic solution and periodic solution when $t = 1$. It can be seen that when c increases, the complex plane image of the solution accelerates to rotate clockwise, and the heart-shaped ring in the middle also increases: (a) complex plane image of the solution for different c when $\phi(\xi)$ takes (23), $t = 1$, $\alpha = 10$, $\omega = c - c^2/4$, and $x \in [-15, 15]$ and (b) complex plane image of the solution for different but similar c when $\phi(\xi)$ takes (23), $t = 1$, and $\omega = c - c^2/4$.

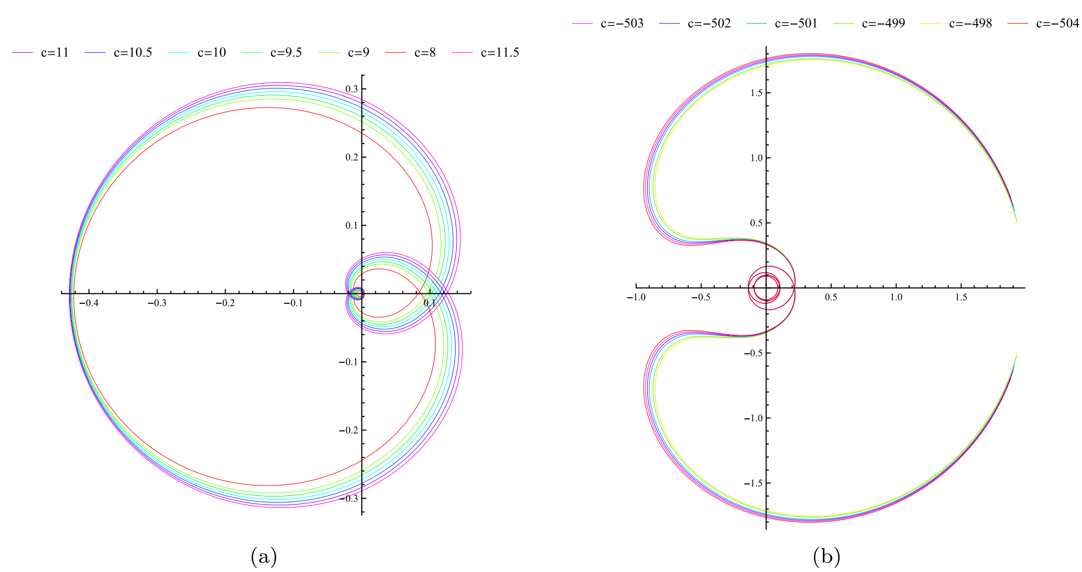

Figure 4: Comparison between the non-periodic solution and periodic solution when $t = 0$. It can be seen that the shape of the complex plane image of the solution is almost the same when Cchanges slightly, for both the periodic and non-periodic solutions: (a) complex plane image of the solution for different c when $\phi(\xi)$ takes (23), $t = 1$, $\alpha = 0$, $\omega = c - c^2/4$, and $x \in [-15, 15]$ and (b) complex plane image of the solution for different but similar c when $\phi(\xi)$ takes (23), $t = 0$, and $\omega = c - c^2/4$.

3 Ill-posedness of the periodic solution

We establish the ill-posedness property of the solution in (21) and similarly for (22). Equation (21) can be transformed into the form

$$\phi(\xi) = \sqrt{2}A_1\left(B_1\cos\left(\frac{1}{2}A_1\xi\right)^2 + C_1\right)^{-\frac{1}{2}},$$

where

$$A_1 = \sqrt{c^2 + 4\omega},$$
$$B_1 = 4\sqrt{-\alpha^2\omega + \alpha c + 1},$$
$$C_1 = -\alpha c - 2\sqrt{-\alpha^2\omega + \alpha c + 1} - 2.$$

Thus, (20) becomes equivalent to

$$\psi(\xi) = \frac{3\alpha A_1}{\sqrt{(B_1 + C_1)C_1}} \arctan\left(\frac{C_1 \tan\left(\frac{1}{2}A_1\xi\right)}{\sqrt{(B_1 + C_1)C_1}}\right) + \frac{1}{2}c\xi.$$

Let

$$d_4 = \frac{1}{2}A_1,$$
$$d_5 = \sqrt{2}A_1,$$

and define

$$h^{[1]}(x) = (B_1 \cos(x)^2 + C_1)^{-\frac{1}{2}}.$$

Then,

$$\phi(x) = d_5 h^{[1]}(d_4 x),$$
$$\psi(x) = \frac{3\alpha A_1}{\sqrt{(B_1 + C_1)C_1}} \arctan\left(\frac{C_1 \tan(d_4\xi)}{\sqrt{(B_1 + C_1)C_1}}\right) + \frac{1}{2}c\xi.$$

Define

$$g^{[1]}(x) = \frac{3\alpha A_1}{\sqrt{(B_1 + C_1)C_1}} \arctan\left(\frac{C_1 \tan(x)}{\sqrt{(B_1 + C_1)C_1}}\right),$$

and

$$F^{[1]}(x) = e^{\mathrm{i}g^{[1]}(x)} h^{[1]}(x).$$

Using (19), we define

$$\varphi^{[1]}_{c,\omega}(x) = Q_{c,\omega}(x, 0) = d_5 e^{\mathrm{i}cx/2} F^{[1]}(d_4 x).$$

We define the Fourier transform on the interval $T_\gamma \coloneqq [-\pi\gamma, \pi\gamma]$ as

$$\mathcal{F}_1(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\gamma}^{\pi\gamma} f(x) e^{-\mathrm{i}x\xi} \mathrm{d}x.$$

Let $\mathrm{Z}_\gamma = [-\gamma, \gamma]$. Then, $H^s(T_\gamma)$ is complete in the space of C^∞ functions with period $T\gamma$ and norm

$$\|f\|_{H^s(T_\gamma)} \coloneqq \|\langle\xi\rangle^s \mathcal{F}_1(f)(\xi)\|_{L^2(\mathrm{Z}_\gamma)}.$$

In the following, we study the ill-posedness of solution (21). Since solution (21) is periodic in $\mathbb{R}$, its norm could be infinite in $H^s(\mathbb{R})$. Therefore, we analyze its ill-posedness over a single period. We first estimate its inner-product norm over an integer period (with a sufficiently large period $\lambda\mathbb{T}$) and then estimate the norm over a single period $\mathbb{T}$.

Proposition 4. *If $\alpha \neq 0$, then the Cauchy problem for equation (2) in $H^s(\mathbb{T})$ can be ill-posed, i.e., the mapping $Q_0 \to Q(t)$ is not uniformly continuous. With the initial condition*

$$Q_0 = \varphi_{c,\omega}^{[1]}(x) = d_5 e^{\mathrm{i}cx/2} F^{[1]}(d_4 x),$$

the solution of equation (2) is ill-posed in $H^s(\mathbb{T})$. Specifically, we have the following:

(I) *If $0 < s < \frac{1}{2}$, then the solution is weakly ill-posed.*

(II) *If $\frac{1}{2} < s < \frac{3}{2}$, then the solution is strongly ill-posed.*

Proof. Using the scaling and time-shifting properties of the Fourier transform, we obtain:

$$\mathcal{F}(\varphi_{c,\omega}^{[1]})(\xi) = \frac{d_5}{d_4}\mathcal{F}(F^{[1]})\left(\frac{\xi}{d_4} - \frac{c}{2d_4}\right).$$

