Research Article

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Cycle integrals and rational period functions for $\Gamma_0^+(2)$ and $\Gamma_0^+(3)$

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Abstract: For $p \in \{2, 3\}$ and an even integer k, let $W_{k-2}^-(p)$ be the space of period polynomials of weight $k-2$ on $\Gamma_0^+(p)$ with eigenvalue -1 under the Fricke involution. We determine the dimension formula for $W_{k-2}^-(p)$ and construct an explicit basis for it using period functions for weakly holomorphic modular forms. Furthermore, for a quadratic form Q, we define the function $F^-(z, Q)$ on the complex upper half-plane as a generating function of the cycle integrals of the canonical basis elements for the space of weakly holomorphic modular forms of weight k and eigenvalue -1 under the Fricke involution on $\Gamma_0(p)$. We also show that $F^-(z, Q)$ is a modular integral on $\Gamma_0^+(p)$. Our approach focuses on calculating cycle integrals within $\Gamma_0(p)$ rather than $\Gamma_0^+(p)$, which allows us to overcome certain technical challenges. This study extends earlier work by Choi and Kim (*Rational period functions and cycle integrals in higher level cases*, J. Math. Anal. Appl. **427** (2015), no. 2, 741–758) which focused on eigenvalue +1, providing new insights by examining eigenvalue −1 cases in the theory of rational period functions and cycle integrals in this setting.

Keywords: rational period functions, period polynomials, weakly holomorphic modular forms, cycle integrals

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1 Introduction and statement of results

For a positive integer p, let $\Gamma_0(p)$ be the Hecke congruence subgroup consisting of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries satisfying $c \equiv 0 \pmod p$. Further, let $\Gamma_0^+(p)$ be the group generated by $\Gamma_0(p)$ and the Fricke involution $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$. Let k be an even integer and ε_C be the character on $\Gamma_0^+(p)$ defined as

$$\varepsilon_C(\gamma) = 1 \quad \text{for } \gamma \in \Gamma_0(p) \quad \text{and} \quad \varepsilon_C(W_p) = C, \quad \text{where } C \in \{\pm 1\}.$$

For a meromorphic function f on the complex upper half plane $\mathbb{H}$, we define the action $|_{k,\varepsilon_C}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^+(p)$ as

$$(f|_{k,\varepsilon_C}\gamma)(z) = \varepsilon_C(\gamma)(cz + d)^{-k} f(\gamma z).$$

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Let $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $U = TW_p = \begin{pmatrix} \sqrt{p} & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$. For $p \in \{1, 2, 3\}$, we consider a rational function $q(z)$ satisfying

$$q\,|_{k,\varepsilon_C} W_p + q = 0 \quad \text{and} \quad \sum_{n=0}^{n_p-1} q\,|_{k,\varepsilon_C} U^n = 0, \tag{1}$$

where $n_p = \begin{cases} 3 & \text{if } p = 1, \\ 2p & \text{if } p = 2, 3. \end{cases}$ Such a function $q(z)$ is called a *rational period function of weight* k *for* $\Gamma_0^+(p)$ and denote by $\mathrm{RPF}_{k,\varepsilon}(p)$ the set of all such functions for $C = \varepsilon 1$ (see [1–4]). Here, we note that, when $p = 1$, W_1 belongs to $\Gamma_0^+(1) = \mathrm{SL}_2(\mathbb{Z})$, and hence, we only need to consider the case $C = +1$.

For negative k, if the rational period functions are polynomials, we call them *period polynomials of weight* k *for* $\Gamma_0^+(p)$ and denote by $W_{-k}^{\varepsilon}(p)$ the set of all such period polynomials for $C = \varepsilon 1$. That is,

$$W_{-k}^{\varepsilon}(p) = \left\{ P \in \mathbb{C}[z] : \deg P \le -k,\ P|_{k,\varepsilon_C} W_p + P = 0 = \sum_{n=0}^{n_p-1} P|_{k,\varepsilon_C} U^n \right\}.$$

We define a *modular integral on* $\Gamma_0^+(p)$ *of weight* k to be a holomorphic function f on $\mathbb{H}$ that has a Fourier expansion at ∞:

$$f(z) = \sum_{n=n_0}^{\infty} a_n e^{2\pi i n z} \quad \text{with some } n_0 \in \mathbb{Z}$$

and satisfies

$$f\,|_{k,\varepsilon_C} W_p = f + Cq \tag{2}$$

for a rational function $q(z)$, where $C \in \{+1, -1\}$. Then, $q(z) \in \mathrm{RPF}_{k,\varepsilon}(p)$ (see [1–5]).

The study of period polynomials and rational period functions has been a significant area of research in the theory of modular forms. Knopp [1,2] initiated the study of rational period functions and their relation with modular integrals on $\mathrm{SL}_2(\mathbb{Z})$. Over the years, the theory of period polynomials and rational period functions has been significantly developed by many authors, and one of the significant advancements in the study of rational period functions was made by Duke et al. [6]. They provided an effective basis for the space of period polynomials $W_{-k}^{+}(1)$ using weakly holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$ and by using cycle integrals, explicitly constructed modular integrals for rational period functions for $\mathrm{SL}_2(\mathbb{Z})$ arising from indefinite binary quadratic forms. Indeed, Knopp [1,2] already showed that rational period functions for $\mathrm{SL}_2(\mathbb{Z})$ have modular integrals. But his construction is very difficult to compute. Choi and Kim [3,5] extended the results of Duke et al. to the space $W_{-k}^{+}(p)$ and modular integrals on $\Gamma_0^+(p)$ for $p \in \{2, 3\}$ and $C = 1$. This article aims to further extend these results by studying the space $W_{-k}^{-}(p)$ and rational period functions for $\Gamma_0^+(p)$ for $p \in \{2, 3\}$ and $C = -1$.

For any even integer k and a prime p, let $M_k(p)$ (resp. $S_k(p)$) be the space of holomorphic modular forms (resp. cusp forms) of weight k on $\Gamma_0(p)$. For $\varepsilon \in \{+, -\}$, we denote by $M_k^{\varepsilon}(p)$ the subspace of $M_k(p)$ consisting of all eigenforms f of the Fricke involution $|_k W_p$ such that the eigenvalue of f is $\varepsilon 1$, that is,

$$M_k^{+}(p) := \{f \in M_k(p) : f\,|_k W_p = f\}, \quad M_k^{-}(p) := \{f \in M_k(p) : f\,|_k W_p = -f\}.$$

Similarly, we can also define two subspaces $S_k^{+}(p)$ and $S_k^{-}(p)$ of $S_k(p)$ as

$$S_k^{+}(p) := \{f \in S_k(p) : f\,|_{k,} W_p = f\}, \quad S_k^{-}(p) := \{f \in S_k(p) : f\,|_k W_p = -f\}.$$

Here $|_k$ is the usual slash operator which is given by $(f\,|_k \gamma)(z) = (\det\gamma)^{k/2}(cz + d)^{-k} f(\gamma z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. Further, let $M_k^{!}(p)$ be the space of weakly holomorphic modular forms (i.e., meromorphic with poles only at the cusps) of weight k for $\Gamma_0(p)$ and let $M_k^{!,\varepsilon}(p)$ be the subspace of $M_k^{!}(p)$ consisting of all eigenforms f of the action $|_k W_p$ such that the eigenvalue of f is $\varepsilon 1$. Then, it can be easily seen that

$$M_k^{!}(p) = M_k^{!,+}(p) \oplus M_k^{!,-}(p).$$

It is well-known that each $f \in M_k^{!,\varepsilon}(p)$ has a Fourier expansion at the cusp ∞ of the form

$$f(z) = \sum_{n \geq n_0} a_f(n) q^n \quad (q = e^{2\pi i z}).$$

We set $\operatorname{ord}_\infty f = n_0$ if n_0 is the smallest integer such that $a_f(n_0) \neq 0$. We additionally define the space $S_k^{!,\varepsilon}(p)$ by the subspace of $M_k^{!,\varepsilon}(p)$ consisting of weakly holomorphic modular forms with zero constant term in the Fourier expansion at the cusp ∞. It is known [7,8] that when the genus of $\Gamma_0^+(p)$ is zero, the space $M_k^{!,\varepsilon}(p)$ has a canonical basis: Let m_k^ε denote the maximal order of a nonzero $f \in M_k^{!,\varepsilon}(p)$ at ∞. For every integer $m \geq -m_k^\varepsilon$, there exists a unique weakly holomorphic modular form $f_{k,m}^\varepsilon \in M_k^{!,\varepsilon}(p)$ with Fourier expansion of the form at the cusp ∞:

$$f_{k,m}^\varepsilon = q^{-m} + \sum_{n > m_k^\varepsilon} a_k(m,n) q^n$$

and together they form a basis for $M_k^{!,\varepsilon}(p)$.

