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Research Article

Wan-Ming Guo*

Zeros distribution and interlacing property for certain polynomial sequences

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Abstract: In this article, we first prove that the Hankel determinant of order three of the polynomial sequence $\{P_n(x) = \sum_{k\geq 0} P(n,k)x^k\}_{n\geq 0}$ is weakly (Hurwitz) stable, where $P(n,k)$ satisfies the recurrence relation

$$P(n,k) = (a_1 n + a_2)P(n-1,k) + (b_1 n + b_2)P(n-1,k-1),$$

with $P(n,k) = 0$ wherever $k \notin \{0,1,\ldots,n\}$. The stability of a polynomial is closely associated with the interlacing property, which is based on the Hermite-Biehler theorem. We also show the interlacing property of the polynomial sequence $(U_n(x))_{n\geq 0}$, which satisfies the following recurrence relation:

$$U_n(x) = (\alpha_n x + \beta_n)U_{n-1}(x) + (u_n x^2 + v_n x)U'_{n-1}(x)$$

based on the Hermite-Biehler theorem. As applications, we obtain the weak (Hurwitz) stability of the Hankel determinant of order three for the row polynomials of the (unsigned) Stirling numbers of the first kind, the Whitney numbers of the first kind, and show the interlacing property of Eulerian polynomials, Bell polynomials, and Dowling polynomials.

Keywords: Hankel determinant, interlacing property, Hurwitz stability, Hermite-Biehler theorem

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1 Introduction

Let $\mathbb{R}$ (resp. $\mathbb{N}, \mathbb{N}^+, \mathbb{C}$) denote the set of all real numbers (resp. non-negative integers, positive integers, complex numbers), and let x be an indeterminate. We denote by $\mathbb{R}[x]$ the ring of polynomials in the indeterminate x with coefficients in $\mathbb{R}$. A real polynomial $f(x)$ is a polynomial with real coefficients, i.e., $f(x) \in \mathbb{R}[x]$.

For a nonnegative real sequence $a_0, a_1, a_2, \ldots, a_n$, the sequence is called log-convex (resp. log-concave) if for all $k \geq 1$, $a_{k-1}a_{k+1} \geq a_k^2$ (resp. $a_{k-1}a_{k+1} \leq a_k^2$). Log-concave sequences often arise in combinatorics, algebra, geometry, analysis, probability, and statistics and have been extensively investigated, see Stanley [1] and Brenti [2]. Clearly, log-convexity is closely related to log-concavity. The log-convexity was developed by Liu and Wang [3]. Many scholars have extended log-convexity to polynomials, see [3–5]. For a polynomial sequence $(\alpha_n(x))_{n\geq 0}$, it is called x-log-convex if all the coefficients of $\alpha_{n+1}(x)\alpha_{n-1}(x) - \alpha_n^2(x)$ are nonnegative for $n \geq 1$. It is called strongly x-log-convex if all the coefficients of $\alpha_{n+1}(x)\alpha_{m-1}(x) - \alpha_n(x)\alpha_m(x)$ are nonnegative for $n \geq m \geq 1$. From the definition, strong x-log-convexity implies x-log-convexity. However, the reverse does not hold. Many famous combinatorial polynomial sequences, including Bell polynomials, Eulerian polynomials, Narayana polynomials, and Dowling polynomials, are x-log-convex and strongly x-log-convex, respectively (see [3–5]).

*** Corresponding author: Wan-Ming Guo,** School of Mathematical Sciences, Qufu Normal University, Qufu 273165, P. R. China, e-mail: wm_gmath@qfnu.edu.cn, wm_gmath@163.com