Next, we compute the norm under different initial conditions:

$$\begin{aligned}
\|\varphi_{c_1,\omega_1}^{[1]} - \varphi_{c_2,\omega_2}^{[1]}\|^2_{H^s\left(T\frac{\gamma}{d_{41}}\right)} &= \int\limits_{Z\frac{\gamma}{d_{41}}} \langle\xi\rangle^s |\mathcal{F}(\varphi_{c_1,\omega_1}^{[1]})(\xi) - \mathcal{F}(\varphi_{c_2,\omega_2}^{[1]})(\xi)|^2 \mathrm{d}\xi \\
&= \int\limits_{Z\frac{\gamma}{d_{41}}} \langle\xi\rangle^s \left| \frac{d_{51}}{d_{41}}\mathcal{F}(F^{[1]})\left(\frac{\xi}{d_{41}} - \frac{c_1}{2d_{41}}\right) - \frac{d_{52}}{d_{42}}\mathcal{F}(F^{[1]})\left(\frac{\xi}{d_{42}} - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\xi \\
&= d_{41} \int\limits_{Z\frac{\gamma}{d_{41}}} \langle d_{41}\eta\rangle^s \left| \frac{d_{51}}{d_{41}}\mathcal{F}(F^{[1]})\left(\eta - \frac{c_1}{2d_{41}}\right) - \frac{d_{52}}{d_{42}}\mathcal{F}(F^{[1]})\left(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\eta \\
&\simeq P_1^{[1]} + P_2^{[1]} + P_3^{[1]},
\end{aligned} \tag{25}$$

where

$$P_1^{[1]} = (d_{41})^{2s+1} \int\limits_{Z\frac{\gamma}{d_{41}}} \langle\eta\rangle^s \frac{d_{51}^2}{d_{41}^2} \left|\mathcal{F}(F^{[1]})\left(\eta - \frac{c_1}{2d_{41}}\right) - \mathcal{F}(F^{[1]})\left(\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}\right)\right|^2 \mathrm{d}\eta,$$

$$P_2^{[1]} = (d_{41})^{2s+1} \int\limits_{Z\frac{\gamma}{d_{41}}} \langle\eta\rangle^s \frac{d_{51}^2}{d_{41}^2} \left|\mathcal{F}(F^{[1]})\left(\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}\right) - \mathcal{F}(F^{[1]})\left(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\eta,$$

$$P_3^{[1]} = (d_{41})^{2s+1} \int\limits_{Z\frac{\gamma}{d_{41}}} \langle\eta\rangle^s \left|\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2}\right| \left|\mathcal{F}(F^{[1]})\left(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\eta.$$

If N_j $(j = 1, 2)$ and N are large integers, then we have the following approximation:

$$c_j = -N_j \simeq -N, \quad \omega_j = N_j^{\lambda s} - \frac{N_j^2}{4}.$$

Assuming $N_1 < N_2$ without loss of generality, we obtain

$$d_{4j} = \frac{1}{2}N_j^{\frac{1}{2}\lambda s}, \quad d_{5j} = \sqrt{2}N_j^{\frac{1}{2}\lambda s}, \quad |d_{41} - d_{42}| \simeq |N_1 - N_2|N^{\frac{1}{2}\lambda s - 1}.$$

With the estimation

$$\left(\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2}\right)^2 = 0,$$

we conclude that

$$P_3^{[1]} = 0.$$

Let $\eta \simeq \frac{c_j}{2d_{4j}}$. If γ is a positive integer, let $\gamma \geq N$ and $\gamma \simeq N$. Considering the Fourier transform on the unit sphere $\eta \in B_1(N^{1-\frac{1}{2}\lambda s})$, we apply the mean-value theorem and the Cauchy-Schwarz inequality:

$$\begin{aligned}
P_1^{[1]} &\simeq N^{\frac{1}{2}\lambda s(2s+1)} \int\limits_{Z_{\frac{\gamma}{d_{41}}}} \langle\eta\rangle^s \left| \mathcal{F}(F^{[1]})\left(\eta - \frac{c_1}{2d_{41}}\right) - \mathcal{F}(F^{[1]})\left(\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}\right) \right|^2 \mathrm{d}\eta \\
&\simeq N^{\frac{1}{2}\lambda s(2s+1)} N^{2s\left(1-\frac{1}{2}\lambda s\right)} \int\limits_{Z_{\frac{\gamma}{d_{41}}}} \left| \int\limits_{\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} (\mathcal{F}(F^{[1]}))'(\zeta)\mathrm{d}\zeta \right|^2 \mathrm{d}\eta \\
&\leq N^{\frac{1}{2}\lambda s+2s} \int\limits_{Z_{\frac{\gamma}{d_{41}}}} \left| \int\limits_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} d\beta \right| \left| \int\limits_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \mathrm{d}\zeta \right| \mathrm{d}\eta \\
&\simeq N^{\frac{1}{2}\lambda s+2s} \left| 1 - \frac{d_{41}}{d_{42}} \right| \int\limits_{Z_{\frac{\gamma}{d_{41}}}} |\eta| \left| \int\limits_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \mathrm{d}\zeta \right| \mathrm{d}\eta \\
&\simeq N^{\frac{1}{2}\lambda s+2s-1} |N_1 - N_2| \int\limits_{Z_{\frac{\gamma}{d_{41}}}} |\eta| \left| \int\limits_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \mathrm{d}\zeta \right| \mathrm{d}\eta.
\end{aligned} \tag{26}$$

By the Fubini theorem,

$$\begin{aligned}
&\int\limits_{Z_{\frac{\gamma}{d_{41}}}} |\eta| \left| \int\limits_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \mathrm{d}\zeta \right| \mathrm{d}\eta \\
&= \int\limits_0^{\frac{\gamma}{d_{41}}} \eta \int\limits_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \mathrm{d}\zeta \mathrm{d}\eta - \int\limits_{-\frac{\gamma}{d_{41}}}^{0} \eta \int\limits_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \mathrm{d}\zeta \mathrm{d}\eta \\
&= \int\limits_{\frac{-c_1}{2d_{41}}}^{\frac{\gamma}{d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \int\limits_{\zeta + \frac{c_1}{2d_{41}}}^{(\zeta + \frac{c_1}{2d_{41}})\frac{d_{42}}{d_{41}}} \eta \mathrm{d}\eta \mathrm{d}\zeta - \int\limits_{-\frac{\gamma}{d_{41}}}^{\frac{-c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \int\limits_{(\zeta + \frac{c_1}{2d_{41}})\frac{d_{42}}{d_{41}}}^{\zeta + \frac{c_1}{2d_{41}}} \eta \mathrm{d}\eta \mathrm{d}\zeta \\
&= \frac{1}{2} \int\limits_{-\frac{c_1}{2d_{41}}}^{\frac{\gamma}{d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \left(\zeta + \frac{c_1}{2d_{41}}\right)^2 \left[\left(\frac{d_{42}}{d_{41}}\right)^2 - 1\right] \mathrm{d}\zeta \\
&\quad - \frac{1}{2} \int\limits_{-\frac{\gamma}{d_{41}}}^{-\frac{c1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \left(\zeta + \frac{c_1}{2d_{41}}\right)^2 \left[1 - \left(\frac{d_{42}}{d_{41}}\right)^2\right] \mathrm{d}\zeta \\
&= \frac{1}{2} \int\limits_{Z_{\frac{\gamma}{d_{41}}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \left(\zeta + \frac{c_1}{2d_{41}}\right)^2 \left[\left(\frac{d_{42}}{d_{41}}\right)^2 - 1\right] \mathrm{d}\zeta.
\end{aligned} \tag{27}$$

Note that

$$\frac{{d_{41}}^2 - {d_{42}}^2}{{d_{41}}^2} = \frac{N_1^{\lambda s} - N_2^{\lambda s}}{N_1^{\lambda s}} \simeq \frac{(N_1 - N_2)N^{\lambda s-1}}{N^{\lambda s}} = \frac{N_1 - N_2}{N}.$$