As previously mentioned, this article focuses on studying the rational period functions for $\Gamma_0^+(p)$ and modular integrals on $\Gamma_0^+(p)$ in the specific cases where $p \in \{2,3\}$ and $C = -1$. In this case, the functional equation (1) can be written as

$$q\,|_k W_p - q = 0 \quad \text{and} \quad \sum_{n=0}^{2p-1} (-1)^n q\,|_k U^n = 0,$$

while the functional equation (2) for modular integrals becomes

$$f + f\,|_k W_p = q$$

using the usual slash operator.

The following is our first main result that gives the dimension of the space $W_{k-2}^-(p)$ for $p \in \{2,3\}$ and its relation to the minus space $S_k^-(p)$ of cusp forms.

Theorem 1.1. *Let $p \in \{2,3\}$ and let k be a positive even integer greater than 2. Then, we have*

$$\dim W_{k-2}^-(p) = 2 \dim S_k^-(p) + 1.$$

In addition to determining the dimension of the space $W_{k-2}^-(p)$ of period polynomials, our second result introduces a method to construct an explicit basis for this space. For the basis construction, we utilize the Eichler integral of the canonical basis, where the Eichler integral is defined as follows: Suppose that $k > 2$. For $f \in \sum a_f(n) q^n \in M_k^{!,\varepsilon}(p)$, we define the *Eichler integral* of f by

$$\mathcal{E}_f(z) \coloneqq \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_f(n) n^{1-k} q^n.$$

In addition, the period function for f is defined by

$$\begin{aligned} r^+(f;\, z) &\coloneqq c_k(\mathcal{E}_f - \mathcal{E}_f\,|_{2-k} W_p)(z), \\ r^-(f;\, z) &\coloneqq c_k(\mathcal{E}_f + \mathcal{E}_f\,|_{2-k} W_p)(z), \end{aligned}$$

where $c_k = -\frac{\Gamma(k-1)}{(2\pi i)^{k-1}}$. We note that for $f \in S_k^{!,\varepsilon}(p)$, the function $r^\varepsilon(f;\, z)$ is a polynomial in z of degree at most $k-2$ with coefficients in $\mathbb{C}$. Indeed, this fact follows from Bol's identity. Specifically, for $f \in M_{2-k}^{!,\varepsilon}(p)$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$, we have

$$D^{k-1}(f\,|_{2-k}\gamma) = (D^{k-1} f)|_k \gamma.$$

This identity implies that $D^{k-1}(M_{2-k}^{!,\varepsilon}(p)) \subseteq S_k^{!,\varepsilon}(p)$.

Theorem 1.2. *Let $p \in \{2,3\}$, $k > 2$ be an even integer, and $t = \dim S_k^-(p)$. Then, the set*

$$\{(\sqrt{p}\,z)^{k-2} + 1\} \cup \{r^-(f_{k,m}^-;\, z) | 0 < |m| \leq t\}$$

forms a basis for the space $W_{k-2}^-(p)$.

To state our third result, consider a positive integer D congruent to a square modulo $4p$. In this context, $\mathcal{Q}_{D,p}$ denotes the set of integral binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2 = [a,b,c]$, where p divides a, and the discriminant $D = b^2 - 4ac$. The group $\Gamma_0(p)$ acts on $\mathcal{Q}_{D,p}$ through the operation $Q \mapsto gQ$, where $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(p)$ and $Q \in \mathcal{Q}_{D,p}$. This action is defined by

$$(gQ)(x,y) = (Q \circ g^{-1})(x,y) = Q(\delta x - \beta y, -\gamma x + \alpha y).$$

The set of classes $\Gamma_0(p)\backslash\mathcal{Q}_{d,p}$ is finite. Furthermore, if $g = \pm\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(p)$, then $g^{-1}Q(\tau,1) = Q(g\tau,1)(\gamma\tau+\delta)^2$. From now on, assume that D is positive and not a perfect square. Note that $\Gamma_0(p)_Q \subseteq \Gamma_0^+(p)_Q$, where $\Gamma_0(p)_Q = \{g \in \Gamma_0(p) | gQ = Q\}$ and $\Gamma_0^+(p)_Q = \{g \in \Gamma_0^+(p) | gQ = Q\}$.

For $Q = [a,b,c] \in \mathcal{Q}_{D,p}$, let S_Q be the oriented semi-circle defined by $a|\tau|^2 + (\mathrm{Re}(\tau))b + c = 0$, with counterclockwise orientation for $a > 0$ and clockwise for $a < 0$. Note that $S_{gQ} = gS_Q$ for any $g \in \Gamma_0(p)$. Let h_Q be a generator of $\Gamma_0(p)_Q$. (For a detailed definition of h_Q, see Remark 5.2.) For a weakly holomorphic modular form $f \in M_k^!(p)$ and $Q \in \mathcal{Q}_{d,p}$, we define

$$r_Q(f) := \int_{C_Q} f(\tau)\mathrm{d}\tau_Q,$$

where $C_Q = C_Q(\tau_0)$ is the directed arc on S_Q from $\tau_0 \in S_Q$ to $h_Q\tau_0$, with the same orientation as S_Q, and $\mathrm{d}\tau_Q := Q(\tau,z)^{\frac{k}{2}-1}\mathrm{d}\tau$.

While the following result is parallel to [3, Theorem 1.5(i)(ii)], it has an application (see Corollary 1.5) which is specific to the case under consideration and has no direct analog in the setting adopted in [3].

Theorem 1.3. *Let $p \in \{2,3\}$. For any even integer k and $Q \in \mathcal{Q}_{D,p}$, we define the function $F^-(z,Q)$ by*

$$F^-(z,Q) := \sum_{m \ge -m_k^-} r_Q(f_{k,m}^-)e^{2\pi imz}.$$

Then, the following are true

- *The function $F^-(z,Q)$ is holomorphic on $\mathbb{H}$ and satisfies*

$$F^-(z,Q) + F^-(z,Q)|_{2-k}W_p = \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}} \mathrm{sgn}(c)(az^2+bz+c)^{\frac{k}{2}-1} - \sum_{\substack{[a,b,c]\in(W_pQ)\\ ac<0}} \mathrm{sgn}(c)(az^2+bz+c)^{\frac{k}{2}-1}, \tag{3}$$

  where $(Q) = \{gQ | g \in \Gamma_0(p)\}$ denotes the class containing Q.
- *For $k > 2$, let $\Psi_Q(z)$ be the polynomial given by*

$$\Psi_Q(z) := \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}} \mathrm{sgn}(c)(az^2+bz+c)^{\frac{k}{2}-1} - \sum_{\substack{[a,b,c]\in(W_pQ)\\ ac<0}} \mathrm{sgn}(c)(az^2+bz+c)^{\frac{k}{2}-1}.$$

  Then, $\Psi_Q(z)$ belongs to the space $W_{k-2}^-(p)$ and can be expressed as

$$\Psi_Q(z) = r_Q(f_{k,0}^-)(1 + (\sqrt{p}z)^{k-2}) + \sum_{0<|m|\le t} (-m)^{k-1} r_Q(f_{k,m}^-)\psi_{k,m}(z)$$

  with $\psi_{k,m}(z) = \frac{1}{c_k} r^-(f_{k,m}^-; z)$.

Remark 1.4. We note that $F^-(z,Q)$ is a modular integral on $\Gamma_0(p)$ since the right-hand side of (3) is a rational function. If the right-hand side of (3) is 0, one might think that $F^-(z,Q)$ is a weakly holomorphic modular form on $\Gamma_0(p)$. However, as described in the following corollary, we can confirm that under certain conditions, not only is the right-hand side of (3) equal to 0, but also all the Fourier coefficients of $F^-(z,Q)$, i.e., the cycle integrals $r_Q(f_{k,m}^-)$, become 0 as well.

The following corollary shows that for each $m \geq -m_k^-$, we have $r_Q(f_{k,m}^-) = 0$ under some condition.