For a polynomial sequence $\alpha = (a_n(x))_{n\geq 0}$, we define its Hankel matrix and Hankel determinant of order p by

$$H(\alpha) = [a_{i+j}(x)]_{i,j\geq 0} = \begin{bmatrix} a_0(x) & a_1(x) & a_2(x) & a_3(x) & \cdots \\ a_1(x) & a_2(x) & a_3(x) & a_4(x) & \cdots \\ a_2(x) & a_3(x) & a_4(x) & a_5(x) & \cdots \\ a_3(x) & a_4(x) & a_5(x) & a_6(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

and

$$H_p(\alpha) = \begin{vmatrix} a_n(x) & a_{n+1}(x) & \cdots & a_{n+p-1}(x) \\ a_{n+1}(x) & a_{n+2}(x) & \cdots & a_{n+p}(x) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+p-1}(x) & a_{n+p}(x) & \cdots & a_{n+2p-2}(x) \end{vmatrix},$$

respectively. We say that for a polynomial sequence, the subdeterminant of order two is obtained by selecting any two rows and two columns from its Hankel matrix and arranging the elements in their original order according to the intersection points of these rows and columns. It is not hard to see that a polynomial sequence is x-log-convex if all the coefficients of its consecutive subdeterminants of order two in its Hankel matrix are nonnegative for any $n \geq 0$ (resp. strongly x-log-convex if all the coefficients of its subdeterminants of order two in its Hankel matrix are nonnegative).

In this article, we are mainly concerned with the distribution of zeros of the Hankel determinant for polynomial sequences. In addition, we prove the interlacing property of certain polynomial sequences using the operator theory. The (Hurwitz) stability is an important property of the distribution of zeros. We say that a real polynomial is (Hurwitz) stable if all its zeros lie in the open left half of the complex plane, and it is weakly (Hurwitz) stable if all of its zeros lie in the closed left half of the complex plane. It is clear that the coefficients of any (Hurwitz) stable polynomial have the same sign. For a real polynomial sequence with positive leading coefficients, the weak (Hurwitz) stability of the consecutive subdeterminants of order two in its Hankel matrix, or of the subdeterminants of order two, implies that the polynomial sequence is x-log-convex or strongly x-log-convex.

For real polynomial sequences, many results have focused on the (Hurwitz) stability of the subdeterminants of order two in their Hankel matrices. For example, Fisk [6] and Zhu [7] independently proved that the consecutive subdeterminants of order two in the Hankel matrices of Bell polynomials and Eulerian polynomials are weakly (Hurwitz) stable. Recently, Liu and Yan [8] obtained that the subdeterminants of order two in the Hankel matrices of Bell polynomials, Dowling polynomials, ordered Bell polynomials, ordered Dowling polynomials, and associated Lah polynomials are weakly (Hurwitz) stable.

Fisk [6] conjectured that the Hankel determinants of order three for Bell polynomials, Eulerian polynomials, and Narayana polynomials have zeros distributed in all quadrants and that the Laguerre polynomials are weakly (Hurwitz) stable. So far, no study has examined the distribution of zeros for the Hankel determinants of order three of combinatorial polynomial sequences.

Inspired by the above works, we consider the distribution of zeros of the Hankel determinant of order three of polynomial sequences from the perspective of recurrence relations.

Let $[P(n,k)]_{n,k\geq 0}$ be a triangle of nonnegative numbers satisfying the following recurrence relation:

$$P(n,k) = (a_1 n + a_2)P(n-1,k) + (b_1 n + b_2)P(n-1,k-1), \tag{1}$$

with $P(n,k) = 0$ wherever $k \notin \{1, 2, \ldots, n\}$. Let $P_n(x) = \sum_{k=0}^{n} P(n,k)x^k$ be the row polynomials of the triangle $[P(n,k)]_{n,k\geq 0}$. Many famous combinatorial numbers, such as the (unsigned) Stirling numbers of the first kind $\left[{n \atop k}\right]$ and the Whitney numbers of the first kind $w_m(n,k)$, satisfy the above recurrence. It is natural to consider the following problem.

Problem 1.1. Let $P(n,k)$ be defined as in (1). Under what conditions, the Hankel determinant of order three of $(P_n(x))_{n\geq 0}$:

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) & P_{n+2}(x) \\ P_{n+1}(x) & P_{n+2}(x) & P_{n+3}(x) \\ P_{n+2}(x) & P_{n+3}(x) & P_{n+4}(x) \end{vmatrix}$$

is weakly (Hurwitz) stable.

