Research Article

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Global existence and extinction for a fast diffusion *p*-Laplace equation with logarithmic nonlinearity and special medium void

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Abstract: This article is devoted to the global existence and extinction behavior of the weak solution to a fast diffusion p-Laplace equation with logarithmic nonlinearity and special medium void. By applying energy estimates approach, Hardy-Littlewood-Sobolev inequality, and some ordinary differential inequalities, the global existence result is proved and the sufficient conditions on the occurrence of the extinction and nonextinction phenomena are given.

Keywords: fast diffusion p-Laplace equation, global existence, extinction, nonextinction, logarithmic nonlinearity

MSC 2020: 35K20, 35K55

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an open bounded domain containing the coordinate origin $x = 0$, with smooth boundary $\partial\Omega$, $|x| = \sqrt{\sum_{i=1}^{N} x_i^2}$ for $x = (x_1, \ldots, x_N) \in \Omega$. In this article, we take an interest in the global existence and extinction behavior of the solutions for a fast diffusion p-Laplace equation with logarithmic nonlinearity and special medium void

$$\begin{cases} |x|^{-s}u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

with

$$f(u) = \begin{cases} |u|^{q-2}u\log|u|, & \text{if } u \neq 0; \\ 0, & \text{if } u = 0, \end{cases}$$

where p, $q \in (1,2)$, and $s \geq 0$ are three parameters, and the initial datum $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is nontrivial. By the assumption on the parameter q, one knows that $\lim_{|u|\to 0^+} f(u) = 0$, and hence, $f(u) \in C(\mathbb{R})$. From a physical point of view, problem (1.1) can be used to describe the flow of a compressible fluid in a homogeneous isotropic rigid porous medium. In this content, $u(x,t)$ is the density of the fluid, $|x|^{-s}$ acts

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as the volumetric moisture content, $|\nabla u|^{p-2}\nabla u$ presents the momentum velocity, and $f(u)$ denotes the source. The readers can refer to [1] for more details of the background of problem (1.1).

In the recent years, mathematicians have focused their attention on the parabolic models with logarithmic nonlinearities and have obtained lots of achievements [2–9]. In particular, Chen et al. [10] dealt with problem (1.1) with $s = 0$ and $p = q = 2$. Under some suitable hypotheses on the initial datum, they proved the existence of the uniformly bounded global solution and the infinite time blow-up phenomenon by utilizing the potential well method and logarithmic Sobolev inequality. Le and Le [11,12] studied the existence and nonexistence of the global weak solutions of problem (1.1) with $s = 0$ and $p, q > 2$. Exactly speaking, they concluded that if $p > q$, then for any $u_0(x) \in W_0^{1,p}(\Omega)$, problem (1.1) admits a global solution; if $p \leq q$, then there exists a weak solution of problem (1.1), which is global provided that $u_0(x)$ belongs to some specific stable sets, and blows up in finite time provided that $u_0(x)$ belongs to some specific unstable sets. Compared with the case $s = 0$, there are fewer results for problem (1.1) with $s \neq 0$. Deng and Zhou [13] considered problem (1.1) with $s \in [0, 2)$ and $p = q = 2$. They gave some threshold results on the global existence or infinite time blow-up of the solutions in the cases of subcritical and critical initial energies. Liao and Tan [14] generalized the results in [13] to the case $2 \leq p < q < \frac{Np}{N-p}$. Unlike the semilinear case (i.e., $p = 2$) in which the blow-up solutions blow up at $t = +\infty$, Liao and Tan [14] pointed out that the blow-up solutions will blow up in finite time when $p > 2$, and got the upper and lower bounds of the blow-up time.