Corollary 1.5. *Let $p \in \{2, 3\}$ and D be a positive integer that is not a perfect square but is congruent to a square modulo $4p$. For a quadratic form $Q = [a, b, c] \in \mathcal{Q}_{D,p}$ and an even integer k, let $F^-(z, Q)$ be the function defined in Theorem 1.3. Denote $v = \gcd(a, b, c)$, $a' = a/v$, $b' = b/v$, $c' = c/v$, and $d' = D/v^2$. If $p|D$ and the Diophantine equation $(pt)^2 - d'u^2 = 4p$ has an integer solution (t, u) with $a'u \in p\mathbb{Z}$, then we have*

$$r_Q(f_{k,m}^-) = 0$$

for each $m \geq -m_k^-$.

This work extends the understanding of period polynomials with eigenvalue −1 under the Fricke involution and modular integrals, building upon previous studies such as [6] and [3]. The proofs of Theorem 1.2 and 1.3 are based on the main ideas presented in [6], while our research specifically expands on the results of [3] to the case of eigenvalue −1 under the Fricke involution. A key distinction of this article from previous work, particularly that of Choi and Kim [3], is our approach to calculating cycle integrals within $\Gamma_0(p)$ rather than $\Gamma_0^+(p)$. This shift requires a more refined computational method, enabling us to determine the exact values of cycle integrals. Our method allows for a more precise and comprehensive understanding of the relationship between cycle integrals and rational period functions in this setting.

The rest of the article is organized as follows. In Section 2, we provide examples illustrating our main results, including numerical calculations. Section 3 is dedicated to proving the dimension formula for $W_{k-2}^-(p)$ stated in Theorem 1.1. In Section 4, we focus on constructing the basis for $W_{k-2}^-(p)$, as described in Theorem 1.2. Finally, the proofs of Theorem 1.3 and the proof of Corollary 1.5 are presented in Section 5.

2 Numerical examples

Let $p = 2$ and $k = 10$. In this case, the space $S_{10}^-(2)$ is one dimensional and spanned by

$$\Delta_2^-(z) := (\eta(z)\eta(2z))^8(2E_2(2z) - E_2(z)),$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$ denotes the Dedekind eta function and $E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n$ is the normalized Eisenstein series of weight 2. Using [8], one can construct the first three basis elements of the space $M_{10}^{!,-}(2)$ as follows:

$$\begin{aligned} f_{10,-1}^-(z) &= \Delta_2^-(z) \\ f_{10,0}^-(z) &= \Delta_2^-(z)(j_2^+(z) - 16) \\ f_{10,1}^-(z) &= \Delta_2^-(z)(j_2^+(z)^2 - 16j_2^+(z) - 8332) \end{aligned}$$

and their q-expansions are of the form

$$\begin{aligned} f_{10,-1}^-(z) &= q + 16q^2 - 156q^3 + 256q^4 + 870q^5 + \dots, \\ f_{10,0}^-(z) &= 1 + 3960q^2 + 168960q^3 + 2094840q^4 + 16625664q^5 + \dots, \\ f_{10,1}^-(z) &= q^{-1} + 131904q^2 + 21947754q^3 + 1145058304q^4 + 30480293440q^5 + \dots. \end{aligned}$$

Here, $j_2^+(z)$ denotes the Hauptmodul for $\Gamma_0^+(2)$ described by

$$\begin{aligned} j_2^+(z) &= \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} + 24 + 4096\left(\frac{\eta(2z)}{\eta(z)}\right)^{24} \\ &= q^{-1} + 4372q + 96256q^2 + 1240002q^3 + 10698752q^4 + 74428120q^5 + \dots. \end{aligned}$$

According to Theorem 1.1, the space $W_8^-(2)$ is three-dimensional, and by Theorem 1.2, it is spanned by the polynomials $1 + 16z^8$, $\psi_{10,-1}(z)$, and $\psi_{10,1}(z)$, where $\psi_{10,m}(z)$ is given by $\psi_{10,m}(z) = r^-(f_{10,m}^-; z)$.

Using the nth Fourier coefficients of $f_{10,m}^-$ for $n \le 500$, one can explicitly compute the coefficients of the polynomials $\psi_{10,m}$. The coefficients of each polynomial are given in Table 1. (The coefficients presented in this table are rounded off.)

For each $f \in M_{10}^{!,+}(2)$ and $Q \in \mathcal{Q}_{d,2}$, it follows from the definition of the cycle integral that

$$r_Q(f) = \int_{C_Q} f(\tau)Q(\tau,1)^4 d\tau, \tag{4}$$

where $C_Q = C_Q(\tau_0)$ is the directed arc on S_Q from $\tau_0 \in S_Q$ to $g_Q\tau_0$.

We consider two cases.

Case (i) $D = 8$ and $Q = [2, 0, -1] \in \mathcal{Q}_{8,2}$: In this case, we find that the smallest positive solution (t_0, u_0) of the Diophantine equation $(2t)^2 - 8u^2 = 8$ is given by $(t_0, u_0) = (2, 1)$. Thus, employing [3, Theorem 1.3 (i)], we see that the group $\Gamma_0^+(2)_Q/\{\pm 1\}$ is generated by $\frac{1}{\sqrt{2}}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \in \Gamma_0(2)W_2$, and hence, h_Q in Remark 5.2 is given by

$$h_Q = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}^2 \in \Gamma_0(2).$$

In (4), we take $\tau_0 = i$ and use the parametrization

$$\tau = \frac{1}{\sqrt{2}}e^{i\theta} \quad \left(\frac{\pi}{2} \le \theta \le \cos^{-1}\left(-\frac{12\sqrt{2}}{17}\right)\right).$$

Then, we can compute that the values of $r_Q(f_{10,m}^-)$ are equal to zero, as expected from Corollary 1.5.

Case (ii) $D = 17$ and $Q = [2, 1, -2] \in \mathcal{Q}_{17,2}$: In this case, we find that the smallest positive solution (t_1, u_1) of the Pell equation $t^2 - 17u^2 = 4$ is given by $(t_1, u_1) = (66, 16)$. Thus, employing [3, Theorem 1.3 (ii)], we see that the group $\Gamma_0^+(2)_Q/\{\pm 1\}$ is generated by

$$h_Q = \begin{pmatrix} 41 & -32 \\ -32 & 25 \end{pmatrix} \in \Gamma_0(2).$$

Table 1: Period polynomials $\psi_{10,m}$

m	$\psi_{10,m}$
–1	$a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + a_6z^6 + a_7z^7 + a_8z^8$
	$a_0 = 1.02451271598830368183811479$
	$a_1 = 6.5565078719932328445697618i$
	$a_2 = -21.1732627970916094246543723$
	$a_3 = -45.8955551039526299119883328i$
	$a_4 = 74.1064197898206329862903031$
	$a_5 = 91.7911102079052598239766656i$
	$a_6 = -84.6930511883664376986174892 3$
	$a_7 = -52.4520629759458627565580946 10i$
	$a_8 = 16.3922034558128589094098366260$
1	$b_0 + b_1z + b_2z^2 + b_3z^3 + b_4z^4 + b_5z^5 + b_6z^6 + b_7z^7 + b_8z^8$
	$b_0 = -62.2887001330506838361664 71$
	$b_1 = -320.2879802949301854074972 3i$
	$b_2 = 1111.2998027497141326141070 7$
	$b_3 = 2242.0158620645112978524806 2i$
	$b_4 = -3889.5493096239994641493747 6$
	$b_5 = -4484.0317241290225957049612 31i$
	$b_6 = 4445.1992109988565304564282 926$
	$b_7 = 2562.3038423594414832599778 4605i$
	$b_8 = -996.6192021288109413786635 4050659$

In (4), we take $\tau_0 = -\frac{1}{4} + \frac{\sqrt{17}}{4}i$ and use the parametrization

$$\tau = -\frac{1}{4} + \frac{\sqrt{17}}{4}e^{i\theta} \quad \left(\frac{\pi}{2} \le \theta \le \cos^{-1}\left(-\frac{528\sqrt{17}}{2{,}177}\right)\right).$$

Then, one can estimate the values of $r_Q(f^-_{10,m})$ which are listed in Table 2.