In addition, the weak (Hurwitz) stability of polynomials is closely related to the interlacing property, which is based on the Hermite-Biehler theorem. Many scholars have studied the interlacing property using the Hermite-Biehler theorem, see [9–11]. In this article, we consider the interlacing property of the polynomial sequences $(U_n(x))_{n\geq 0}$, which satisfy the following recurrence relation:

$$U_n(x) = (\alpha_n x + \beta_n)U_{n-1}(x) + (u_n x^2 + v_n x)U'_{n-1}(x),$$

based on the Hermite-Biehler theorem. Many famous combinatorial polynomial sequences, such as Eulerian polynomials, Bell polynomials, and Dowling polynomials, satisfy the recurrence.

The organization of this article is as follows. In the next section, we present a sufficient condition concerning Problem 1.1. Then, based on the Hermite-Biehler theorem and the theory of linear operators preserving Hurwitz stability, we prove that $U_{n-1}(x)$ interlaces $U_n(x)$. Next, we present some applications for our results. In the final section, we propose some questions regarding the distribution of zeros for the Hankel determinant of order three of polynomial sequences.

2 Preliminaries

A real polynomial $f(x)$ is said to be *standard* if either its leading coefficient is positive or it is identically zero. Let $f(x)$ and $g(x)$ be standard polynomials with real zeros, where the zeros of $f(x)$ are $\theta_1 \leq \ldots \leq \theta_n$, and the zeros of $g(x)$ are $\xi_1 \leq \ldots \leq \xi_m$. We say that $g(x)$ interlaces $f(x)$ if $\deg f(x) = \deg g(x) + 1$ and the zeros of $f(x)$ and $g(x)$ satisfy $\theta_1 \leq \xi_1 \leq \theta_2 \leq \ldots \leq \xi_{n-1} \leq \theta_n$. We say that $g(x)$ alternates left of $f(x)$ if $\deg f(x) = \deg g(x)$ and their zeros satisfy $\xi_1 \leq \theta_1 \leq \xi_2 \leq \ldots \leq \xi_n \leq \theta_n$. We use the notation $g(x) \preccurlyeq_{\text{int}} f(x)$ for "$g(x)$ interlaces $f(x)$," $g(x) \preccurlyeq_{alt} f(x)$ for "$g(x)$ alternates left $f(x)$," and $g(x) \preccurlyeq f(x)$ for either "$g(x) \preccurlyeq_{\text{int}} f(x)$" or "$g(x) \preccurlyeq_{alt} f(x)$." For notational convenience, let $a \preccurlyeq bx + c$ for any constants a, b, c and $f(x) \preccurlyeq 0$, $0 \preccurlyeq f(x)$ for any real polynomial $f(x)$ with only real zeros.

The next result is classical.

Theorem 2.1. (Hermite-Biehler theorem) *Let $F(x) = f(x^2) + xg(x^2) \in \mathbb{R}[x]$ be standard. Then, F is weakly (Hurwitz) stable if and only if both f and g are standard polynomials with only nonpositive zeros, and $g \preccurlyeq f$.*

Let $\mathbb{C}[z]$ denote the set of all polynomials in z with complex coefficients, and let $\mathbb{C}_m[z] := \{f \in \mathbb{C}[z] : \deg f \leq m\}$. The notion of Hurwitz stability has been extended to polynomials in more than one variable. Let $\mathbb{C}[z_1, z_2, \ldots, z_m]$ denote the set of polynomials in the variables $z_1, z_2, \ldots, z_m$ over $\mathbb{C}$. A polynomial $Q(z_1, z_2, \ldots, z_m) \in \mathbb{C}[z_1, z_2, \ldots, z_m]$ is weakly (Hurwitz) stable (resp. Hurwitz stable) if $Q(z_1, z_2, \ldots, z_m) \neq 0$ for all $(z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$ with $\mathrm{Re} z_i > 0$ (resp., $\mathrm{Re} z_i \geq 0$) for $1 \leq i \leq m$. The following is a classical result concerning linear operators that preserve the Hurwitz stability of multivariate polynomials [12].

Theorem 2.2. *Let $m \in \mathbb{N}$ and let $T : \mathbb{C}_m[z] \to \mathbb{C}[z]$ be a linear operator. Then, T preserves weak Hurwitz stability if and only if*

(*a*) *T has a range of dimensions at most one and is of the form $T(f) = \alpha(f)P$, where α is a linear functional on $\mathbb{C}_m[z]$ and P is a weakly Hurwitz stable polynomial*, or

(*b*) *the polynomial*

$$T[(zw+1)^m] = \sum_{k=0}^{m} \binom{m}{k} T(z^k)(w^k),$$

called the algebraic symbol of T is weakly Hurwitz stable in two variables z, w.