We say that the solution $u(x, t)$ of problem (1.1) possesses finite time extinction property (i.e., $u(x, t)$ vanishes in finite time) if there exists a $T \in (0, +\infty)$ such that $u(x, t)$ is nontrivial for $t \in (0, T)$ but $u(x, t) \equiv 0$ for $t > T$. As one of the most remarkable properties that distinguish nonlinear parabolic problems from the linear ones, finite time extinction can be used to explain certain natural phenomena, such as, the depletion of the reactants in a chemical reaction process, the extinction of the species in biopopulation dynamics, and so on. There exist a number of works on the extinction and nonextinction properties of the weak solutions to problem (1.1) in the case of power nonlinearities and its generalization [15–26], but there is a little known about the one with logarithmic nonlinearity. Very recently, Deng and Zhou [27] considered problem (1.1) with $s = 0$ and $p, q \in (1, 2)$, by using integral norm estimates approach and some ordinary differential inequalities, they obtained the sufficient conditions on the extinction and nonextinction behaviors of the global weak solutions.

As far as we know, there are no results on the extinction and nonextinction of the global solutions to the parabolic model with both special medium void and logarithmic nonlinearity. How to evaluate the effects of the singular potential $|x|^{-s}$ and the logarithmic nonlinearity $|u|^{q-2}u\log|u|$ on the occurrence of the extinction phenomenon of the global solution to problem (1.1)? The main goal of the present article is to give an answer to this question. The main results of this article are stated as follows.

Theorem 1.1. *Assume that* $1 < p < 2$, $1 < q < 2$, $s \geq 0$, *and* $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. *Then the weak solution* $u(x, t)$ *of problem* (1.1) *exists globally.*

Theorem 1.2. *Assume that* $\max\{1, s\} < p < q < 2$. *If the initial data* $u_0(x)$ *satisfies*

$$2\max\left\{\left(\int_\Omega |x|^{-s}u_{0+}^{pa+2}\right)^{\frac{q+\beta-p}{pa+2}}, \left(\int_\Omega |x|^{-s}u_{0-}^{pa+2}\right)^{\frac{q+\beta-p}{pa+2}}\right\} \leq C_4C_5^{-1} \tag{1.2}$$

with some $\beta \in (0, 2-q)$, $a \geq \max\left\{-\frac{1}{p}, \frac{2(N-p)-p(N-s)}{p(p-s)}\right\}$. *Then the weak solution* $u(x, t)$ *of problem* (1.1) *vanishes in finite time, where* C_4 *and* C_5 *are two positive constants given by* (3.12) *and* (3.13), *respectively.*

Theorem 1.3. *Assume that* $1 < q \leq p < 2$ *and* $s \geq 0$. *If*

$$\int_\Omega |x|^{-s}u_0^2\,\mathrm{d}x > 0,$$

and

$$\begin{cases} E(u_0) \leq 0, & \text{when } p = q; \\ E(u_0) < 0, & \text{when } p > q. \end{cases} \tag{1.3}$$

Then the weak solution of problem (1.1) *cannot vanish in finite time, where*

$$E(u_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p \mathrm{d}x - \frac{1}{q}\int_\Omega |u_0|^q \log|u_0| \mathrm{d}x + \frac{1}{q^2}\int_\Omega |u_0|^q \mathrm{d}x.$$

The remainder of this article is organized as follows. In Section 2, some definitions and notations are given, and some useful auxiliary lemmas are collected. In Section 3, the global existence result is proved and the conditions on the occurrence of the extinction and nonextinction behaviors are discussed. The proofs of Theorems 1.1, 1.2, and 1.3 will be given in Section 3. Finally, a conclusion of this article and the future scope of our work are provided in Section 4.

2 Preliminaries

In this section, we first introduce some notations and definitions. For a given function f, we define its positive part f_+ and negative part f_- as the forms

$$f_+ = \max\{f, 0\}, \quad f_- = \max\{-f, 0\}.$$

Then $f = f_+ - f_-$ and $|f| = f_+ + f_-$. For given $p \in [1, +\infty)$, we introduce the Hilbert space

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), \quad i = 1, 2, \ldots, N\right\},$$

endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \sqrt[p]{\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}}.$$

Denote

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u\,|_{\partial\Omega} = 0\}.$$

Due to Poincaré's inequality, one can know that

$$\|\nabla u\|_{L^p(\Omega)} = \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

is an equivalent norm to $\|u\|_{W^{1,p}(\Omega)}$ in $W_0^{1,p}(\Omega)$.