Then, by (26) and (27), we have

$$\begin{aligned} {P_1}^{[1]} &\le N^{\frac{1}{2}\lambda s+2s-1}|N_1 - N_2| \int_{Z_{\frac{y}{d_{41}}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \left(\zeta + \frac{c_1}{2d_{41}}\right)^2 \left[1 - \left(\frac{d_{42}}{d_{41}}\right)^2\right] \mathrm{d}\zeta \\ &\simeq N^{\frac{1}{2}\lambda s+2s-2} \, |N_1 - N_2|^2 \int_{Z_{\frac{y}{d_{41}}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \left(\zeta + \frac{c_1}{2d_{41}}\right)^2 \mathrm{d}\zeta \\ &\simeq N^{\frac{1}{2}\lambda s+2s-2} \, |N_1 - N_2|^2 N^{2\left(1-\frac{1}{2}\lambda s\right)} \, \|(\mathcal{F}(F^{[1]}))'(\zeta)\|^2_{L^2\left(Z_{\frac{y}{d_{41}}}\right)} \\ &= N^{-\frac{1}{2}\lambda s+2s} \, |N_1 - N_2|^2 \|(\mathcal{F}(F^{[1]}))'(\zeta)\|^2_{L^2\left(Z_{\frac{y}{d_{41}}}\right)}. \end{aligned} \tag{28}$$

Similar to the computation of P_1,

$$\begin{aligned} {P_2}^{[1]} &\simeq (d_{41})^{2s+1}\left(\frac{d_{42}}{d_{41}}\right)^{2s+1} \int_{Z_{\frac{y}{d_{41}}}} |\eta|^{2s} \left|\mathcal{F}(F^{[1]})\left(\eta - \frac{c_1}{2d_{41}}\right) - \mathcal{F}(F^{[1]})\left(\eta - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\eta \\ &\simeq N^{\frac{1}{2}\lambda s(2s+1)} N^{2s\left(1-\frac{1}{2}\lambda s\right)} \int_{Z_{\frac{y}{d_{41}}}} \left| \int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} (\mathcal{F}(F^{[1]}))'(\zeta)\mathrm{d}\zeta \right|^2 \mathrm{d}\eta \\ &\leqslant N^{\frac{1}{2}\lambda s+2s} \int_{Z_{\frac{y}{d_{41}}}} \left| \int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} \mathrm{d}\zeta \right| \left| \int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \mathrm{d}\zeta \right| \mathrm{d}\eta \\ &\simeq N^{\frac{1}{2}\lambda s+2s} \left| \frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}} \right| \int_{Z_{\frac{y}{d_{41}}}} \int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} |(\mathcal{F}(F^{[1]}))'(\zeta)|^2 \mathrm{d}\zeta \mathrm{d}\eta \\ &\simeq N^{\frac{1}{2}\lambda s+2s} \left| \frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}} \right|^2 \|(\mathcal{F}(F^{[1]}))'\|^2_{L^2\left(Z_{\frac{y}{d_{41}}}\right)} \\ &\simeq N^{-\frac{1}{2}\lambda s+2s} \, |N_2 - N_1|^2 \|(\mathcal{F}(F^{[1]}))'\|^2_{L^2\left(Z_{\frac{y}{d_{41}}}\right)}. \end{aligned} \tag{29}$$

Note that

$$\|(\mathcal{F}(F^{[1]}))'\|^2_{L^2(\mathbb{R})} = \|xh^{[1]}(x)\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty} \frac{2x^2}{B_1\cos(2x) + B_1 + 2C_1}\mathrm{d}x = +\infty.$$

This implies that the estimates of the supremum in (28) and (29) for $P_1^{[1]}$ diverge to infinity, because $|F|$ is a function with period π.

Moreover, using the Plancherel theorem, we can estimate F as follows:

$$\begin{aligned} \|\mathcal{F}(F^{[1]})\|^2_{L^2(Z_{\frac{y}{d_{41}}})} &= \|h^{[1]}(x)\|^2_{L^2(T_{\frac{y}{d_{41}}})} \\ &= \int_{-\frac{y\pi}{d_{41}}}^{\frac{y\pi}{d_{41}}} \frac{2\mathrm{d}x}{B_1\cos(2x) + B_1 + 2C_1} \end{aligned} \tag{30}$$

$$
\begin{aligned}
&= \int_{0}^{\frac{\gamma\pi}{d_{41}}} \frac{2\mathrm{d}x}{B_1\cos(2x) + B_1 + 2C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{2\mathrm{d}x}{B_1\cos(2x) + B_1 + 2C_1} \\
&\le \int_{0}^{\frac{\gamma\pi}{d_{41}}} \frac{2\mathrm{d}x}{B_1 + 2C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{\mathrm{d}x}{C_1} \\
&= \frac{2\gamma\pi}{(B_1 + 2C_1)d_{41}} + \frac{\gamma\pi}{C_1 d_{41}} \\
&\simeq \frac{\gamma}{d_{41}} N^{-1}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|(\mathcal{F}(F^{[1]}))'\|^2_{L^2\left(\mathbb{Z}_{\frac{\gamma\pi}{d_{41}}}\right)} &= \|xh^{[1]}(x)\|^2_{L^2\left(T_{\frac{\gamma\pi}{d_{41}}}\right)} \\
&= \int_{-\frac{\gamma\pi}{d_{41}}}^{\frac{\gamma\pi}{d_{41}}} \frac{2x^2\mathrm{d}x}{B_1\cos(2x) + B_1 + 2C_1} \\
&= \int_{0}^{\frac{\gamma\pi}{d_{41}}} \frac{2x^2\mathrm{d}x}{B_1\cos(2x) + B_1 + 2C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{2x^2\mathrm{d}x}{B_1\cos(2x) + B_1 + 2C_1} \\
&\le \int_{0}^{\frac{\gamma\pi}{d_{41}}} \frac{2x^2\mathrm{d}x}{B_1 + 2C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{x^2\mathrm{d}x}{C_1} \\
&= \frac{2\gamma^3\pi^3}{3(B_1 + 2C_1){d_{41}}^3} + \frac{\gamma^3\pi^3}{3C_1{d_{41}}^3} \\
&\simeq \frac{\gamma^3}{{d_{41}}^3} N^{-1}.
\end{aligned}
\tag{31}
$$

Similarly, we can estimate the lower bound of $(\mathcal{F}(F))'$,

$$
\begin{aligned}
\|(\mathcal{F}(F^{[1]}))'\|^2_{L^2\left(\mathbb{Z}_{\frac{\gamma}{d_{41}}}\right)} &= \int_{0}^{\frac{\gamma\pi}{d_{41}}} \frac{2x^2\mathrm{d}x}{B_1\cos(2x) + B_1 + 2C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{2x^2\mathrm{d}x}{B_1\cos(2x) + B_1 + 2C_1} \\
&\ge \int_{0}^{\frac{\gamma\pi}{d_{41}}} \frac{x^2\mathrm{d}x}{B_1 + C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{x^2\mathrm{d}x}{B_1 + 2C_1} \\
&= \frac{\gamma^3\pi^3}{3(B_1 + C_1){d_{41}}^3} + \frac{\gamma^3\pi^3}{3(B_1 + 2C_1){d_{41}}^3} \\
&\simeq \frac{\gamma^3}{{d_{41}}^3} N^{-1}.
\end{aligned}
\tag{32}
$$

Equations (31) and (32) indicate that $(\mathcal{F}(F))'$ is in a scale of $\frac{\gamma^3}{{d_{41}}^3}N^{-1}$. Combining (25), (28), (29), (31), and (32), we have the estimate in $H^s\left(T_{\frac{\gamma}{d_{41}}}\right)$

$$\|\varphi^{[1]}_{c_1,\omega_1}-\varphi^{[1]}_{c_2,\omega_2}\|^2_{H^s\left(T_{\frac{\gamma}{d_{41}}}\right)}\le N^{-\frac{1}{2}\lambda s+2s-1}\,|N_2-N_1|^2\frac{\gamma^3}{{d_{41}}^3}.$$

We integrate the function in one period and choose the period as follows:

$$T_{\frac{1}{d_{41}}}=\left[-\frac{\pi}{d_{41}},\ \frac{\pi}{d_{41}}\right],\quad Z_{\frac{1}{d_{41}}}=\left[-\frac{1}{d_{41}},\frac{1}{d_{41}}\right].$$

Then, we have the estimate in $H^s\left(T_{\frac{1}{d_{41}}}\right)$

$$\|\varphi^{[1]}_{c_1,\omega_1}-\varphi^{[1]}_{c_2,\omega_2}\|^2_{H^s\left(T_{\frac{1}{d_{41}}}\right)}\le N^{-2\lambda s+2s+1}\,|N_2-N_1|^2.$$

If $-2\lambda s+2s+1<0$, let $b=|-2\lambda s+2s+1|$. Then, we can control the distance between solitons

$$N_2-N_1=\delta N^{\varepsilon s},\tag{33}$$

to control the norm

$$\|\varphi^{[1]}_{c_1,\omega_1}-\varphi^{[1]}_{c_2,\omega_2}\|^2_{H^s\left(T_{\frac{1}{d_{41}}}\right)}\le \tilde{C}\delta^2N^{2\varepsilon s-b}\le\tilde{C}\delta^2,$$

where $\tilde{C}>0$ is a constant, and ε and δ are any real values larger than 0.