3 Main results

Now, we are in the position to provide a sufficient condition for Problem 1.1.

Theorem 3.1. *Let $[P(n,k)]_{n,k\geq 0}$ be defined as in* (1), *and let $P_n(x)=\sum_{k=0}^{n}P(n,k)x^k$ be the row polynomials of the triangle. Suppose that $a_1, b_1 \geq 0$, $P_n(x)$ is weakly (Hurwitz) stable for $n \geq 1$, and $P_0(x)$ is either a constant or weakly (Hurwitz) stable. Then, the Hankel determinant of order three of $(P_n(x))_{n\geq 0}$ is weakly (Hurwitz) stable.*

Proof. From the recurrence relation (1), for $n \geq 1$, we have

$$P_n(x) = (a_1 n + a_2 + (b_1 n + b_2)x)P_{n-1}(x). \tag{2}$$

Let $s_n = a_1 n + a_2$, $t_n = b_1 n + b_2$. Using this property and the properties of the determinants, for the Hankel determinant of order three of $(P_n(x))_{n\geq 0}$, we have

$$\begin{aligned}
&\begin{vmatrix} P_n(x) & P_{n+1}(x) & P_{n+2}(x) \\ P_{n+1}(x) & P_{n+2}(x) & P_{n+3}(x) \\ P_{n+2}(x) & P_{n+3}(x) & P_{n+4}(x) \end{vmatrix} \\
&= P_n(x)P_{n+1}(x)P_{n+2}(x)\begin{vmatrix} 1 & s_{n+1}+t_{n+1}x & (s_{n+1}+t_{n+1}x)(s_{n+2}+t_{n+2}x) \\ 1 & s_{n+2}+t_{n+2}x & (s_{n+2}+t_{n+2}x)(s_{n+3}+t_{n+3}x) \\ 1 & s_{n+3}+t_{n+3}x & (s_{n+3}+t_{n+3}x)(s_{n+4}+t_{n+4}x) \end{vmatrix} \\
&= P_n(x)P_{n+1}(x)P_{n+2}(x) \\
&\quad \begin{vmatrix} 1 & s_{n+1}+t_{n+1}x & (s_{n+1}+t_{n+1}x)(s_{n+2}+t_{n+2}x) \\ 0 & s_{n+2}-s_{n+1}+(t_{n+2}-t_{n+1})x & (s_{n+2}+t_{n+2}x)(s_{n+3}+t_{n+3}x)-(s_{n+1}+t_{n+1}x)(s_{n+2}+t_{n+2}x) \\ 0 & s_{n+3}-s_{n+1}+(t_{n+3}-t_{n+1})x & (s_{n+3}+t_{n+3}x)(s_{n+4}+t_{n+4}x)-(s_{n+1}+t_{n+1}x)(s_{n+2}+t_{n+2}x) \end{vmatrix} \\
&= P_n(x)P_{n+1}(x)P_{n+2}(x) \\
&\quad \times \begin{vmatrix} s_{n+2}-s_{n+1}+(t_{n+2}-t_{n+1})x & (s_{n+2}+t_{n+2}x)(s_{n+3}+t_{n+3}x)-(s_{n+1}+t_{n+1}x)(s_{n+2}+t_{n+2}x) \\ s_{n+3}-s_{n+1}+(t_{n+3}-t_{n+1})x & (s_{n+3}+t_{n+3}x)(s_{n+4}+t_{n+4}x)-(s_{n+1}+t_{n+1}x)(s_{n+2}+t_{n+2}x) \end{vmatrix} \\
&= P_n(x)P_{n+1}(x)P_{n+2}(x) \\
&\quad \times \begin{vmatrix} s_{n+2}-s_{n+1}+(t_{n+2}-t_{n+1})x & (s_{n+2}+t_{n+2}x)(s_{n+3}+t_{n+3}x)-(s_{n+1}+t_{n+1}x)(s_{n+2}+t_{n+2}x) \\ s_{n+3}-s_{n+2}+(t_{n+3}-t_{n+2})x & (s_{n+3}+t_{n+3}x)(s_{n+4}+t_{n+4}x)-(s_{n+3}+t_{n+3}x)(s_{n+2}+t_{n+2}x) \end{vmatrix} \\
&= P_n(x)P_{n+1}(x)P_{n+2}(x)\begin{vmatrix} a_1+b_1x & (s_{n+2}+t_{n+2}x)(s_{n+3}-s_{n+1}+(t_{n+3}-t_{n+1})x) \\ a_1+b_1x & (s_{n+3}+t_{n+3}x)(s_{n+4}-s_{n+2}+(t_{n+4}-t_{n+2})x) \end{vmatrix} \\
&= P_n(x)P_{n+1}(x)P_{n+2}(x)\begin{vmatrix} a_1+b_1x & (s_{n+2}+t_{n+2}x)(2a_1+2b_1x) \\ a_1+b_1x & (s_{n+3}+t_{n+3}x)(2a_1+2b_1x) \end{vmatrix} \\
&= P_n(x)P_{n+1}(x)P_{n+2}(x)(a_1+b_1x)^2[2(s_{n+3}+t_{n+3}x)-2(s_{n+2}+t_{n+2}x)] \\
&= 2P_{n+1}(x)P_n(x)P_{n+2}(x)(a_1+b_1x)^3.
\end{aligned}$$