It is worth noting that the first equation in problem (1.1) is singular at the points where $|\nabla u| = 0$ when $1 < p < 2$, and hence, there is no classical solution in general. Throughout this article, we work with the weak solution of problem (1.1) in the sense of the following definition.

Definition 2.1. Let $T > 0$. A function $u = u(x, t)$ defined in $\Omega \times (0, T)$ is called a weak solution of problem (1.1) if $u \in C([0, T); W_0^{1,p}(\Omega))$, $|x|^{-\frac{s}{2}} u_t \in L^2(0, T; L^2(\Omega))$, $u(x, 0) = u_0(x)$, and for any $\varphi \in W_0^{1,p}(\Omega)$, it holds that

$$\int_0^T\int_\Omega |x|^{-s} u_t \varphi \mathrm{d}x + \int_0^T\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \mathrm{d}x = \int_0^T\int_\Omega |u|^{q-2} u \log|u| \varphi \mathrm{d}x. \tag{2.1}$$

Similar to Theorem 2.3 of [14] (or Theorem 3.2 of [11]), the local existence of the weak solution of problem (1.1) can be guaranteed by the Galerkin approximation method.

Next, we collect some useful auxiliary lemmas, which play a key role in the later proofs of our main results.

Lemma 2.1. (Hardy-Littlewood-Sobolev inequality, see [28]) *Assume that* $N \geq 2$, $1 < \mu < N$, $0 \leq \vartheta \leq \mu$ *and* $\sigma = \frac{\mu(N-\vartheta)}{N-\mu}$. *Then, there exists a positive constant* $\kappa = \kappa(\mu, \vartheta, N)$ *such that*

$$\int_\Omega \frac{|u(x)|^\sigma}{|x|^\vartheta} \mathrm{d}x \leq \kappa \left(\int_\Omega |\nabla u|^\mu \mathrm{d}x \right)^{\frac{N-\vartheta}{N-\mu}} \tag{2.2}$$

holds for any $u \in W_0^{1,\mu}(\Omega)$, *where* $\Omega \subset \mathbb{R}^N$ *is a bounded domain.*

Lemma 2.2. (see [29]) *Assume that* $0 < k < p \leq 1$. *Let* $y(t)$ *be the solution of the following inequality*:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} + Cy^k \leq \gamma y^p, & t > 0, \\ y(0) = y_0 > 0, \end{cases}$$

with $C > 0$ *and* $0 < \gamma < \frac{Cy_0^{k-p}}{2}$. *Then, there are two positive constants* η *and* ξ *such that*

$$0 \leq y(t) \leq \xi e^{-\eta t}$$

holds for any $t \geq 0$.

Lemma 2.3. (see [30]) *Assume that* θ, δ, *and* χ *are three positive constants. Let* $y(t)$ *be a nonnegative absolutely continuous function satisfying*

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \delta y^\theta(t) \geq \chi, \quad t > 0.$$

Then, we have

$$y(t) \geq \min\left\{ y(0), \left(\frac{\chi}{\delta}\right)^{\frac{1}{\theta}} \right\}.$$

Lemma 2.4. (see [31]) *Assume that* σ *and* ρ *are two positive constants. Then we have*

$$\Psi^\sigma \log\Psi \leq (e\rho)^{-1}\Psi^{\sigma+\rho} \quad \text{for all } \Psi > 0$$

and

$$|\Psi^\sigma \log\Psi| \leq (e\sigma)^{-1} \quad \text{for all } 0 < \Psi < 1.$$