Here, each coefficient was rounded off to 30 digits. Meanwhile, one has

$$\{[a,b,c] \in (Q) | ac < 0\} = \{[2,1,-2], [-2,1,2], [2,-3,-1], [-2,-3,1], [4,1,-1], [-4,1,1]\},$$
$$\{[a,b,c] \in (W_2Q) | ac < 0\} = \{[-4,-1,1], [4,-1,-1], [2,-1,-2], [-2,-1,2], [2,3,-1], [-2,3,1]\}.$$

Thus, we find that

$$\sum_{\substack{[a,b,c]\in(Q)\\ ac<0}} \operatorname{sgn}(c)(az^2+bz+c)^4 - \sum_{\substack{[a,b,c]\in(W_2Q)\\ ac<0}} \operatorname{sgn}(c)(az^2+bz+c)^4 = 96z - 672z^3 + 1344z^5 - 768z^7.$$

Thus, one should have an equality

$$r_Q(f^-_{10,0})(1+16z^8) - r_Q(f^-_{10,-1})\psi_{10,1}(z) + r_Q(f^-_{10,1})\psi_{10,-1}(z) = 96z - 672z^3 + 1344z^5 - 768z^7,$$

which can be verified numerically from the values listed in Tables 1 and 2.

3 Proof of Theorem 1.1

We begin this section by noting that the matrix $\mu = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfies the following properties:

- $\mu W_p \mu = -W_p$,
- $\mu U^\zeta \mu = W_p U^{2p-\zeta} W_p$ for every integer $0 < \zeta < 2p$,
- For a polynomial $P(z)$ of degree at most $k-2$, the equality $(-1)^{k/2}P(-z) = -P(z)|_{2-k}\mu = 0$ holds if and only if $P(z) = 0$.

Then, for a period polynomial $P(z)$, we have the following proposition:

Proposition 3.1. *Let $P(z) \in W^-_{k-2}(p)$ be a period polynomial for $\Gamma^+_0(p)$. Then, the polynomial $P(-z)$ obtained by substituting $-z$ for z in $P(z)$ also belongs to the space $W^-_{k-2}(p)$.*

Proof. Using the properties of the matrix μ, we can show that $(-1)^{k/2}P(-z) = -(P|_{2-k}\mu)(z)$ satisfies the defining conditions of the space $W^-_{k-2}(p)$. First, we have

$$P|_{2-k}\mu - (P|_{2-k}\mu)|_{2-k}W_p = 0, \tag{5}$$

which follows from the fact that $P|_{2-k}\mu W_p \mu = P|_{2-k}W_p = P = P|_{2-k}\mu^2$. Furthermore, by using the property $\mu U^q \mu = W_p U^{2p-q} W_p$ and a straightforward calculation, we obtain

$$\sum_{n=0}^{2p-1} (-1)^n P|_{2-k}\mu U^n = 0. \tag{6}$$

Table 2: Values of $r_Q(f^-_{10,m})$

m	$r_Q(f^-_{10,m})$
–1	$4.02737428315417140776069345524i$
0	$-34.2976390565387500531878410 38i$
1	$-211.380753611733371105167237 6i$

Equations (5) and (6) together imply that $P\,|_{2-k}\mu \in W^{-}_{k-2}(p)$, which completes the proof. □

Let us consider two important subspaces of the space $W^{-}_{k-2}(p)$:

$$W^{-+}_{k-2} \coloneqq \{P \in W^{-}_{k-2}(p) : P(z) = P(-z)\},$$
$$W^{--}_{k-2} \coloneqq \{P \in W^{-}_{k-2}(p) : P(z) = -P(-z)\}.$$

In other words, W^{-+}_{k-2} is the subspace of even period polynomials in $W^{-}_{k-2}(p)$, while W^{--}_{k-2} is the subspace of odd period polynomials.

Proposition 3.2. *The space $W^{-}_{k-2}(p)$ can be decomposed as a direct sum of its even and odd subspaces*:

$$W^{-}_{k-2}(p) = W^{-+}_{k-2} \oplus W^{--}_{k-2}.$$

Proof. Given a period polynomial $P(z) \in W^{-}_{k-2}(p)$, we can express it as a sum of its even and odd parts:

$$P(z) = \frac{P(z) + P(-z)}{2} + \frac{P(z) - P(-z)}{2}.$$

It follows from Proposition 3.1 that $\frac{P(z)+P(-z)}{2} \in W^{-+}_{k-2}$ and $\frac{P(z)-P(-z)}{2} \in W^{--}_{k-2}$. Hence, we obtain the assertion. □

Let us now turn our attention to the following lemma that will be instrumental in proving our main result:

Lemma 3.3. [9, Proposition 3] *For $p \in \{2, 3\}$ and $\varepsilon \in \{+,-\}$, there exist the following isomorphisms*:

$$M^{\varepsilon}_{k}(p) \simeq W^{\varepsilon+}_{k-2} \quad \text{and} \quad S^{\varepsilon}_{k}(p) \simeq W^{\varepsilon-}_{k-2}.$$

With the help of Lemma 3.3, we can now provide a concise proof of Theorem 1.1:

Proof of Theorem 1.1. By Proposition 3.2, the space $W^{-}_{k-2}(p)$ decomposes as a direct sum of its even and odd subspaces:

$$\dim W^{-}_{k-2}(p) = \dim W^{-+}_{k-2} + \dim W^{--}_{k-2}.$$

Applying the isomorphisms from Lemma 3.3, we obtain

$$\dim W^{-}_{k-2}(p) = \dim M^{-}_{k}(p) + \dim S^{-}_{k}(p) = 2 \dim S^{-}_{k}(p) + 1,$$

which is the desired result. □

4 Proof of Theorem 1.2

Proposition 4.1. *For $f \in S^{!,-}_{k}(p)$, the following statements are equivalent*:
- $r^{-}(f;\, z) = \alpha((\sqrt{p}\,z)^{k-2} + 1)$ *for some constant $\alpha \in \mathbb{C}$.*
- $\mathcal{E}_f - \frac{\alpha}{c_k} \in M^{!,-}_{2-k}(p)$.
- $f \in D^{k-1}(M^{!,-}_{2-k}(p))$.

Proof. We will show (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (ii): Assume $r^{-}(f;\, z) = \alpha((\sqrt{p}\,z)^{k-2} + 1)$ for some $\alpha \in \mathbb{C}$. Then, $\mathcal{E}_f - \frac{\alpha}{c_k} = -(\mathcal{E}_f - \frac{\alpha}{c_k})|_{2-k}W_p$. From the definition of $\mathcal{E}_f$, we also have $\mathcal{E}_f - \frac{\alpha}{c_k} = (\mathcal{E}_f - \frac{\alpha}{c_k})|_{2-k}T$. Since $\Gamma^{+}_{0}(p)$ is generated by $\pm T$ and $\pm W_p$, any $\gamma \in \Gamma_0(p)$ can be expressed as a product of an even number of $\pm W_p$ and any number of $\pm T$. Therefore, $\mathcal{E}_f - \frac{\alpha}{c_k} \in M^{!,-}_{2-k}(p)$.

(ii) ⇒ (i): Suppose $\mathcal{E}_f - \frac{\alpha}{c_k} \in M^{!,-}_{2-k}(p)$ for some $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned}\frac{1}{c_k} r^-(f;\, z) &= (\mathcal{E}_f + \mathcal{E}_f\,|_{2-k}\, W_p)(z)\\ &= \frac{\alpha}{c_k} + \left(\mathcal{E}_f - \frac{\alpha}{c_k}\right) + \left(\frac{\alpha}{c_k} + \left(\mathcal{E}_f - \frac{\alpha}{c_k}\right)\right)\Bigg|_{2-k} W_p\\ &= \frac{\alpha}{c_k}((\sqrt{p}\,z)^{k-2} + 1).\end{aligned}$$

(i) ⇒ (iii): Assume $r^-(f;\, z) = \alpha((\sqrt{p}\,z)^{k-2} + 1)$ for some $\alpha \in \mathbb{C}$. By (i) ⇒ (ii), we have $\mathcal{E}_f - \frac{\alpha}{c_k} \in M^{!,-}_{2-k}(p)$. Then, $f = D^{k-1}\mathcal{E}_f = D^{k-1}(\mathcal{E}_f - \frac{\alpha}{c_k}) \in D^{k-1}(M^{!,-}_{2-k}(p))$.