Since $a_1, b_1 \geq 0$, the polynomials $P_n(x)$, $P_{n+1}(x)$, $P_{n+2}(x)$ are weakly (Hurwitz) stable, and the product of two stable polynomials is also stable; then, the Hankel determinant of order three of $(P_n(x))_{n\geq 0}$ is weakly (Hurwitz) stable. □

Next, we prove the interlacing property of the polynomial sequence $(U_n(x))_{n\geq 0}$, which satisfies the following recurrence relation:

$$U_n(x) = (\alpha x + \beta)U_{n-1}(x) + (ux^2 + vx)U'_{n-1}(x), \tag{3}$$

where α (resp., β, u, v) denotes α_n (resp., β_n, u_n, v_n). Liu and Wang [13] have proven the interlacing property for such sequences using recurrence relations. They demonstrated that if $ux^2 + vx \leq 0$, for $x \leq 0$, then $U_{n-1}(x) \preccurlyeq U_n(x)$. In this article, based on the Hermite-Biehler theorem and the theory of linear transformations preserving Hurwitz stability, we present a new approach to proving the interlacing of polynomial sequence $(U_n(x))_{n\geq 0}$. We only assume that the zeros of $U_n(x)$ are in $(-\infty, 0)$ for $n \geq 2$. By Theorem 2.1, to prove that $U_{n-1}(x)$ interlaces $U_n(x)$, we consider the weak (Hurwitz) stability of the polynomial $F_n(x)$, denoted by

$$F_n(x) = U_n(x^2) + xU_{n-1}(x^2)$$

for $n \geq 2$. We have the following main result.

Theorem 3.2. *For $n \geq 2$, suppose that*
(1) $(nu + 2\alpha) > 0$, $\alpha \geq 0$, $\beta \geq 0$, $u \leq 0$, $v \geq 0$, *and* $(v\alpha - 3u\beta)n - 4\alpha\beta + 2 \geq 0$; *or*
(2) $nu + 2\alpha = 0$, $\alpha \geq 0$, $\beta \geq 0$, *and* $v \geq 0$.

Then, the polynomial $F_n(x)$ is weakly (Hurwitz) stable, and $U_{n-1}(x) \preccurlyeq U_n(x)$.

Proof. By the recurrence relation (3), we have

$$F_n(x) = \left[(\alpha x^2 + x + \beta) + \frac{1}{2}(ux^3 + vx)\frac{\mathrm{d}}{\mathrm{d}x}\right]U_{n-1}(x^2) = T(U_{n-1}(x^2)),$$

where $T = (\alpha x^2 + x + \beta) + \frac{1}{2}(ux^3 + vx)\frac{\mathrm{d}}{\mathrm{d}x}$ is a linear operator on $\mathbb{C}_n[x]$. Since the zeros of $U_n(x)$ are in $(-\infty, 0)$, $U_n(x^2)$ is weakly (Hurwitz) stable.