3 Proofs of the main results

Proof of Theorem 1.1. Let $u(x,t) \in C([0,T]; W_0^{1,p}(\Omega))$ fulfilling $|x|^{-\frac{s}{2}} u_t \in L^2(0,T; L^2(\Omega))$ and $u(x,0) = u_0(x)$ be a weak solution of problem (1.1). Since Ω is a bounded domain in $\mathbb{R}^N$, there is a ball $B(0,R)$ of radius $R = \max_{x=(x_1,x_2,\ldots,x_N)\in\overline{\Omega}} \left(\sum_{i=1}^N x_i^2 \right)^{\frac{1}{2}}$ centered at $x = 0$ such that $\Omega \subseteq B(0,R)$. Recalling that $q \in (1,2)$, one can choose $\beta \in (0, 2-q)$ such that $q + \beta \in (1,2)$. For this selected β, by Lemma 2.4, one has

$$\log|u| \leq \frac{1}{e\beta} |u|^\beta.$$

Taking the test function $\varphi = u(x,t)$ in (2.1), one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^2\mathrm{d}x + \int_\Omega |\nabla u|^p\mathrm{d}x = \int_\Omega |u|^q\log|u|\mathrm{d}x \le \frac{1}{e\beta}\int_\Omega |u|^{q+\beta}\mathrm{d}x. \tag{3.1}$$

Making use of Hölder's inequality, one obtains

$$\begin{aligned}\int_\Omega |u|^{q+\beta}\mathrm{d}x &\le \left(\int_\Omega |x|^{\frac{s(q+\beta)}{2-q-\beta}}\mathrm{d}x\right)^{\frac{2-q-\beta}{2}}\left(\int_\Omega |x|^{-s}u^2\mathrm{d}x\right)^{\frac{q+\beta}{2}}\\ &\le \left(\int_{B(0,R)} |x|^{\frac{s(q+\beta)}{2-q-\beta}}\mathrm{d}x\right)^{\frac{2-q-\beta}{2}}\left(\int_\Omega |x|^{-s}u^2\mathrm{d}x\right)^{\frac{q+\beta}{2}}\\ &= \underbrace{\left(\frac{\omega_N(2-q-\beta)}{2N+(s-N)(q+\beta)}R^{\frac{2N+(s-N)(q+\beta)}{2-q-\beta}}\right)^{\frac{2-q-\beta}{2}}}_{C_0}\left(\int_\Omega |x|^{-s}u^2\mathrm{d}x\right)^{\frac{q+\beta}{2}},\end{aligned} \tag{3.2}$$

where ω_N is the surface area of the unit sphere $\partial B(0,1)$. Combining (3.1) and (3.2), one arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^2\mathrm{d}x \le \frac{2C_0}{e\beta}\left(\int_\Omega |x|^{-s}u^2\mathrm{d}x\right)^{\frac{q+\beta}{2}},$$

which implies that

$$\int_\Omega |x|^{-s}u^2\mathrm{d}x \le \left[\frac{C_0(2-q-\beta)}{e\beta}t + \left(\int_\Omega |x|^{-s}u_0^2\mathrm{d}x\right)^{\frac{2-q-\beta}{2}}\right]^{\frac{2}{2-q-\beta}}. \tag{3.3}$$

Substituting (3.3) into (3.2) suggests that

$$\int_\Omega |u|^{q+\beta}\mathrm{d}x \le C_0\left[\frac{C_0(2-q-\beta)}{e\beta}t + \left(\int_\Omega |x|^{-s}u_0^2\mathrm{d}x\right)^{\frac{2-q-\beta}{2}}\right]^{\frac{q+\beta}{2-q-\beta}}. \tag{3.4}$$