Similarly, we can estimate the lower bound of $\mathcal{F}(F^{[1]})$

$$\begin{aligned}\|\mathcal{F}(F^{[1]})\|^2_{L^2\left(Z_{\frac{\gamma\pi}{d_{41}}}\right)}&=\int_0^{\frac{\gamma\pi}{d_{41}}}\frac{2\mathrm{d}x}{B_1\cos(2x)+B_1+2C_1}+\int_{-\frac{\gamma\pi}{d_{41}}}^{0}\frac{2\mathrm{d}x}{B_1\cos(2x)+B_1+2C_1}\\&\ge\int_0^{\frac{\gamma\pi}{d_{41}}}\frac{\mathrm{d}x}{B_1+C_1}+\int_{-\frac{\gamma\pi}{d_{41}}}^{0}\frac{\mathrm{d}x}{B_1+2C_1}\\&\simeq\frac{\gamma}{d_{41}}N^{-1}.\end{aligned}\tag{34}$$

Combining (30) and (34), we then compute the norm of $\varphi^{[1]}_{c,\omega}$ in $H^s\left(T_{\frac{\gamma}{d_{41}}}\right)$

$$\begin{aligned}\|\varphi^{[1]}_{c,\omega}\|^2_{H^s\left(T_{\frac{\gamma}{d_{41}}}\right)}&\simeq d_4^{2s+1}\frac{d_5^2}{d_4^2}\int_{Z_{\frac{\gamma}{d_{41}}}}|\eta|^{2s}|\mathcal{F}(F^{[1]})\left(\eta-\frac{c}{2d_4}\right)|^2\mathrm{d}\eta\\&\simeq N^{\left(\frac{1}{2}\lambda s\right)(2s+1)}N^{2s\left(1-\frac{1}{2}\lambda s\right)}\|h^{[1]}(x)\|^2_{L^2\left(T_{\frac{\gamma}{d_{41}}}\right)}\\&\simeq N^{\frac{1}{2}s\lambda+2s-1}\frac{\gamma}{d_{41}}.\end{aligned}\tag{35}$$

We then have the estimate of the norm of $\varphi^{[1]}_{c,\omega}$ in $H^s\left(T_{\frac{1}{d_{41}}}\right)$

$$\|\varphi^{[1]}_{c,\omega}\|^2_{H^s\left(T_{\frac{1}{d_{41}}}\right)}\simeq N^{2s-1}.\tag{36}$$

Assume that the solution is $Q_{c_j,\omega_j}(x,T)$ at $t=T$. By the translational invariant of the traveling wave solution and (36), we have

$$\|Q_{c_j,\omega_j}(x,T)\|^2_{H^s\left(T_{\frac{1}{d_{41}}}\right)} = \|\varphi^{[1]}_{c_j,\omega_j}\|^2_{H^s\left(T_{\frac{1}{d_{41}}}\right)} \simeq N^{2s-1}.$$

On the other hand,

$$Q_{c_j,\omega_j}(x,T) = e^{-i\omega_j T}e^{i\psi(x-c_jT)}d_{5j}h^{[1]}(d_{4j}(x-c_j)T).$$

Restricting $Q_{c_j,\omega_j}(x,T)$ on the sphere $B_{(d_{4j})^{-1}}(Tc_j)$, we can choose c_j and ω_j to determine the phase. Then, combining (30) and (34), we have

$$\begin{aligned}\|Q_{c_1,\omega_1}(x,T)-Q_{c_2,\omega_2}(x,T)\|^2_{L^2\left(T_{\frac{\gamma}{d_{41}}}\right)} &\simeq \|Q_{c_1,\omega_1}(x,T)\|^2_{L^2\left(T_{\frac{\gamma}{d_{41}}}\right)} + \|Q_{c_2,\omega_2}(x,T)\|^2_{L^2\left(T_{\frac{\gamma}{d_{41}}}\right)}\\ &\simeq d_{5j}{}^2\int_{-\frac{\gamma\pi}{d_{41}}}^{\frac{\gamma\pi}{d_{41}}} h^{[1]2}(d_{4j}(x-c_jT))\mathrm{d}x\\ &\simeq \frac{1}{d_{4j}}d_{5j}{}^2\|h^{[1]}(x)\|^2_{L^2\left(T_{\frac{\gamma}{d_{41}}}\right)}\\ &\simeq N^{\frac{1}{2}\lambda s-1}\frac{\gamma}{d_{41}},\end{aligned}$$

So

$$\begin{aligned}\|Q_{c_1,\omega_1}(x,T)-Q_{c_2,\omega_2}(x,T)\|^2_{H^s\left(T_{\frac{\gamma}{d_{41}}}\right)} &= \int_{-\frac{\gamma}{d_{41}}}^{\frac{\gamma}{d_{41}}}(1+|\mu|^2)^s|\hat{Q}_{c_1,\omega_1}(\mu)-\hat{Q}_{c_2,\omega_2}(\mu)|^2\mathrm{d}\mu\\ &\geq N^{2s}\|Q_{c_1,\omega_1}(x,T)-Q_{c_2,\omega_2}(x,T)\|^2_{L^2\left(T_{\frac{\gamma}{d_{41}}}\right)}\\ &\simeq N^{2s+\frac{1}{2}\lambda s-1}\frac{\gamma}{d_{41}}.\end{aligned}$$

Furthermore,

$$\|Q_{c_1,\omega_1}(x,T)-Q_{c_2,\omega_2}(x,T)\|^2_{H^s\left(T_{\frac{1}{d_{41}}}\right)} \geq N^{2s-1}.$$

In the following, we study the separability of the wave packet. We select c_1 and c_2 such that the wave exhibits separability on the scale of $N^{-\frac{1}{2}\lambda s}$. Specifically, we need to choose N such that $N^{\frac{1}{2}\lambda s+\varepsilon s} \gg (T\delta)^{-1}$. Given the phase difference $N_1 - N_2 = \delta N^{\varepsilon s}$ from (32), we obtain:

$$T(c_2-c_1) = T(N_2-N_1) \gg \max\left(\frac{1}{d_{41}}, \frac{1}{d_{42}}\right) \simeq N^{-\frac{1}{2}\lambda s}.$$

As a summary,

$$\begin{aligned}&\|Q_{c_1,\omega_1}(x,0)-Q_{c_2,\omega_2}(x,0)\|^2_{H^s\left(T_{\frac{1}{d_{41}}}\right)} \leq \delta,\\ &Q_{c_j,\omega_j}(x,T)\in H^s\left(T_{\frac{1}{d_{41}}}\right), \quad \|Q_{c_1,\omega_1}(x,T)-Q_{c_2,\omega_2}(x,T)\|^2_{H^s\left(T_{\frac{1}{d_{41}}}\right)} \geq \varepsilon.\end{aligned}$$

Then, this force $\lambda > 0$ and $s > 0$ to be

$$\left\{(\lambda, s)|1 - \frac{1}{2}s\lambda > 0,\ \ -2\lambda s + 2s + 1 < 0, 2s - 1 < 0\right\}.$$

We then have the range of the weak ill-posedness index s:

$$0 < s < \frac{1}{2}.$$

Similarly, if we solve the system

$$\left\{(\lambda, s)|1 - \frac{1}{2}s\lambda > 0,\ \ -2\lambda s + 2s + 1 < 0, 2s - 1 > 0\right\},$$

we obtain the strong ill-posedness index s as follows:

$$\frac{1}{2} < s < \frac{3}{2}. \qquad \square$$

4 Ill-posedness of the non-periodic solution

In this section, we study the ill-posedness of (23) and (24). Since equation (23) is not periodic in $\mathbb{R}$ and has finite energy in $H^s(\mathbb{R})$, we can apply the Fourier transform over the entire real line. We define the Fourier transform of a Lebesgue integrable function $f : \mathbb{R} \to \mathbb{C}$ as

$$\mathcal{F}_2(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-\mathrm{i}x\xi}\mathrm{d}x,$$

where ξ is any real number.