Let $x = a + bi$, $y = c + di$, where $a, c > 0$ and $b, d \in \mathbb{R}$. Then, $xy + 1 \neq 0$. Since

$$T[(xy + 1)^n] = (xy + 1)^n \frac{xy[(nu + 2\alpha)x^2 + 2x + 2\beta + nv] + 2(\alpha x^2 + x + \beta)}{2(xy + 1)},$$

it suffices to prove that

$$xy[(nu + 2\alpha)x^2 + 2x + 2\beta + nv] + 2(\alpha x^2 + x + \beta) \neq 0.$$

Assuming that it is not the case, when $nu + 2\alpha \neq 0$, $(nu + 2\alpha)x^2 + 2x + 2\beta + nv = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1 - (nu + 2\alpha)(2\beta + nv)}}{nu + 2\alpha}$. When $nu + 2\alpha > 0$, $\beta \geq 0$, and $v \geq 0$, we have $\mathrm{Re}\left(\frac{-1 \pm \sqrt{1 - (nu + 2\alpha)(2\beta + nv)}}{nu + 2\alpha}\right) \leq 0$. It follows that $(nu + 2\alpha)x^2 + 2x + 2\beta + nv \neq 0$, so that

$$\begin{aligned} y &= -\frac{2(\alpha x^2 + x + \beta)}{[(nu + 2\alpha)x^3 + 2x^2 + (2\beta + nv)x]} \\ &= -\frac{2(\alpha x^2 + x + \beta)((nu + 2\alpha)\bar{x}^3 + 2\bar{x}^2 + (2\beta + nv)\bar{x})}{|(nu + 2\alpha)x^3 + 2x^2 + (2\beta + nv)x|^2} \\ &= M(\alpha x^2 + x + \beta)((nu + 2\alpha)\bar{x}^3 + 2\bar{x}^2 + (2\beta + nv)\bar{x}), \end{aligned} \tag{4}$$

where $M = -\frac{2}{|(nu + 2\alpha)x^3 + 2x^2 + (2\beta + nv)x|^2} < 0$.

Note that when $nu + 2\alpha > 0$, $\beta \geq 0$, $\alpha \geq 0$, $u \leq 0$, $v \geq 0$, and $(v\alpha - 3u\beta)n - 4\alpha\beta + 2 \geq 0$, we have $\mathrm{Re}((\alpha x^2 + x + \beta)((nu + 2\alpha)\bar{x}^3 + 2\bar{x}^2 + (2\beta + nv)\bar{x})) > 0$, which contradicts $\mathrm{Re}\, y = c > 0$. Consequently,

$$xy[(nu + 2\alpha)x^2 + 2x + 2\beta + nv] + 2(\alpha x^2 + x + \beta) \neq 0.$$

Therefore, $T[(xy + 1)^n]$ is weakly (Hurwitz) stable in x and y. By Theorem 2.2, $F_n(x)$ is weakly (Hurwitz) stable.

Conversely, when $nu + 2\alpha = 0$, $(nu + 2\alpha)x^2 + 2x + 2\beta + nv = 0 \Rightarrow x = -\frac{2\beta + nv}{2}$. When $\beta \geq 0$, $v \geq 0$, the real parts of the zeros of $2x + 2\beta + nv = 0$ are not positive. It follows that $2x + 2\beta + nv \neq 0$, so that

$$\begin{aligned} y &= -\frac{2(\alpha x^2 + x + \beta)}{[2x^2 + (2\beta + nv)x]} \\ &= -\frac{2(\alpha x^2 + x + \beta)(2\bar{x}^2 + (2\beta + nv)\bar{x})}{|2x^2 + (2\beta + nv)x|^2} \\ &= M(\alpha x^2 + x + \beta)(2\bar{x}^2 + (2\beta + nv)\bar{x}), \end{aligned} \tag{5}$$

where $M = -\frac{2}{|2x^2 + (2\beta + nv)x|^2} < 0$.