Taking the test function $\varphi = u_t(x,t)$ in (2.1), and using Lemma 2.4, Cauchy's inequality and Hölder's inequality lead us to that

$$\begin{aligned}&\int_\Omega |x|^{-s}(u_t)^2\mathrm{d}x + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p\mathrm{d}x\\ &\quad= \int_\Omega |u|^{q-2}u\log|u|u_t\mathrm{d}x\\ &\quad= \int_{\Omega_1=\{x\in\Omega:|u|\ge1\}} |u|^{q-2}u\log|u|u_t\mathrm{d}x + \int_{\Omega_2=\{x\in\Omega:|u|<1\}} |u|^{q-2}u\log|u|u_t\mathrm{d}x\\ &\quad\le \frac{1}{e\beta}\int_{\Omega_1}|u|^{q+\beta-1}|u_t|\mathrm{d}x + \frac{1}{e(q-1)}\int_{\Omega_2}|u_t|\mathrm{d}x\\ &\quad\le \left(\frac{\varepsilon_1}{e\beta}+\frac{\varepsilon_2}{e(q-1)}\right)\int_\Omega |x|^{-s}(u_t)^2\mathrm{d}x + \frac{1}{4\varepsilon_2 e(q-1)}\int_\Omega |x|^s\mathrm{d}x + \frac{1}{4\varepsilon_1 e\beta}\left(\int_\Omega |x|^{\frac{s(q+\beta)}{2-q-\beta}}\mathrm{d}x\right)^{\frac{2-q-\beta}{q+\beta}}\left(\int_\Omega |u|^{q+\beta}\mathrm{d}x\right)^{\frac{2(q+\beta-1)}{q+\beta}}.\end{aligned} \tag{3.5}$$

If ε_1 and ε_2 are sufficiently small such that $\frac{\varepsilon_1}{e\beta} + \frac{\varepsilon_2}{e(q-1)} \le 1$. Then from (3.5), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \mathrm{d}x \le C_1\left(\int_\Omega |u|^{q+\beta}\mathrm{d}x\right)^{\frac{2(q+\beta-1)}{q+\beta}} + C_2, \tag{3.6}$$

where

$$C_1 = \frac{p}{4\varepsilon_1 e\beta}\left(\int_{B(0,R)} |x|^{\frac{s(q+\beta)}{2-q-\beta}}\mathrm{d}x\right)^{\frac{2-q-\beta}{q+\beta}} = \frac{p}{4\varepsilon_1 e\beta}C_0^{\frac{2}{q+\beta}}$$

and

$$C_2 = \frac{p}{4\varepsilon_2 e(q-1)}\int_{B(0,R)} |x|^s \mathrm{d}x = \frac{p}{4\varepsilon_2 e(q-1)}\cdot\frac{\omega_N}{N+s}R^{N+s}.$$

Summing up (3.4) and (3.6), one immediately obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \mathrm{d}x \le C_1 C_0^{\frac{2(q+\beta-1)}{q+\beta}}\left[\frac{C_0(2-q-\beta)}{e\beta}t + \left(\int_\Omega |x|^{-s}u_0^2\mathrm{d}x\right)^{\frac{2-q-\beta}{2}}\right]^{\frac{2(q+\beta-1)}{2-q-\beta}} + C_2. \tag{3.7}$$

Integrating (3.7) with respect to the time variable over the interval $(0, T)$ yields that

$$\begin{aligned}\int_\Omega |\nabla u|^p \mathrm{d}x \le{}& \frac{e\beta C_1}{q+\beta}C_0^{\frac{q+\beta-2}{q+\beta}}\left[\frac{C_0(2-q-\beta)}{e\beta}T + \left(\int_\Omega |x|^{-s}u_0^2\mathrm{d}x\right)^{\frac{2-q-\beta}{2}}\right]^{\frac{2(q+\beta-1)}{2-q-\beta}+1}\\ &+ C_2 T + \int_\Omega |\nabla u_0|^p \mathrm{d}x - \frac{e\beta C_1}{q+\beta}C_0^{\frac{q+\beta-2}{q+\beta}}\left(\int_\Omega |x|^{-s}u_0^2\mathrm{d}x\right)^{\frac{q+\beta}{2}},\end{aligned}$$