Let $\langle x\rangle := (1 + |x|^2)^{\frac{1}{2}}$. Then, the Sobolev space $H^s(\mathbb{R})$ is a complete space of C^∞ functions with the norm

$$\|f\|_{H^s(\mathbb{R})} := \|\langle\xi\rangle^s\mathcal{F}_2(f)(\xi)\|_{L^2(\mathbb{R})}.$$

Note that

$$\mathrm{sech}(x) = 2(\mathrm{e}^x + \mathrm{e}^{-x})^{-1}.$$

Then, the solution in (23) can be transformed to

$$\phi(\xi) = \sqrt{2}\left(\frac{\mathrm{e}^{-\sqrt{-c^2-4\omega}\,x}\sqrt{-\omega\alpha^2 + \alpha c + 1} + \mathrm{e}^{\sqrt{-c^2-4\omega}\,x}\sqrt{-\omega\alpha^2 + \alpha c + 1} + \alpha c + 2}{-c^2 - 4\omega}\right)^{-\frac{1}{2}}.$$

Let

$$A_2 = \sqrt{-c^2 - 4\omega}, \quad B_2 = \sqrt{-\omega\alpha^2 + \alpha c + 1}, \quad C_2 = \alpha c + 2.$$

Then, (23) is transformed to

$$\phi(\xi) = \sqrt{2}\left(\frac{\mathrm{e}^{A_2\xi}B_2}{A_2^2} + \frac{\mathrm{e}^{-A_2\xi}B_2}{A_2^2} + \frac{C_2}{A_2^2}\right)^{-\frac{1}{2}}.$$

At the same time, (20) is in the form of

$$\psi(\xi) = -3\arctan\left(\frac{2\mathrm{e}^{A_2\xi}B_2 + C_2}{\alpha A_2}\right) + \frac{1}{2}c\xi.$$

Let

$$d_4 = A_2,$$
$$d_5 = \sqrt{2}\sqrt{\frac{B_2}{A_2^2}},$$

and

$$h^{[2]}(x) = \left(\mathrm{e}^{A_2 x} + \mathrm{e}^{-A_2 x} + \frac{C_2}{B_2}\right)^{-\frac{1}{2}}.$$

Then,

$$\phi(x) = d_5 h^{[2]}(d_4 x).$$

Let

$$g^{[2]}(x) = -3\arctan\left(\frac{2\mathrm{e}^{A_2 x}B_2 + C_2}{\alpha A_2}\right),$$

and

$$F^{[2]}(x) = e^{\mathrm{i}g^{[2]}(x)}h^{[2]}(x).$$

By (19), we define

$$\varphi_{c,\omega}^{[2]}(x) = Q_{c,\omega}(x,0) = d_5 e^{\mathrm{i}cx/2}F^{[2]}(d_4 x),$$

The transformation formula in $\mathbb{R}$ is

$$\mathcal{F}_2(\varphi_{c,\omega}^{[2]})(\xi) = \frac{d_5}{d_4}\mathcal{F}_2(F^{[2]})\left(\frac{\xi}{d_4} - \frac{c}{2d_4}\right).$$

Proposition 5. *If $\alpha \neq 0$, then the Cauchy problem for (2) can be ill-posed in $H^s(T_\lambda)$. Specifically, the mapping $Q_0 \to Q(t)$ is not uniformly continuous. Given the initial condition*

$$Q_0 = \varphi_{c,\omega}^{[2]}(x) = d_5 e^{\mathrm{i}cx/2}F^{[2]}(d_4 x),$$

the solution of (2) can also be ill-posed in $H^s(\mathbb{R})$. More precisely, we have the following:

(*I*) *If $0 < s < \frac{1}{2}$, then the solution is weakly ill-posed.*

(*II*) *If $0 < s < 1$, then the solution is strongly ill-posed.*

Proof. Similar to (25), we have

$$\begin{aligned}\|\varphi_{c_1,\omega_1}^{[2]} - \varphi_{c_2,\omega_2}^{[2]}\|_{H^s(\mathbb{R})}^2 &= \int_{\mathbb{R}} \langle\xi\rangle^s |\mathcal{F}_2(\varphi_{c_1,\omega_1}^{[2]})(\xi) - \mathcal{F}_2(\varphi_{c_2,\omega_2}^{[2]})(\xi)|^2 \mathrm{d}\xi \\ &\simeq P_1^{[2]} + P_2^{[2]} + P_3^{[2]},\end{aligned} \tag{37}$$

where

$$P_1^{[2]} = (d_{41})^{2s+1}\int_{\mathbb{R}} \langle\eta\rangle^s \frac{d_{51}^2}{d_{41}^2}\left|\mathcal{F}_2(F^{[2]})\left(\eta - \frac{c_1}{2d_{41}}\right) - \mathcal{F}_2(F^{[2]})\left(\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}\right)\right|^2 \mathrm{d}\eta,$$

$$P_2^{[2]} = (d_{41})^{2s+1}\int_{\mathbb{R}} \langle\eta\rangle^s \frac{d_{51}^2}{d_{41}^2}\left|\mathcal{F}_2(F^{[2]})\left(\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}\right) - \mathcal{F}_2(F^{[2]})\left(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\eta,$$

$$P_3^{[2]} = (d_{41})^{2s+1}\int_{\mathbb{R}} \langle\eta\rangle^s \left|\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2}\right| \mathcal{F}_2(F^{[2]})\left(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}}\right)\Bigg|^2 \mathrm{d}\eta.$$

Assuming $\eta \simeq \frac{c_j}{2d_{4j}}$, we perform the Fourier transformation on the unit sphere $\eta \in B_1(N^{1-\frac{1}{2}\lambda s})$. We can estimate $P_1^{[2]}$, as with (26), to obtain

$$\begin{aligned} P_1^{[2]} &\simeq (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2}N^{2s\left(1-\frac{1}{2}\lambda s\right)}\int_{\mathbb{R}}\left|\int_{\frac{d_{41}}{d_{42}}\eta-\frac{c_1}{2d_{41}}}^{\eta-\frac{c_1}{2d_{41}}}\mathcal{F}_2'(F^{[2]})(\beta)\mathrm{d}\beta\right|^2\mathrm{d}\eta \\ &\leq (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2}N^{2s\left(1-\frac{1}{2}\lambda s\right)}\int_{\mathbb{R}}\left|\int_{\eta\frac{d_{41}}{d_{42}}-\frac{c1}{2d_{41}}}^{\eta-\frac{c1}{2d_{41}}}\mathrm{d}\alpha\right|\left|\int_{\eta\frac{d_{41}}{d_{42}}-\frac{c1}{2d_{41}}}^{\eta-\frac{c1}{2d_{41}}}|\mathcal{F}_2'(F^{[2]})(\beta)|^2\mathrm{d}\beta\right|\mathrm{d}\eta \\ &\simeq (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2}N^{2s\left(1-\frac{1}{2}\lambda s\right)}\left|1-\frac{d_{41}}{d_{42}}\right|\int_{\mathbb{R}}|\eta|\left|\int_{\eta\frac{d_{41}}{d_{42}}-\frac{c1}{2d_{41}}}^{\eta-\frac{c1}{2d_{41}}}|\mathcal{F}_2'(F^{[2]})(\beta)|^2\mathrm{d}\beta\right|\mathrm{d}\eta. \end{aligned} \tag{38}$$