(iii) ⇒ (i): Suppose $f = D^{k-1}g$ for some $g \in M^{!,-}_{2-k}(p)$. Note that $\mathcal{E}_f(z) = \sum_{n\gg-\infty, n\neq 0} a_f(n) n^{1-k} q^n = g(z) - a_g(0)$. Then

$$r^-(f;\, z) = c_k(g(z) - a_g(0) + (g(z) - a_g(0))|_{2-k} W_p) = -c_k a_g(0)((\sqrt{p}\,z)^{k-2} + 1). \qquad \square$$

Lemma 4.2. *Suppose k is a positive even integer. Let p be 1 or a prime for which $\Gamma_0^+(p)$ has genus zero, and let $t = \dim S_k^-(p)$. Consider the quotient space $V = S^{!,-}_k(p)/D^{k-1}(M^{!,-}_{2-k}(p))$. If $t \geq 1$, then V is a $2t$-dimensional vector space with a basis given by*

$$\{[f^-_{k,m}] : 0 < |m| \leq t\}.$$

Here, $[f]$ denotes the equivalence class of f in V.

Proof. Let m_k^- be the maximal vanishing order of weakly holomorphic modular forms in $M^{!,-}_k(p)$. It is known from [10] that

$$m^-_{2-k} = -m_k^- - 1 = -t - 1.$$

The space $S^{!,-}_k(p)$ is spanned by $\{f^-_{k,m} : m \neq 0,\ m \geq -t\}$. The functions $f^-_{2-k,i} \in M^{!,-}_{2-k}(p)$ have Fourier coefficients satisfying

$$\begin{aligned} f^-_{2-k,t+1} &= q^{-t-1} + O(q^{-t}),\\ f^-_{2-k,t+2} &= q^{-t-2} + O(q^{-t}),\\ &\vdots\\ f^-_{2-k,i} &= q^{-i} + O(q^{-t}),\\ &\vdots\end{aligned}$$

and consequently,

$$\begin{aligned} D^{k-1}f^-_{2-k,t+1} &= c_0 q^{-t-1} + O(q^{-t}) \in S^{!-}_k(p),\\ D^{k-1}f^-_{2-k,t+2} &= c_1 q^{-t-2} + O(q^{-t}) \in S^{!-}_k(p),\\ &\vdots\\ D^{k-1}f^-_{2-k,i} &= c_i q^{-i} + O(q^{-t}) \in S^{!-}_k(p),\\ &\vdots\end{aligned}$$

Thus, $D^{k-1}(M^{!-}_{2-k}(p))$ is spanned by $\{f^-_{k,m} : m \geq t+1\}$. It is now clear that the set $\{[f^-_{k,m}] : 0 < |m| \leq t\}$ is linearly independent in V and spans V. $\square$

Lemma 4.3. *Let $p \in \{2, 3\}$ and $f \in S_k^{!,-}(p)$. Then, the period polynomial $r^-(f;\, z)$ belongs to the space $W_{k-2}^-(p)$.*

Proof. We have

$$\mathcal{E}_f - \mathcal{E}_f\,|_{2-k} W_p - (\mathcal{E}_f - \mathcal{E}_f\,|_{2-k} W_p)|_{2-k} W_p = 0.$$

Further, using the relations $\mathcal{E}_f = \mathcal{E}_f\,|_{2-k} T$ and $TW_p = U$, we can rewrite $\frac{1}{c_k} r^-(f;\, z)$ as $\mathcal{E}_f = \mathcal{E}_f\,|_{2-k} U$. Then

$$\sum_{k=0}^{2p-1} (-1)^k (\mathcal{E}_f - \mathcal{E}_f\,|_{2-k} W_p)|_{2-k} U^k = \sum_{k=0}^{2p-1} (-1)^k (\mathcal{E}_f - \mathcal{E}_f\,|_{2-k} U)|_{2-k} U^k = 0.$$

Thus, $\frac{1}{c_k} r^-(f;\, z)$ belongs to the space $W_{k-2}(p)$, as desired. □

Proof of Theorem 1.2. By Lemma 4.3, we can define a linear map $r^- : S_k^{!,-}(p) \to W_{k-2}^-(p)$ that sends $f \in S_k^{!,-}(p)$ to its period polynomial $r^-(f;\, z)$. Proposition 4.1 implies that r^- induces an injective linear transformation

$$\widetilde{r^-} : S_k^{!,-}(p)/D^{k-1}(M_{2-k}^{!,-}(p)) \to W_{k-2}(p)/\langle(\sqrt{p}\,z)^{k-2} + 1\rangle.$$

Theorem 1.1 and Lemma 4.2 show that the spaces $S_k^{!,-}(p)/D^{k-1}(M_{2-k}^{!,-}(p))$ and $W_{k-2}(p)/\langle(\sqrt{p}\,z)^{k-2} + 1\rangle$ have the same dimension, so $\widetilde{r^-}$ is an isomorphism. Furthermore, Lemma 4.2 ensures that the set

$$\{(\sqrt{p}\,z)^{k-2} + 1\} \cup \{r^-(f_{k,m}^-;\, z) | 0 < |m| < t\}$$

forms a basis for the space $W_{k-2}^-(p)$. □

5 Proofs of Theorem 1.3 and Corollary 1.5

We first note that the stabilizer group $\Gamma_0(p)_Q = \{g \in \Gamma_0(p) | gQ = Q\}$ and $\Gamma_0^+(p)_Q = \{g \in \Gamma_0^+(p) | gQ = Q\}$ is an infinite cyclic group. More precisely, we have the following proposition:

Proposition 5.1. [3, Theorem 1.3] *Let p be one or a prime, D be a positive integer that is not a perfect square, but is congruent to a square modulo 4p. Consider a quadratic form $Q = [a, b, c] \in \mathcal{Q}_{D,p}$ with $v = \gcd(a, b, c)$, $d' = D/v^2$, $a' = a/v$, $b' = b/v$, and $c' = c/v$. We define $g_Q \in \Gamma_0^+(p)$ subgect to the following conditions:*

- *If $p|D$ and the Diophantine equation $(pt)^2 - d'u^2 = 4p$ has an integer solution (t, u) with $a'u \in p\mathbb{Z}$, then we set*

$$g_Q := \frac{1}{\sqrt{p}} \begin{pmatrix} \frac{pt_0 + bu_0}{2} & c'u_0 \\ -a'u_0 & \frac{pt_0 - bu_0}{2} \end{pmatrix} \in \Gamma_0(p) W_p,$$

  where (t_0, u_0) is the smallest positive solution of $(pt)^2 - d'u^2 = 4p$ with $p|a'u_0$.
- *Otherwise, we set*

$$g_Q := \begin{pmatrix} \frac{t_1 + b'u_1}{2} & c'u_1 \\ -a'u_1 & \frac{t_1 - b'u_1}{2} \end{pmatrix} \in \Gamma_0(p),$$

  where (t_1, u_1) is the smallest positive solution of the Pell equation $t^2 - d'u^2 = 4$ with $p|a'u_1$.

Then, g_Q generates $\Gamma_0^+(p)_Q \backslash \{\pm 1\}$.

Remark 5.2. If the condition in Proposition 5.1(i) is satisfied, we define h_Q as $h_Q = g_Q^2$; otherwise, if the condition in Proposition 5.1(ii) holds, we define h_Q as $h_Q = g_Q$. In either case, noting that $\Gamma_0(p)_Q \subseteq \Gamma_0^+(p)_Q$, we can easily deduce that $\Gamma_0(p)_Q$ is an infinite cyclic group generated by h_Q.

The following lemma provides a fundamental relation between the cycle integrals associated with a quadratic form Q and the integrals over the fundamental domain $\mathcal{F}$ for $\Gamma_0(p)$.

Lemma 5.3. *Let p be a prime, $f \in M_k^!(p)$, $\mathcal{F}$ be the standard fundamental domain for $\Gamma_0(p)$, and $Q \in \mathcal{Q}_{D,p}$ with $D > 0$ not a perfect square. For any $\tau_0 \in S_Q$, we have*

$$\int_{C_Q(\tau_0)} f(\tau)\mathrm{d}\tau_Q = \sum_{q\in(Q)} \int_{S_q\cap\mathcal{F}} f(\tau)\mathrm{d}\tau_q,$$

where $(Q) = \{gQ | g \in \Gamma_0(p)\}$.

Lemma 5.3, inspired by [6, Lemma 1], can be obtained by similar arguments as in the proof of [6, Lemma 1]. For the reader's convenience, we provide a proof of Lemma 5.3.