which means that, for any $T \in (0, +\infty)$, the $W^{1,p}$ norm of the weak solution $u(x, t)$ is finite in $[0, T)$. That is to say, the weak solution $u(x, t)$ of problem (1.1) is global in the sense of $W^{1,p}$ norm. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Motivated by the method introduced by DiBenedetto [32], one can select, modulo a Steklov average, the test function $\varphi = |u|^{pa}u_+$ in (2.1) to obtain

$$\begin{aligned}&\frac{1}{pa+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u_+^{pa+2}\mathrm{d}x + \frac{pa+1}{(a+1)^p}\int_\Omega |\nabla u_+^{a+1}|^p \mathrm{d}x\\ &= \int_\Omega |u|^{q-2}u\,|u|^{pa}u_+\log|u|\mathrm{d}x \le \frac{1}{\beta}\int_\Omega u_+^{q+pa+\beta}\mathrm{d}x.\end{aligned} \tag{3.8}$$

Recalling that $\max\{1, s\} < p < q < q+\beta < 2$ and $a \ge \max\left\{-\frac{1}{p}, \frac{2(N-p)-p(N-s)}{p(p-s)}\right\}$, and by virtue of Hölder's inequality and Hardy-Littlewood-Sobolev inequality, one has

$$\begin{aligned}\int_\Omega |x|^{-s}u_+^{pa+2}\mathrm{d}x &\le \left(\int_\Omega |x|^{-s}\mathrm{d}x\right)^{1-\frac{(N-p)(pa+2)}{p(N-s)(a+1)}}\left(\int_\Omega |x|^{-s}(u_+^{a+1})^{\frac{p(N-s)}{N-p}}\mathrm{d}x\right)^{\frac{(N-p)(pa+2)}{p(N-s)(a+1)}}\\ &\le C_3\kappa^{\frac{(N-p)(pa+2)}{p(N-s)(a+1)}}\left(\int_\Omega |\nabla u_+^{a+1}|^p \mathrm{d}x\right)^{\frac{pa+2}{p(a+1)}},\end{aligned}$$

which means that

$$C_3^{-\frac{p(a+1)}{pa+2}} \kappa^{-\frac{N-p}{N-s}} \left(\int_\Omega |x|^{-s} u_+^{pa+2} \mathrm{d}x \right)^{\frac{p(a+1)}{pa+2}} \le \int_\Omega |\nabla u_+^{a+1}|^p \mathrm{d}x, \tag{3.9}$$

where

$$C_3 = \left(\int_{B(0,R)} |x|^{-s} \mathrm{d}x \right)^{1-\frac{(N-p)(pa+2)}{p(N-s)(a+1)}} = \left(\frac{\omega_N}{N-s} R^{N-s} \right)^{1-\frac{(N-p)(pa+2)}{p(N-s)(a+1)}},$$

and $\kappa = \kappa(p, s, N)$ is the embedding constant given in Lemma 2.1. On the other hand, by applying Hölder's inequality again, one obtains

$$\int_\Omega u_+^{q+pa+\beta} \mathrm{d}x \le \left(\int_\Omega |x|^{\frac{s(q+\beta+pa)}{2-q-\beta}} \mathrm{d}x \right)^{\frac{2-q-\beta}{pa+2}} \left(\int_\Omega |x|^{-s} u_+^{pa+2} \mathrm{d}x \right)^{\frac{q+\beta+pa}{pa+2}}. \tag{3.10}$$