Denote

$$I_1^{[2]} = \int_0^\infty \eta \int_{\eta\frac{d_{41}}{d_{42}}-\frac{c1}{2d_{41}}}^{\eta-\frac{c1}{2d_{41}}} |\hat{L}'(\beta)|^2\mathrm{d}\beta\mathrm{d}\eta.$$

By Fubini's theorem, we can change the order of integration

$$\begin{aligned} I_1^{[2]} &= \int_{\frac{-c1}{2d_{41}}}^{\infty}|\mathcal{F}_2'(F^{[2]})(\beta)|^2\int_{\beta+\frac{c1}{2d_{41}}}^{\left(\beta+\frac{c1}{2d_{41}}\right)\frac{d_{42}}{d_{41}}}\eta\mathrm{d}\eta\mathrm{d}\beta \\ &= \frac{1}{2}\int_{-\frac{c1}{2d_{41}}}^{\infty}|\mathcal{F}_2'(F^{[2]})(\beta)|^2\left(\beta+\frac{c_1}{2d_{41}}\right)^2\left[\left(\frac{d_{42}}{d_{41}}\right)^2-1\right]\mathrm{d}\beta. \end{aligned} \tag{39}$$

Moreover, we set

$$I_2^{[2]} = \int_{-\infty}^0 \eta \int_{\eta\frac{d_{41}}{d_{42}}-\frac{c1}{2d_{41}}}^{\eta-\frac{c1}{2d_{41}}} |\mathcal{F}_2'(F^{[2]})(\beta)|^2\mathrm{d}\beta\mathrm{d}\eta.$$

Similarly,

$$I_2^{[2]} = \frac{1}{2}\int_{-\infty}^{-\frac{c1}{2d_{41}}}|\mathcal{F}_2'(F^{[2]})(\beta)|^2\left(\beta+\frac{c_1}{2d_{41}}\right)^2\left[1-\left(\frac{d_{42}}{d_{41}}\right)^2\right]\mathrm{d}\beta. \tag{40}$$

Let N_j $(j = 1, 2)$ and N be large positive integers with the following relation:

$$c_j = N_j \simeq N, \quad \omega_j = -N_j^{\lambda s} - \frac{N_j^2}{4}$$

$$d_{4j} = 2N_j^{\frac{1}{2}\lambda s}, \quad d_{5j} = \left(\frac{\sqrt{\alpha^2(N_j^{\lambda s}+N_j^2/4)+\alpha N_j+1}}{2N_j^{\lambda s}}\right)^{-\frac{1}{2}} \simeq N_j^{-\frac{1}{2}+\frac{1}{2}\lambda s}, \quad \alpha \neq 0.$$

Assume that $N_1 < N_2$. Then, we have

$$|d_{41}-d_{42}| \simeq |N_1-N_2|N^{\frac{1}{2}\lambda s-1}.$$

Similarly,

$$\frac{{d_{41}}^2 - {d_{42}}^2}{{d_{41}}^2} = \frac{4N_1^{\lambda s} - 4N_2^{\lambda s}}{4N_1^{\lambda s}} \simeq \frac{(N_1 - N_2)N^{\lambda s-1}}{N^{\lambda s}} = \frac{N_1 - N_2}{N}$$

and

$$\frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}} = \frac{N_1}{2N_1^{\frac{1}{2}\lambda s}} - \frac{N_2}{2N_2^{\frac{1}{2}\lambda s}} \simeq \frac{N_1 - N_2}{N}.$$

Combining (38)–(40), we obtain

$$\begin{aligned} P_1^{[2]} &\leq N^{\frac{1}{2}\lambda s-2+2s}|N_1 - N_2|(I_1^{[2]} - I_2^{[2]}) \\ &\simeq N^{\frac{1}{2}\lambda s-2+2s}|N_1 - N_2|\left|1 - \left(\frac{d_{42}}{d_{41}}\right)^2\right| \int_{\mathbb{R}} |\mathcal{F}_2'(F^{[2]})(\beta)|^2\left(\beta + \frac{c_1}{2d_{41}}\right)^2 \mathrm{d}\beta \\ &\simeq N^{\frac{1}{2}\lambda s-2+2s}|N_1 - N_2|\left|1 - \left(\frac{d_{42}}{d_{41}}\right)^2\right| N^{2\left(1-\frac{1}{2}\lambda s\right)} \|\mathcal{F}_2'(F^{[2]})\|_{L^2}^2 \\ &= N^{-\frac{1}{2}\lambda s-1+2s}(N_1 - N_2)^2\|\mathcal{F}_2'(F^{[2]})\|_{L^2}^2, \end{aligned} \tag{41}$$

and, similarly,

$$\begin{aligned} P_2^{[2]} &\simeq (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2}\left(\frac{d_{42}}{d_{41}}\right)^{2s+1} \int_{\mathbb{R}} |\eta|^{2s} \left|\mathcal{F}_2(F^{[2]})\left(\eta - \frac{c_1}{2d_{41}}\right) - \mathcal{F}_2(F^{[2]})\left(\eta - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\eta \\ &\simeq \frac{d_{51}^2 {d_{42}}^{2s+1}}{d_{41}^2} N^{2s\left(1-\frac{1}{2}\lambda s\right)} \int_{\mathbb{R}} \left| \int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} \mathcal{F}_2'(F^{[2]})(\alpha)\mathrm{d}\alpha \right|^2 \mathrm{d}\eta \\ &\leq \frac{d_{51}^2 {d_{42}}^{2s+1}}{d_{41}^2} \left|\frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}}\right| N^{2s\left(1-\frac{1}{2}\lambda s\right)} \int_{\mathbb{R}} \int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} |\mathcal{F}_2'(F^{[2]})(\alpha)|^2 \mathrm{d}\alpha \mathrm{d}\eta \\ &\simeq \frac{d_{51}^2 {d_{42}}^{2s+1}}{d_{41}^2} \left|\frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}}\right|^2 N^{2s\left(1-\frac{1}{2}\lambda s\right)} \|\mathcal{F}_2'(F^{[2]})\|_{L^2}^2 \\ &\simeq N^{\frac{1}{2}\lambda s-3+2s} |N_2 - N_1|^2 \|\mathcal{F}_2'(F^{[2]})\|_{L^2}^2. \end{aligned} \tag{42}$$

We compute

$$\frac{{d_{51}}^2}{{d_{41}}^2} - \frac{{d_{52}}^2}{{d_{42}}^2} \simeq \left(\frac{N_1^{-\frac{1}{2}+\frac{1}{2}\lambda s}}{N_1^{\frac{1}{2}\lambda s}}\right)^2 - \left(\frac{N_2^{-\frac{1}{2}+\frac{1}{2}\lambda s}}{N_2^{\frac{1}{2}\lambda s}}\right)^2 \simeq (N_1 - N_2)N^{-2},$$

and so,

$$\begin{aligned} P_3^{[2]} &\simeq (d_{41})^{2s+1}\left(\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2}\right)^2 \int_{\mathbb{R}} |\eta|^{2s} \left|\mathcal{F}_2(F^{[2]})\left(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\eta \\ &\simeq (d_{41})^{2s+1}\left(\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2}\right)^2\left(\frac{d_{42}}{d_{41}}\right)^{2s+1} \int_{\mathbb{R}} |\eta|^{2s} \left|\mathcal{F}_2(F^{[2]})\left(\eta - \frac{c_2}{2d_{42}}\right)\right|^2 \mathrm{d}\eta \\ &\simeq N^{\lambda s^2+\frac{1}{2}\lambda s-4} |N_1 - N_2|^2\left(\frac{c_2}{2d_{42}}\right)^{2s} \|\mathcal{F}_2(F^{[2]})\|^2 \\ &\simeq N^{\frac{1}{2}\lambda s-4+2s}(N_1 - N_2)^2\|F^{[2]}\|_{L^2}^2. \end{aligned} \tag{43}$$