Note that when $\alpha \geq 0$, $\beta \geq 0$, and $v \geq 0$, we have $\mathrm{Re}((\alpha x^2 + x + \beta)(2\bar{x}^2 + (2\beta + nv)\bar{x})) > 0$, which contradicts $\mathrm{Re}y = c > 0$. Consequently,

$$xy[2x + 2\beta + nv] + 2(\alpha x^2 + x + \beta) \neq 0.$$

Therefore, $T[(xy + 1)^n]$ is weakly (Hurwitz) stable in x and y. By Theorem 2.2, $F_n(x)$ is weakly (Hurwitz) stable. □

4 Applications

In this section, we present some applications of the main results.

The (unsigned) Stirling numbers of the first kind, denoted as $\begin{bmatrix} n \\ k \end{bmatrix}$, count the number of ways to form k disjoint cycles from a set of n elements, and they satisfy the following recurrence relation:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix},$$

for $n, k \geq 1$.

The row polynomials of $\begin{bmatrix} n \\ k \end{bmatrix}$ are the rising factorials defined as $\langle x \rangle_n = x(x+1)(x+2)\ldots(x+n-1)$. Then, by Theorem 3.1, we have the following result.

Proposition 4.1. *The Hankel determinant of order three of the rising factorials $\langle x \rangle_n$ is weakly (Hurwitz) stable.*

It is well known that the Whitney numbers of the first kind, $w_m(n, k)$, are the coefficients of the characteristic polynomial of Dowling lattices $Q_n(G)$ [14], and they satisfy the following recurrence relation:

$$w_m(n, k) = (1 + m(n-1))w_m(n-1, k) + w_m(n-1, k-1).$$

Let $w_n(x) = \sum_{k\geq 0} w_m(n, k)x^k$ be the row polynomial of $w_m(n, k)$. Then, by Theorem 3.1, we have the following result.

Proposition 4.2. *The Hankel determinant of order three of $w_n(x)$ is weakly (Hurwitz) stable.*

Now, we are in the position to present some examples of Theorem 3.2.

Example 4.3.

(1) The Eulerian polynomial $A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{des}(\pi)}$ enumerates n-permutations based on the number of descents, where $\mathfrak{S}_n$ is the symmetric group on the set $[n] = \{1, 2, \ldots, n\}$ and $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$. It is well known that [15] $A_n(x)$ has only nonpositive real zeros and satisfies the following recurrence relation for $n \geq 1$:

$$A_n(x) = ((n-1)x + 1)A_{n-1}(x) + x(1-x)A'_{n-1}(x).$$

By Theorem 3.2, we have $A_{n-1}(x) \preccurlyeq A_n(x)$.

(2) The nth type B Eulerian polynomial, $B_n(x) = \sum_{\pi\in\mathfrak{B}_n} x^{\mathrm{des}_B(\pi)}$, is the generating function for signed n permutations based on their type B descent numbers. $\mathfrak{B}_n$ denotes the nth hyperoctahedral group, which is the set of permutations π of $\{\pm1, \pm2, \ldots, \pm n\}$ satisfying $\pi(-j) = -\pi(j)$. For $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{B}_n$, the type B descents are defined as $\mathrm{Des}_B(\pi) = \{j \in \{0, 1, 2, \ldots, n-1\} : \pi_j > \pi_{j+1}\}$ and $\mathrm{des}_B = |\mathrm{Des}_B|$. It is well known that $B_n(x)$ has only nonpositive real zeros and satisfies the following recurrence relation for $n \geq 1$:

$$B_n(x) = ((2n-1)x + 1)B_{n-1}(x) + 2x(1-x)B'_{n-1}(x).$$

By Theorem 3.2, we have $B_{n-1}(x) \preccurlyeq B_n(x)$.

(3) The Bell polynomial is defined by

$$B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k,$$

where $\left\{ {n \atop k} \right\}$ denotes the Stirling numbers of the second kind, which enumerate the number of partitions of a set with n elements into k disjoint nonempty subsets. It satisfies the following recurrence relation for $n \geq 1$:

$$B_n(x) = xB_{n-1}(x) + xB'_{n-1}(x).$$

It is well known that $B_n(x)$ has only nonpositive real zeros [16]. By Theorem 3.2, we have $B_{n-1}(x) \preccurlyeq B_n(x)$.