Letting

$$y(t) := \int_\Omega |x|^{-s} u_+^{pa+2} \mathrm{d}x,$$

and summing up (3.8), (3.9), and (3.10), one arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + C_4 (y(t))^{\frac{p(a+1)}{pa+2}} \le C_5 (y(t))^{\frac{q+\beta+pa}{pa+2}}, \tag{3.11}$$

where

$$C_4 = \frac{(pa+1)(pa+2)}{(a+1)^p} C_3^{-\frac{p(a+1)}{pa+2}} \kappa^{-\frac{N-p}{N-s}} \tag{3.12}$$

and

$$\begin{aligned} C_5 &= \frac{pa+2}{\beta} \left(\int_{B(0,R)} |x|^{\frac{s(q+\beta+pa)}{2-q-\beta}} \mathrm{d}x \right)^{\frac{2-q-\beta}{pa+2}} \\ &= \frac{pa+2}{\beta} \cdot \left[\frac{\omega_N (2-q-\beta)}{2N + spa + (s-N)(q+\beta)} R^{\frac{2N+spa+(s-N)(q+\beta)}{2-q-\beta}} \right]^{\frac{2-q-\beta}{pa+2}}. \end{aligned} \tag{3.13}$$

Remembering that $\{1, s\} < p < q < q + \beta < 2$ and $a \ge \max\left\{ -\frac{1}{p}, \frac{2(N-p) - p(N-s)}{p(p-s)} \right\}$, one can check that

$$0 < \frac{p(a+1)}{pa+2} < \frac{pa+q+\beta}{pa+2} < 1.$$

And then, by our assumption (1.2) and Lemma 2.2, one can claim that there exist two positive constants ξ and η such that, for any $t \ge 0$,

$$0 \le y(t) \le \xi e^{-\eta t}. \tag{3.14}$$

Choosing

$$T_0 > \max\left\{ 0, \frac{1}{\eta} \ln\left[\xi \left(\frac{2C_5}{C_4} \right)^{\frac{pa+2}{q+\beta-p}} \right] \right\},$$

then it follows from (3.11) and (3.14) that

$$\frac{dy}{dt} + \frac{C_4}{2} y^{\frac{p(a+1)}{pa+2}} \le 0, \quad t \ge T_0.$$

Integrating both sides of the aforementioned inequality with respect to the time variable on $[T_0, t]$, one has

$$0 \le y^{\frac{2-p}{pa+2}}(t) \le y^{\frac{2-p}{pa+2}}(T_0) - \frac{C_4(2-p)}{2(pa+2)}(t - T_0),$$

which suggests that there exists a

$$T_0' \in \left[T_0, T_0 + \frac{2(pa+2)}{C_4(2-p)} y^{\frac{2-p}{pa+2}}(T_0)\right]$$

such that

$$\lim_{t \to T_0'^-} y(t) = \lim_{t \to T_0'^-} \int_\Omega |x|^{-s} u_+^{pa+2}(t) dx = 0.$$

Moreover, one can conclude that

$$\int_\Omega |x|^{-s} u_+(t) dx = \int_\Omega |x|^{-s} u_+^{pa+2}(t) dx \to 0$$

as $t \to T_0'^-$ for $a = -\frac{1}{p}$, and

$$\begin{aligned}\int_\Omega |x|^{-s} u_+(t) dx &\le \left(\int_\Omega |x|^{-s} dx\right)^{\frac{pa+1}{pa+2}} \left(\int_\Omega |x|^{-s} u_+^{pa+2}(t) dx\right)^{\frac{1}{pa+2}} \\ &\le \left(\frac{\omega_N}{N-s} R^{N-s}\right)^{\frac{pa+1}{pa+2}} \left(\int_\Omega |x|^{-s} u_+^{pa+2}(t) dx\right)^{\frac{1}{pa+2}} \\ &\to 0\end{aligned}$$

as $t \to T_0'^-$ for $a > -\frac{1}{p}$.