Proof. Let $\overline{\Gamma}_0(p) := \Gamma_0(p)/\{\pm1\}$ and define

$$\widetilde{f}(\tau) = \begin{cases} f(\tau) & \text{if } \tau \in \mathcal{F}, \\ 0 & \text{if } \tau \notin \mathcal{F}. \end{cases}$$

Note that $f(\tau) = \sum_g (\tilde{f}|_k g)(\tau)$ holds on C_Q except for a discrete set of points. This allows us to write

$$\begin{aligned} \int_{C_Q} f(\tau)\mathrm{d}\tau_Q &= \int_{C_Q} \sum_{g\in\overline{\Gamma}_0(p)} (\widetilde{f}|_k g)(\tau)\mathrm{d}\tau_Q \\ &= \sum_{g\in\overline{\Gamma}_0(p)/\overline{\Gamma}_0(p)_Q} \sum_{\sigma\in\overline{\Gamma}_0(p)_Q} \int_{C_Q} (\widetilde{f}|_k g|_k \sigma)(\tau)\mathrm{d}\tau_Q \\ &= \sum_{g\in\overline{\Gamma}_0(p)/\overline{\Gamma}_0(p)_Q} \int_{S_Q} (\widetilde{f}|_k g)(\tau)\mathrm{d}\tau_Q \\ &= \sum_{g\in\overline{\Gamma}_0(p)/\overline{\Gamma}_0(p)_Q} \int_{S_{gQ}} \widetilde{f}(\tau)\mathrm{d}\tau_{gQ}. \end{aligned}$$

The last equality follows from the change of variables $\omega := g\tau$ and the transformation properties of modular forms. More precisely, letting $\omega := g\tau$, we have

$$\mathrm{d}\tau_Q = j(g^{-1}, \omega)^{-k}\mathrm{d}\tau_{gQ}.$$

Hence,

$$\begin{aligned} \int_{S_Q} (\widetilde{f}|_k g)(\tau)\mathrm{d}\tau_Q &= \int_{S_Q} \widetilde{f}(g\tau)j(g,\tau)^{-k}\mathrm{d}\tau_Q \\ &= \int_{gS_Q} \widetilde{f}(\omega)j(g, g^{-1}\omega)^{-k}j(g^{-1},\omega)^{-k}\mathrm{d}\tau_{gQ} \\ &= \int_{gS_Q} \widetilde{f}(\omega)\mathrm{d}\tau_{gQ}. \end{aligned}$$

Therefore, we arrived at the desired result

$$\int_{C_Q} f(\tau)\mathrm{d}\tau_Q = \sum_{q\in(Q)} \int_{S_q\cap\mathcal{F}} f(\tau)\mathrm{d}\tau_q.$$

□

Proposition 5.4. *Let* $f_k^-(\tau) := f_{k,-m_k^-}^-(\tau)$. *Then, the basis elements* $f_{k,m}^-$ *of the space* $M_k^{!,-}(p)$ *satisfy the following properties*:

- *For a fixed* $z \in \mathbb{H}$ *with* $\frac{\mathrm{d}j_p^+(z)}{\mathrm{d}z} \neq 0$, *the residue of the expression* $\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)}$ *at* $\tau = z$ *is given by*

$$\mathrm{Res}_{\tau=z}\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)} = \frac{1}{2\pi i}.$$

- *The expansion*

$$\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)} = \sum_{n\geq -m_k^-} f_{k,n}^-(\tau)e^{2\pi i n z}$$

converges uniformly on compact subsets in τ *for fixed* $z \in \mathbb{H}$ *having sufficiently large* $\mathrm{Im}(z)$ *with* $\mathrm{Im}(z) > \mathrm{Im}(\tau)$.

Proof.

- We first observe from [8, p. 322] and [3, p. 756] that

$$f_k^-(z)f_{2-k}^-(z) = f_{2,1}^+(z) = \frac{\Delta_{p,\delta+2}^+}{\Delta_p^+} = -\frac{1}{2\pi i}\frac{\mathrm{d}j_p^+}{\mathrm{d}z}.$$

Using this identity, a direct computation yields

$$\begin{aligned}\mathrm{Res}_{\tau=z}\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)} &= \lim_{\tau\to z}(\tau-z)\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)}\\ &= \lim_{\tau\to z}(-1)\left(\frac{\mathrm{d}j_p^+}{\mathrm{d}\tau}\bigg|_{\tau=z}\right)^{-1}\cdot\left(-\frac{1}{2\pi i}\right)\cdot\frac{\mathrm{d}j_p^+}{\mathrm{d}z}\\ &= \frac{1}{2\pi i}.\end{aligned}$$

- Recall that

$$f_{k,m}^- = (\Delta_p^+)^{l_k}\Delta_{p,r_k}^- F_{k,m+m_k^-}(j_p^+) = q^{-m} + O(q^{m_k^-+1}),$$

where $F_{k,D}(x)$ is a monic polynomial of degree D in x. Moreover, the modular forms $\Delta_{p,\delta+2}^+$, Δ_{p,r_k}^-, and $\Delta_{p,r_{2-k}}^-$ satisfy the identity $\Delta_{p,\delta+2}^+ = \Delta_{p,r_k}^-\Delta_{p,r_{2-k}}^-$, and the exponents l_k and l_{2-k} are related by $l_{2-k} = -l_k - 1$ (see [8, p. 322]).

Using the Cauchy integral formula, we can express the polynomial $F_{k,m+m_k^-}(\zeta)$ as

$$F_{k,m+m_k^-}(\zeta) = \frac{1}{2\pi i}\oint_{C'}\frac{F_{k,D}(t)}{t-\zeta}\mathrm{d}t = \frac{1}{2\pi i}\oint_{C'}\frac{q^{-m}}{(t-\zeta)(\Delta_p^+)^{l_k}\Delta_{p,r_k}^-}\mathrm{d}t,$$

where C' is a counterclockwise circle centered at ζ in the t-plane. Changing variables $t \mapsto q = e^{2\pi i z}$ and using the identity $q\frac{\mathrm{d}t}{\mathrm{d}q} = -\frac{\Delta_{p,\delta+2}^+(z)}{\Delta_p^+(z)}$ with $t(z) = j_p^+(z)$, we obtain

$$\begin{aligned}F_{k,m+m_k^-}(\zeta) &= \frac{1}{2\pi i}\oint_C\frac{\Delta_{p,\delta+2}^+(z)q^{-m-1}}{(t(z)-\zeta)(\Delta_p^+)^{l_k+1}(z)\Delta_{p,r_k}^-(z)}\mathrm{d}q\\ &= \frac{1}{2\pi i}\oint_C\frac{\Delta_{p,r_{2-k}}^-(z)q^{-m-1}}{(t(z)-\zeta)(\Delta_p^+)^{l_k+1}(z)}\mathrm{d}q,\\ &= \frac{1}{2\pi i}\oint_C\frac{f_{2-k}^-(z)q^{-m-1}}{t(z)-\zeta}\mathrm{d}q,\end{aligned}$$

where C is a counterclockwise circle centered at 0 in the q-plane with some radius. Replacing ζ with $t(z)$ and multiplying by $(\Delta_p^+)^{l_k}(\tau)\Delta_{p,r_k}^-(\tau)$, we arrive at the integral representation

$$f_{k,m}^-(\tau) = \frac{1}{2\pi i}\oint_C \frac{f_k^-(\tau)f_{2-k}^-(z)q^{-m-1}}{t(z) - t(\tau)}\mathrm{d}q. \tag{7}$$

Let $z \in \mathbb{H}$ with sufficiently large $\mathrm{Im}(z)$ and let K be any compact subset of $\{\tau \in \mathbb{H} : \mathrm{Im}(z) > \mathrm{Im}(\tau)\}$. Choose $A > 1$ such that $\mathrm{Im}(z) > A > \mathrm{Im}(\tau)$ for all $\tau \in K$. Changing variables $q \mapsto z$ in (7) and deforming the contour by Cauchy's theorem, we have

$$f_{k,m}^-(\tau) = \int_{-\frac{1}{2}+iA}^{\frac{1}{2}+iA} \frac{f_k^-(\tau)f_{2-k}^-(z)}{t(z) - t(\tau)}e^{-2\pi imz}\mathrm{d}z.$$