Let $K = C_2/B_2$, where $\tilde{C}$ is a constant. Thus, we have

$$\begin{aligned}\|F^{[2]}\|_{L^2}^2 &= \|h^{[2]}(x)\|_{L^2}^2\\ &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\mathrm{e}^x + \mathrm{e}^{-x} + K}\\ &\le \int_0^{+\infty}\frac{\mathrm{d}x}{\mathrm{e}^x} + \int_{-\infty}^0\frac{\mathrm{d}x}{\mathrm{e}^{-x}}\\ &\le \tilde{C}.\end{aligned} \tag{44}$$

Similarly, we have

$$\begin{aligned}\|\mathcal{F}_2'(F^{[2]})\|^2 &= \|xh^{[2]}(x)\|^2\\ &= \int_{-\infty}^{+\infty}\frac{x^2\mathrm{d}x}{\mathrm{e}^x + \mathrm{e}^{-x} + K}\\ &\le \int_0^{+\infty}\frac{x^2\mathrm{d}x}{\mathrm{e}^x} + \int_{-\infty}^0\frac{x^2\mathrm{d}x}{\mathrm{e}^{-x}}\\ &\le \tilde{C}.\end{aligned} \tag{45}$$

If $1 - \frac{1}{2}\lambda s > 0$, substituting (45) with (41) and (42),

$$P_1^{[2]} \le \tilde{C}N^{-\frac{1}{2}\lambda s - 1 + 2s}(N_1 - N_2)^2,$$
$$P_2^{[2]} \le \tilde{C}N^{\frac{1}{2}\lambda s - 3 + 2s}(N_1 - N_2)^2.$$

Combining (43) and (44), the following holds:

$$P_3^{[2]} \le \tilde{C}N^{\frac{1}{2}\lambda s - 4 + 2s}(N_1 - N_2)^2.$$

If $-\frac{1}{2}\lambda s - 1 + 2s < 0$, $\frac{1}{2}\lambda s - 3 + 2s < 0$, and $\frac{1}{2}\lambda s - 4 + 2s < 0$, (37) satisfies

$$\|\varphi_{c_1,\omega_1}^{[2]} - \varphi_{c_2,\omega_2}^{[2]}\|_{H^s}^2 \le \frac{\tilde{C}(N_1 - N_2)^2}{N^b},$$

where

$$b = \mathrm{Min}\left\{\left|-\frac{1}{2}\lambda s - 1 + 2s\right|, \left|\frac{1}{2}\lambda s - 3 + 2s\right|, \left|\frac{1}{2}\lambda s - 4 + 2s\right|\right\}.$$

Then, we can control the distance between solitons

$$N_2 - N_1 = \delta N^{\varepsilon s}$$

to control the norm

$$\|\varphi_{c_1,\omega_1}^{[2]} - \varphi_{c_2,\omega_2}^{[2]}\|_{H^s}^2 \le \tilde{C}\delta^2 N^{2\varepsilon s - b} \le \tilde{C}\delta^2. \tag{46}$$

According to (44), there is an upper bound of $\|F^{[2]}\|_{L^2}^2$. Moreover, the lower bound of it can be estimated as follows:

$$\begin{aligned}\|F^{[2]}\|_{L^2}^2 &= \|h^{[2]}(x)\|_{L^2}^2\\ &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\mathrm{e}^x + \mathrm{e}^{-x} + K}\end{aligned}$$

$$= \begin{cases} \dfrac{2}{\sqrt{K^2-4}}\operatorname{arctanh}\left(\dfrac{K}{\sqrt{K^2-4}}\right), & K > 2, \\ 1, & K = 2, \\ \dfrac{1}{\sqrt{-K^2+4}}\left(-2\arctan\left(\dfrac{K}{\sqrt{-K^2+4}}\right)+\pi\right), & 0 \le K < 2, \end{cases}$$
$$\ge \tilde{C}(\alpha) > 0,$$

where $\tilde{C}(\alpha)$ is a constant that depends on α.

So, we have

$$\begin{aligned} \|\varphi_{c,\omega}\|_{H^s}^2 &\simeq d_4^{2s+1}\frac{d_5^2}{d_4^2}\int_{-\infty}^{+\infty}|\eta|^{2s}\left|\mathcal{F}_2(F^{[2]})\left(\eta-\frac{c}{2d_4}\right)\right|^2\mathrm{d}\eta \\ &\simeq N^{\frac{1}{2}\lambda s(2s+1)}N^{-1}N^{2s\left(1-\frac{1}{2}\lambda s\right)}\|F^{[2]}\|_{L^2}^2 \\ &\simeq N^{\frac{1}{2}\lambda s+2s-1}. \end{aligned}$$

In the following deduction, we consider the time-dependent solution (i.e., the solution $Q_{c_j,\omega_j}^{[2]}(x,T)$ at $t=T$). As we know, the solitary wave satisfies the translational invariance property. Thus, we have

$$\|Q_{c_j,\omega_j}^{[2]}(\cdot,T)\|_{H^s}^2 = \|\varphi_{c_j,\omega_j}\|_{H^s}^2 \simeq N^{\frac{1}{2}\lambda s+2s-1}.$$

By

$$\|Q_{c_1,\omega_1}^{[2]}(\cdot,T)-Q_{c_2,\omega_2}^{[2]}(\cdot,T)\|_{H^s}^2 = \int_{\mathbb{R}}(1+|\mu|^2)^s|\hat{Q}_{c_1,\omega_1}^{[2]}(\mu)-\hat{Q}_{c_2,\omega_2}^{[2]}(\mu)|^2\mathrm{d}\mu,$$

we have

$$\|Q_{c_1,\omega_1}^{[2]}(\cdot,T)-Q_{c_2,\omega_2}^{[2]}(\cdot,T)\|_{H^s}^2 \ge N^{2s}\|Q_{c_1,\omega_1}^{[2]}(\cdot,T)-Q_{c_2,\omega_2}^{[2]}(\cdot,T)\|_{L^2}^2. \tag{47}$$

In addition, we noted that

$$Q_{c_j,\omega_j}^{[2]}(x,T) = e^{-i\omega_j T}e^{i\psi(x-c_jT)}d_{5j}h^{[2]}(d_{4j}(x-c_j)T),$$

so we restrict $Q_{c_j,\omega_j}^{[2]}(T)$ on the sphere $B_{(d_{6j})^{-1}}(Tc_j)$. At the same time, different values of c_j and ω_j can be used to avoid the superposition of peaks, such that

$$\begin{aligned} \|Q_{c_1,\omega_1}^{[2]}(\cdot,T)-Q_{c_2,\omega_2}^{[2]}(\cdot,T)\|_{L^2}^2 &\simeq \|Q_{c_1,\omega_1}^{[2]}(\cdot,T)\|_{L^2}^2+\|Q_{c_2,\omega_2}^{[2]}(\cdot,T)\|_{L^2}^2 \\ &\simeq d_{5j}{}^2\int h^{[2]2}(d_{4j}(x-c_j)T)\mathrm{d}x \\ &\simeq \frac{1}{d_{4j}}d_{5j}{}^2\|F^{[2]}\|_{L^2}^2 \\ &\simeq N^{\frac{1}{2}\lambda s-1}. \end{aligned} \tag{48}$$

By (47) and (48), we have

$$\|Q_{c_1,\omega_1}^{[2]}(\cdot,T)-Q_{c_2,\omega_2}^{[2]}(\cdot,T)\|_{H^s}^2 \ge N^{\frac{1}{2}\lambda s+2s-1}.$$

We now consider the possibility of dispersion for the soliton solution. If $Q_{c_j,\omega_j}^{[2]}(T)$ is within $B_{(d_{4j})^{-1}}(Tc_j)$ for $j=1,2$, we can select c_1 and c_2 to ensure dispersion at the scale of $N^{-\frac{1}{2}\lambda s}$. Specifically, we choose N such that $N^{\frac{1}{2}\lambda s+\varepsilon s} \gg (T\delta)^{-1}$. Simultaneously, we maintain $N_1 - N_2 = \delta N^{\varepsilon s}$, as used in (45). This yields

$$T(c_2-c_1) = T(N_2-N_1) \gg \max\left(\frac{1}{d_{41}},\frac{1}{d_{42}}\right) \simeq N^{-\frac{1}{2}\lambda s}.$$