(4) The Dowling lattice $Q_n(G)$ is a class of geometric lattices introduced by Dowling [17]. The Dowling polynomial $D_n(m, x) = \sum_{k=0}^{n} W_m(n, k)x^k$ was introduced by Benoumhani [18], where $W_m(n, k)$ denotes the Whitney numbers of the second kind. It satisfies the following recurrence relation for $n \geq 1$:

$$D_m(n; x) = (x + 1)D_m(n - 1; x) + mxD'_m(n - 1; x).$$

It is known that $D_n(m, x)$ has nonpositive real zeros [16]. By Theorem 3.2, when $m \geq 1$, we have $D_m(n - 1; x) \preccurlyeq D_m(n; x)$.

(5) It is known that the flower triangle $[F_{n,k}]_{n,k\geq 0}$ satisfies the recurrence relation (see [19, A156920])

$$F_{n,k} = (1 + k)F_{n-1,k} + (2n - 2k + 1)F_{n-1,k-1},$$

where $F_{0,0} = 1$ and $F_{n,k} = 0$ wherever $k \notin \{1, 2, \ldots, n\}$. Then, its row polynomial $F_n(x)$ satisfies the recurrence relation

$$F_{n+1}(x) = [(2n + 1)x + 1]F_n(x) + x(1 - 2x)F'_n(x).$$

The polynomial $F_n(x)$ has only nonpositive real zeros. By Theorem 3.2, we have $F_{n-1}(x) \preccurlyeq F_n(x)$.

(6) Let $d_{n,k}$ be the number of the augmented André permutations in $\mathfrak{S}_n$ with $k - 1$ left peaks. Denote

$$D_n(x) = \sum_{k\geq 1} d_{n,k}x^k.$$

It is known that

$$d_{n+1,k} = kd_{n,k} + (n - 2k + 3)d_{n,k-1},$$

where $d_{1,1} = 1$. Note that

$$D_{n+1}(x) = (n + 1)xD_n(x) + x(1 - 2x)D'_n(x),$$

and the degree of $D_n(x)$ is $\lceil \frac{n}{2} \rceil$. For further details, see [19, A094503] and Foata and Scützenberger [20]. The polynomial $D_n(x)$ has only nonpositive real zeros. By Theorem 3.2, we have $D_{n-1}(x) \preccurlyeq D_n(x)$.

5 Open problems

In this article, we mainly provide a sufficient condition for the Hankel determinant of order three of $P_n(x) = \sum_{k=0}^{n} P(n, k)x^k$ to be weakly (Hurwitz) stable, where $P(n, k)$ satisfies the following recurrence relation:

$$P(n, k) = (a_1n + a_2)P(n - 1, k) + (b_1n + b_2)P(n - 1, k - 1),$$

with $P(n, k) = 0$ wherever $k \notin \{1, 2, \ldots, n\}$. It is natural to consider the distribution of zeros of the Hankel determinant of order three for row polynomials of combinatorial sequences that satisfy other recursive relations.

Question 1: For combinatorial sequences that satisfy the recurrence relation

$$P(n, k) = (a_1k + a_2)P(n - 1, k) + (b_1k + b_2)P(n - 1, k - 1),$$

with $P(n, k) = 0$ wherever $k \notin \{1, 2, \ldots, n\}$, what is the distribution of the zeros for the Hankel determinant of order three of $P_n(x) = \sum_{k=0}^{n} P(n, k)x^k$?

Question 2: For combinatorial sequences that satisfy the recurrence relation

$$P(n,k) = (a_1n + a_2k + a_3)P(n-1,k) + (b_1n + b_2k + b_3)P(n-1,k-1),$$

with $P(n,k) = 0$ wherever $k \notin \{1, 2, \ldots, n\}$, what is the distribution of the zeros for the Hankel determinant of order three of $P_n(x) = \sum_{k=0}^{n} P(n,k)x^k$?

Many famous combinatorial sequences, such as Eulerian numbers, Stirling numbers of the second kind, and Whitney numbers of the second kind, satisfy the above recurrence relations. It is interesting to study the distribution of zeros for the Hankel determinant of order three of their row polynomials.

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