On the other hand, by using the similar way, one can show that $\int_\Omega |x|^{-s} u_-(t) dx$ will also vanish in finite time. Thus,

$$\int_\Omega |x|^{-s} |u(t)| dx = \int_\Omega |x|^{-s} u_+(t) dx + \int_\Omega |x|^{-s} u_-(t) dx$$

will vanish in finite time. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. Denote

$$E(u(t)) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{1}{q} \int_\Omega |u|^q \log|u| dx + \frac{1}{q^2} \int_\Omega |u|^q dx.$$

A direct calculation shows that

$$\frac{dE(u(t))}{dt} = -\int_\Omega |x|^{-s} (u_t)^2 dx,$$

which implies that

$$E(u(t)) = E(u_0) - \int_0^t \int_\Omega |x|^{-s} (u_\tau)^2 dx d\tau. \tag{3.15}$$

Set

$$M(t) = \frac{1}{2}\int_\Omega |x|^{-s}u^2\mathrm{d}x.$$

Taking the derivative of $M(t)$ with respect to t, and using (3.15) lead us to

$$\begin{aligned} M'(t) &= -pE(u(t)) + \frac{q-p}{q}\int_\Omega |u|^q\log|u|\mathrm{d}x + \frac{p}{q^2}\int_\Omega |u|^q\mathrm{d}x \\ &= -pE(u_0) + \int_0^t\int_\Omega |x|^{-s}(u_\tau)^2\mathrm{d}x\mathrm{d}\tau + \frac{q-p}{q}\int_\Omega |u|^q\log|u|\mathrm{d}x + \frac{p}{q^2}\int_\Omega |u|^q\mathrm{d}x \\ &\geq -pE(u_0) + \frac{q-p}{q}\int_\Omega |u|^q\log|u|\mathrm{d}x. \end{aligned} \tag{3.16}$$

If $p = q$. Then from (3.16), one can immediately see that, for any $t \geq 0$,

$$M(t) \geq M(0) - pE(u_0)t. \tag{3.17}$$

Keeping in mind that

$$M(0) = \int_\Omega |x|^{-s}u_0^2\mathrm{d}x > 0$$

and $E(u_0) \leq 0$, then (3.17) gives us that $M(t) > 0$, which means that the solution $u(x,t)$ of problem (1.1) cannot vanish in finite time.

If $p > q$. Then by Lemma 2.4 and Hölder's inequality, one has

$$\begin{aligned} M'(t) &\geq -pE(u_0) + \frac{q-p}{q}\int_\Omega |u|^q\log|u|\mathrm{d}x \\ &\geq -pE(u_0) - \frac{p-q}{e\beta q}\int_\Omega |u|^{q+\beta}\mathrm{d}x \\ &\geq -pE(u_0) - \frac{C_0(p-q)}{e\beta q}(M(t))^{\frac{q+\beta}{2}}. \end{aligned} \tag{3.18}$$

Remembering that

$$M(0) = \int_\Omega |x|^{-s}u_0^2\mathrm{d}x > 0$$

and

$$E(u_0) < 0,$$

by combining (3.18) with Lemma 2.3, one can conclude that

$$M(t) \geq \min\left\{M(0), \left[-\frac{e\beta pqE(u_0)}{C_0(p-q)}\right]^{\frac{2}{q+\beta}}\right\} > 0.$$

Thus, the solution $u(x,t)$ of problem (1.1) cannot vanish in finite time. The proof of Theorem 1.3 is complete. □

4 Conclusions

In this article, we deal with the global existence and extinction behavior of the solution for a fast diffusion p-Laplace equation with logarithmic nonlinearity and special medium void. By analyzing the effect of the singular potential $|x|^{-s}$ and the logarithmic nonlinearity $|u|^{q-2}u\log|u|$ on the global existence and extinction behaviors of the solution, along with the modified energy estimates approach, Hardy-Littlewood-Sobolev inequality and some ordinary differential inequalities, the global existence of the solution is proved and sufficient conditions on the occurrence of the extinction and nonextinction phenomena are obtained.

Our next work is to study the numerical extinction and nonextinction phenomena of the parabolic problems like (1.1). We hope to give some numerical examples and applications for our theoretical researches in the near future.

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