Based on the aforementioned analysis, to achieve weak ill-posedness, λ and s ($s > 0$) must satisfy the following conditions:

$$\left\{(\lambda, s)|1 - \frac{1}{2}\lambda s > 0,\ -\frac{1}{2}\lambda s + 2s < 1,\ \frac{1}{2}\lambda s + 2s < 1,\ \frac{1}{2}\lambda s + 2s < 3,\ \frac{1}{2}\lambda s + 2s < 4\right\},$$

which indicates that the ill-posedness index is

$$0 < s < \frac{1}{2}.$$

Similarly, in order to obtain the strong ill-posedness of the solution, the set of λ and s is as follows:

$$\left\{(\lambda, s)|1 - \frac{1}{2}\lambda s > 0,\ -\frac{1}{2}\lambda s + 2s < 1, \frac{1}{2}\lambda s + 2s > 1, \frac{1}{2}\lambda s + 2s < 3, \frac{1}{2}\lambda s + 2s < 4\right\},$$

which indicates that the range of s is

$$0 < s < 1. \qquad \square$$

Proof of Theorem 2. With Propositions 4 and 5, Theorem 2 is now proved. □

Proof of Theorem 1. If $f(x)$ is square-integrable in T_γ, the Plancherel theorem states that

$$\int\limits_{T_\gamma} |f(x)|^2 \mathrm{d}x = \int\limits_{Z_\gamma} |\mathcal{F}(f)(\xi)|^2 \mathrm{d}\xi,$$

which can be used to estimate the relationship between Q and $\mathbf{S}$ as follows:

$$\begin{aligned}
||Q||^2_{H^s(T_\gamma)} &= \int\limits_{Z_\gamma} \langle\xi\rangle^s |\mathcal{F}_1(Q)(\xi)|^2 \mathrm{d}\xi \\
&\simeq \int\limits_{Z_\gamma} \langle\xi\rangle^s (|S_{1x}|^2 + |S_{2x}|^2 + |S_{3x}|^2) \mathrm{d}\xi \\
&\simeq \int\limits_{Z_\gamma} \langle\xi\rangle^s (|\mathcal{F}_1(|S_{1x}|)(\xi)|^2 + |\mathcal{F}_1(|S_{2x}|)(\xi)|^2 + |\mathcal{F}_1(|S_{3x}|)(\xi)|^2) \mathrm{d}\xi \\
&\simeq \int\limits_{Z_\gamma} \langle\xi\rangle^s |\xi|^2 (|\mathcal{F}_1(|S_1|)(\xi)|^2 + |\mathcal{F}_1(|S_2|)(\xi)|^2 + |\mathcal{F}_1(|S_3|)(\xi)|^2) \mathrm{d}\xi \\
&\simeq ||\mathbf{S}||^2_{H^{\frac{s}{2}+1}_{\mathbb{S}^2}(T_\gamma)}.
\end{aligned}$$

Hence, we have

$$||Q||^2_{H^s(T_\gamma)} \simeq ||\mathbf{S}||^2_{H^{\frac{s}{2}+1}_{\mathbb{S}^2}(T_\gamma)}. \tag{49}$$

$\mathbf{S}_j$ ($j = 1, 2$) falls on the sphere, and $\mathbf{S}_j \cdot \mathbf{S}_j = 1$. Furthermore, the components of the vector $\mathbf{S}_j$ are non-intersecting traveling wave solutions. Hence, it may be assumed that $S_{1,i_x} \simeq S_{2,i_x}$ ($i = 1, 2, 3$). Then, we have

$$\begin{aligned}
||Q_1 - Q_2||^2_{H^s(T_\gamma)} &= \int\limits_{Z_\gamma} \langle\xi\rangle^s |\mathcal{F}_1(Q_1)(\xi) - \mathcal{F}_1(Q_2)(\xi)|^2 \mathrm{d}\xi \\
&\simeq \int\limits_{\mathbb{R}} \langle\xi\rangle^s (|\mathcal{F}_1(Q_1)(\xi)|^2 + |\mathcal{F}_1(Q_2)(\xi)|^2 - 2|\mathcal{F}_1(Q_1)(\xi)||\mathcal{F}_1(Q_2)(\xi)|) \mathrm{d}\xi \\
&\simeq \int\limits_{\mathbb{R}} \langle\xi\rangle^s (|Q_1|^2 + |Q_2|^2 - 2|Q_1||Q_2|) \mathrm{d}\xi \\
&\simeq \int\limits_{\mathbb{R}} \langle\xi\rangle^s (||S_{1,1_x} - S_{2,1_x}|^\wedge|^2 + ||S_{1,2_x} - S_{2,2_x}|^\wedge|^2 + ||S_{3,2_x} - S_{3,2_x}|^\wedge|^2) \mathrm{d}\xi
\end{aligned}$$

$$\simeq \int_{\mathbb{R}} \langle\xi\rangle^s |\xi|^2 (|\mathcal{F}_1(|S_{1,1} - S_{2,1}|)(\xi)|^2 + |\mathcal{F}_1(|S_{1,2} - S_{2,2}|)(\xi)|^2 + |\mathcal{F}_1(|S_{1,3} - S_{2,3}|)(\xi)|^2) \mathrm{d}\xi$$
$$\simeq \|\mathbf{S}_1 - \mathbf{S}_2\|^2_{H^{s+1}_{\mathbb{S}^2}(T_\gamma)},$$

which indicates

$$\|\mathbf{S}_1 - \mathbf{S}_2\|^2_{H^{s+1}_{\mathbb{S}^2}(T_\gamma)} \simeq \|Q_1 - Q_2\|^2_{H^s(T_\gamma)}. \tag{50}$$

Similar to (49) and (50), the non-periodic case admits the following isometric isomorphism relationship:

$$\|Q\|^2_{H^s(\mathbb{R})} \simeq \|\mathbf{S}\|^2_{H^{s+1}_{\mathbb{S}^2}(\mathbb{R})}, \quad \|\mathbf{S}_1 - \mathbf{S}_2\|^2_{H^{s+1}_{\mathbb{S}^2}(\mathbb{R})} \simeq \|Q_1 - Q_2\|^2_{H^s(\mathbb{R})}. \tag{51}$$

With the equivalence relationship (49)–(51), and Theorem 2, we complete the proof of Theorem 1. □

5 Conclusions

In this article, we studied two distinct types of two-parameter solitary wave solutions for the DCHS equation (1). Using the derivative Schrödinger equation, we constructed these solutions and analyzed their ill-posedness in both the periodic space $H^s_{\mathbb{S}^2}(\mathbb{T})$ and the non-periodic space $H^s_{\mathbb{S}^2}(\mathbb{R})$. Although different spaces were used to evaluate the two solutions, the range of the weak ill-posedness index was identical: $1 < s < \frac{3}{2}$. Notably, $s = \frac{1}{2}$ emerges as a critical index that determines the ill-posedness behavior. However, the strong ill-posedness indices differed between the two cases. By extending the analysis used in the weak cases, we found that the periodic and non-periodic solutions cannot be well-posed in a bounded subset of $H^s_{\mathbb{S}^2}$ with indices $\frac{3}{2} < s < \frac{5}{2}$ and $1 < s < 2$, respectively.

In both weak and strong cases, it is important to note that our discussion focused exclusively on the setting where $\alpha \neq 0$. Whether the ill-posedness index range remains the same as α approaches zero ($\lim_{\alpha\to 0}$) remains an open question, and further investigation is needed in our future work. Additionally, since the solutions considered in this article assume $c_1 = c_2 = 0$, another interesting avenue for future research is to explore the ill-posedness properties when $c_1, c_2 \neq 0$.

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