We now move the contour of integration downward to a height $A' < \mathrm{Im}(\tau)$ for all $\tau \in K$. We can take $A' > 0$ such that

$$G(\tau, z) := \frac{f_k^-(\tau)f_{2-k}^-(z)}{t(z) - t(\tau)}e^{-2\pi imz}$$

has no poles on $\{t + iA' : -\frac{1}{2} \le t < \frac{1}{2}\}$ as a function of z. As we do this, each pole τ_0 of $G(\tau, z)$ in the region

$$R = \left\{z \in \mathbb{H} : -\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2} \text{ and } A' < \mathrm{Im}(z) < A\right\}$$

contributes a term $2\pi i \cdot \mathrm{Res}_{z=\tau_0}G(\tau, z)$ to the equation. Note that the poles of $G(\tau, z)$ occur only when z is equivalent to τ under the action of $\Gamma_0^+(p)$, and there are only finitely many such poles in R. To calculate the residues, we can use the following alternative formula for $G(\tau, z)$:

$$\begin{aligned} G(\tau, z) &= \frac{e^{-2\pi imz}(\Delta_p^+(\tau))^{l_k}\Delta_{p,r_k}^-(\tau)\Delta_{p,r_{2-k}}^-(z)}{(\Delta_p^+(z))^{l_k+1}(t(z) - t(\tau))} \\ &= -\frac{1}{2\pi i}e^{-2\pi imz}\frac{(\Delta_p^+(\tau))^{l_k}\Delta_{p,r_k}^-(\tau)\left(\frac{\mathrm{d}}{\mathrm{d}z}(t(z) - t(\tau))\right)}{(\Delta_p^+(z))^{l_k}\Delta_{p,r_k}^-(z)(t(z) - t(\tau))}, \end{aligned}$$

which follows from the identity

$$\frac{\Delta_{p,\delta+2-k}^-\Delta_{p,r_k}^-}{\Delta_p^+} = \frac{\Delta_{p,r_{2-k}}^-\Delta_{p,r_k}^-}{\Delta_p^+} = \frac{-1}{2\pi i}\cdot\frac{\mathrm{d}j_p^+}{\mathrm{d}z} = \frac{-1}{2\pi i}\frac{\mathrm{d}t}{\mathrm{d}z}.$$

Using the fact that

$$2\pi i \lim_{z\to\gamma\tau}(z - \gamma\tau)G(\tau, z) = \begin{cases} -e^{-2\pi im\tau} & \text{if } \gamma = \pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}, \\ \pm e^{-2\pi im\gamma\tau}j(\gamma, \tau)^{-k} & \text{if } \gamma \neq \pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}, \end{cases}$$

for any $n \in \mathbb{Z}$, we obtain the equation

$$\int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} G(\tau, z)\mathrm{d}z = f_{k,m}^-(\tau) - e^{-2\pi im\tau} \pm \sum_\gamma j(\gamma, \tau)^{-k}e^{-2\pi im\gamma\tau},$$

where the sum runs over some finite set of $\gamma \in \Gamma_0^+(p)\backslash\left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} \middle| n \in \mathbb{Z}\right\}$ satisfying $\gamma\tau \in R$.

Multiplying both sides of the equation by $e^{-2\pi m v}$ yields

$$\left| e^{-2\pi m v} \int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} G(\tau, z)\mathrm{d}z \right| = \left| e^{-2\pi m v} f_{k,m}^{-}(\tau) - e^{-2\pi i m u} \pm \sum_{\gamma} j(\gamma, \tau)^{-k} e^{-2\pi i m \gamma\tau - 2\pi m v} \right|. \tag{8}$$

The function $\left| \frac{f_k^{-}(\tau) f_{2-k}^{-}(z)}{t(z) - t(\tau)} \right|$ is continuous on the set

$$\Omega = K \times \left\{ t + iA' : -\frac{1}{2} \le t \le \frac{1}{2} \right\},$$

and therefore bounded on Ω. Consequently,

$$|G(\tau, z)| = |e^{-2\pi i m z}| \frac{|f_k^{-}(\tau) f_{2-k}^{-}(z)|}{|t(z) - t(\tau)|} \le e^{2\pi m A'} M$$

for all $(\tau, z) \in \Omega$ and for some constant $M \in \mathbb{R}_{>0}$. This implies

$$\left| e^{-2\pi m v} \int_{-\frac{1}{2}+A'i}^{\frac{1}{2}+A'i} G(\tau, z)\mathrm{d}z \right| \le e^{-2\pi m (v - A')} M \le M$$

since $v > A'$ when $m \ge 0$. Moreover, for the sum $\sum_\gamma j(\gamma, \tau)^{-k} e^{-2\pi i m \gamma\tau - 2\pi m v}$ appearing in (8), we can show that

$$\begin{aligned} \left| \sum_{\gamma} j(\gamma, \tau)^{-k} e^{-2\pi i m \gamma\tau - 2\pi m v} \right| &\le \sum_{\gamma} |j(\gamma, \tau)|^{-k} e^{-2\pi m (v - \mathrm{Im}(\gamma\tau))} \\ &< \sum_{\gamma} |j(\gamma, \tau)|^{-k} e^{-2\pi m (v - A)} \\ &\le N \cdot e^{-2\pi m (v - A)} \end{aligned}$$

for some constant $N \in \mathbb{R}_{>0}$ for all $\tau \in K$. Combining this with (8) yields

$$e^{-2m\pi v} |f_{k,m}^{-}(\tau)| \le 1 + M + N \cdot e^{-2\pi m (v - A)}$$

for all $\tau \in K$, which implies

$$|f_{k,m}^{-}(\tau)| \le e^{2m\pi v}(1 + M) + e^{-2\pi m (v - A)} N \tag{9}$$

for some constants M and N independent of m and for any $\tau \in K$. From bound (9), it follows that

$$|f_{k,m}^{-}(\tau) e^{2\pi i m z}| \le e^{-2m\pi(\mathrm{Im}(z) - v)}(1 + M) + e^{-2\pi m (\mathrm{Im}(z) - A)} N.$$

Since $\mathrm{Im}(z) - v > 0$ for any $v \in K$ and $\mathrm{Im}(z) - A > 0$, the Weierstrass M-test implies that $\sum_{m \ge -m_k^{-}} f_{k,m}^{-}(\tau) e^{2\pi i m z}$ converges uniformly on K. Moreover, by [8, Theorem 4.2],

$$\sum_{m \ge -m_k^{-}} f_{k,m}^{-}(\tau) e^{2\pi i m z} = \frac{f_{2-k}^{-}(z) f_k^{-}(\tau)}{(j_p^{+}(z) - j_p^{+}(\tau))},$$

which completes the proof. □

Proof of Theorem 1.3 (i). By Proposition 5.4 (ii), for $\mathrm{Im}(z)$ sufficiently large, we have

$$\begin{aligned}F^-(z,Q) &= \sum_{m\geq -m_k^-} r_Q(f_{k,m}^-)e^{2\pi imz} = \sum_{m\geq -m_k^-}\left(\int_{C_Q} f_{k,m}^-(\tau)\mathrm{d}\tau_Q\right)e^{2\pi imz}\\ &= \int_{C_Q}\left(\sum_{m\geq -m_k^-} f_{k,m}^-(\tau)e^{2\pi imz}\right)\mathrm{d}\tau_Q\\ &= \int_{C_Q}\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)}\mathrm{d}\tau_Q,\end{aligned}$$

where C_Q can be any smooth curve joining an arbitrary point $\tau_0\in\mathbb{H}$ to $h_Q\tau_0$. As in the proof [6, Theorem 3], one can show that $F^-(z,Q)$ extends analytically to $\mathbb{H}$ with a convergent Fourier expansion.

Let $p\in\{2,3\}$ and $\mathcal{D}$ be the closure of $\mathcal{F}^+\cup\mathcal{F}'$ where $\mathcal{F}^+$ is the fundamental domain for $\Gamma_0^+(p)$ described in [11] and $\mathcal{F}'$ is its image under $z\mapsto -\frac{1}{pz}$. Then, $\mathcal{F}^+\cup\mathcal{F}'$ is a fundamental domain for $\Gamma_0(p)$. By Lemma 5.3,

$$F^-(z,Q) = \sum\int_{A_q}\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)}\mathrm{d}\tau_Q,$$

where the sum is over all $q\in(Q)$ such that $S_q\cap\mathrm{int}(\mathcal{D})\neq\varnothing$, with $A_q = S_q\cap\mathcal{D}$.

We note from [3, (11)] that

$$S_{[a,b,c]}\cap\mathcal{D}\neq\varnothing \iff ac<0 \quad\text{or}\quad \left(ac>0 \quad\text{and}\quad \left|\frac{b}{2}\right| > \left|\frac{a}{p}+c\right|\right). \tag{10}$$

As in [3, p. 750], each arc A_q corresponding to the first condition in the right-hand side of (10) is deformed to a curve B_q within $\mathcal{D}$ having the same endpoints as A_q, but leaving z and $-\frac{1}{pz}$ in the same connected component C determined by B_q. By Proposition 5.4 (i),

$$\begin{aligned}\int_{A_q-B_q}\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)}\mathrm{d}\tau_q &= -\mathrm{sgn}(a)(2\pi i)\,\mathrm{Res}_{\tau=z}\left(\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)}(a\tau^2+b\tau+c)^{\frac{k}{2}-1}\right)\\ &= -\mathrm{sgn}(a)(az^2+bz+c)^{\frac{k}{2}-1}.\end{aligned}$$

For A_q corresponding to the second condition on the right-hand side of (10), z and $-\frac{1}{pz}$ are already in the same connected component [3, p. 757]. Therefore,

$$F^-(z,Q) = \sum_{\substack{[a,b,c]\in(Q)\\ ac>0,\left|\frac{b}{2}\right|>\left|\frac{a}{p}+c\right|}}\int_{A_q}\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)}\mathrm{d}\tau_q + \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}}\int_{B_q}\frac{f_k^-(\tau)f_{2-k}^-(z)}{j_p^+(z)-j_p^+(\tau)}\mathrm{d}\tau_q - \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}}\mathrm{sgn}(a)(az^2+bz+c)^{\frac{k}{2}-1}.$$

As in [3, p. 757], for some open neighborhood of z,

$$F^-(z,Q) + F^-(z,Q)|_{2-k}W_p = \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}}\mathrm{sgn}(c)(az^2+bz+c)^{\frac{k}{2}-1} + \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}}\mathrm{sgn}(c)\left(cpz^2-bz+\frac{a}{p}\right)^{\frac{k}{2}-1} \tag{11}$$

By the identity theorem, (11) holds for any $z\in\mathbb{H}$.

Let $Q' = W_pQ = W_p[a,b,c]$. Then, $Q' = [cp,-b,a/p] = [A,B,C]$, so

$$\begin{aligned}\sum_{\substack{[a,b,c]\in(Q)\\ ac<0}}\mathrm{sgn}(c)\left(cpz^2-bz+\frac{a}{p}\right)^{\frac{k}{2}-1} &= \sum_{\substack{[A,B,C]\in(W_pQ)\\ AC<0}}\mathrm{sgn}(A)(Az^2+Bz+C)^{\frac{k}{2}-1}\\ &= -\sum_{\substack{[a,b,c]\in(W_pQ)\\ ac<0}}\mathrm{sgn}(c)(az^2+bz+c)^{\frac{k}{2}-1}\end{aligned}$$

Hence, by (11), the theorem follows. □

Proof of Theorem 1.3 (ii). We first note that in Theorem 1.3 (i) $F^{-}(z,Q)$ is a modular integral. This implies that the polynomial $\Psi_Q(z)$ is a period polynomial for a modular integral $F^{-}(z,Q)$, and hence, $\Psi_Q(z)\in W_{k-2}^{-}(p)$ for $p\in\{2,3\}$ and an even integer $k>2$ (see [1, 4,5]). Thus, by Theorem 1.2, there exist some complex number a_i such that

$$\Psi_Q(z)=a_0(1+(\sqrt{p}z)^{k-2})+\sum_{0<|m|\le t}a_m\psi_{k,m}(z),$$

with $t=\dim S_k^{-}(p)$ and $\psi_{k,m}(z):=\frac{1}{c_k}r^{-}(f_{k,m}^{-};z)$.

Note that $F_{k,m}(z):=\mathcal{E}_{f_{k,m}^{-}}(z)$ is a modular integral for period polynomials $\psi_{k,m}(z)$, that is,

$$\psi_{k,m}(z)=\frac{1}{c_k}r^{-}(f_{k,m}^{-};z)=(\mathcal{E}_{f_{k,m}^{-}}+\mathcal{E}_{f_{k,m}^{-}}|_{2-k}W_p)(z).$$

If we define $G(z)=F^{-}(z,Q)-\sum_{0<|m|\le t}a_mF_{k,m}(z)-a_0$, then

$$G(z+1)=G(z),$$
$$G(z)+G(z)|_{2-k}W_p=\Psi_Q(z)-a_0(q+(\sqrt{p}z)^{k-2})-\sum_{0<|m|\le t}a_m\psi_{k,m}(z)=0,$$

which implies $G\in M_{2-k}^{!,-}(p)$.

If G is nonzero, then we have $\operatorname{ord}_\infty G\ge -t$, which contradicts the fact $m_{2-k}^{-}=-1-t$. Thus $G=0$, and so

$$F^{-}(z,Q)=a_0+\sum_{0<|m|\le t}a_mF_{k,m}(z).$$

Comparing Fourier coefficients, we obtain that

$$a_0=r_Q(f_{k,0}^{-})$$

and for nonzero m with $-t\le m\le t$,

$$a_m=(-m)^{k-1}r_Q(f_{k,-m}^{-}).$$

Therefore,

$$\Psi_Q(z)=r_Q(f_{k,0}^{-})(1+(\sqrt{p}z)^{k-2})+\sum_{0<|m|\le t}(-m)^{k-1}r_Q(f_{k,-m}^{-})\psi_{k,m}(z)$$

as desired. □

Proof of Corollary 1.5. To prove the assertion, we first find the condition for the right-hand side of (3) to be 0.

For g_Q, the generator of $\Gamma_0^{+}(p)_Q/\{\pm1\}$ defined in Proposition 5.1, we first claim that if $g_Q\in W_p\Gamma_0(p)$, then $(Q)=(W_pQ)$, and if $g_Q\in\Gamma_0(p)$, then $(Q)\ne(W_pQ)$. Assume $g_Q=W_p\gamma$ for some $\gamma\in\Gamma_0(p)$. For any $\delta\in\Gamma_0(p)$, we have

$$\delta Q=\delta g_QQ=\delta W_p\gamma Q=\delta\gamma' W_pQ$$

for some $\gamma'\in\Gamma_0(p)$. Hence, $(Q)\subseteq(W_pQ)$. Now, for any $\delta\in\Gamma_0(p)$,

$$\delta(W_pQ)=\delta W_pQ=\delta W_pg_QQ=\delta W_pW_p\gamma Q=\delta\gamma Q\in(Q).$$

Therefore, $(W_pQ)\subseteq(Q)$, and hence, $(Q)=(W_pQ)$. Conversely, suppose $(Q)=(W_pQ)$. Then, $\gamma Q=W_pQ$ for some $\gamma\in\Gamma_0(p)$, implying $\gamma^{-1}\circ W_p\in W_p\Gamma_0(p)$ and $\gamma^{-1}W_pQ=Q$. However, since $g_Q\in\Gamma_0(p)$, we have $\gamma^{-1}W_p\notin W_p\Gamma_0(p)$, which is a contradiction. Thus, $(Q)\ne(W_pQ)$. Therefore, if $g_Q\in W_p\Gamma_0(p)$, then the right-hand side of (3) becomes zero, and otherwise, the left-hand side of (3) is a rational function.

Note that since $\Gamma_0^{+}(p)$ is generated by $\pm T$ and $\pm W_p$, any $\gamma\in\Gamma_0(p)$ can be written as $\gamma=\mu_1\mu_2\ldots\mu_r$ with each $\mu_i\in\{\pm T,\pm W_p\}$ and an even number of μ_i equal to $\pm W_p$. Therefore, as shown above, the fact that the right-hand side of (3) is 0 is sufficient to confirm that $F^{-}(z,Q)$ is a weakly holomorphic modular form of weight $2-k$ on $\Gamma_0(p)$. Noting $\operatorname{ord}_\infty F^{-}(z,Q)$ must be less than equal to m_k^{-} and $m_{2-k}^{-}=-m_k^{-}-1$, we have $F^{-}(z,Q)=0$. Now, using Proposition 5.1, the assertion follows. □

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