Research Article

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# Riemannian invariants for warped product submanifolds in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$ and their applications

**Abstract:** This article investigates the geometric and topologic of warped product submanifolds in Riemannian warped product $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. In this respect, we obtain the first Chen inequality that involves extrinsic invariants like the length of the warping functions and the mean curvature. This inequality involves two intrinsic invariants (sectional curvature and $\delta$-invariant). In addition, an integral bound is provided for the Bochner operator formula of compact warped product submanifolds in terms of the Ricci curvature gradient. We aim to apply this theory to many structures and obtain Dirichlet eigenvalues for problem applications. Some new results regarding the vanishing mean curvature are presented as a partial solution, and this can be considered for the well-known problem given by Chern.

## 1 Introduction and motivations

As the warping function determines the properties of the warped product manifold, their geometry and physics are rich and varied. Studying these objects requires a fundamental understanding of how this function behaves. Recently, there has been a surge of interest in warped product manifolds, driven partly by their wide range of applications and their relationship with other mathematical fields. Thus, warped product manifolds can be applied in many fields of geometry and physics. For example, certain types of black hole spacetime can be modeled with warped product manifolds in general relativity. The study of vector bundles on algebraic varieties causes them to arise in algebraic geometry. The use of such manifolds has been used in topology for the construction of examples of exotic manifolds without smooth structures [1].

Many intrinsic invariants exist for (sub)manifolds in contemporary research. A significant physical and geometrical aspect of product manifolds extends beyond Hermitian geometry. According to Einstein's general relativity, their respective metrics determine the topology of three-dimensional space and one-dimensional time. In addition to Kaluza-Klein, brane, and gauge theories, product manifolds can be used to solve complex problems. It is also well-known that many researchers have investigated submanifolds and product manifolds. We need to give the new forms of Riemannian invariants that are distinct from "classical" invariants to

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address these issues. Furthermore, the new intrinsic invariants for submanifolds should be connected to the essential extrinsic invariants in a general optimum way. This is why Chen [2] introduced the notion of delta-invariants on Riemannian manifolds. It was the first author, in the 1990s, who introduced a new family of curvature functions on submanifolds. It is roughly described by an isometric immersion of a Riemannian manifold into a real space form that creates the least tension within the space around each point. According to Chen, the ideal submanifold is one that meets the equality condition, and numerous inequalities are derived as invariants of this submanifold. The ideal submanifolds of real and complex space forms have been described by Chen [2], Chen et al. [3]. Furthermore, Petrovic, Dillen, Verstraelen, Tripathi, and Ozgur studied conformally flat, semisymmetric, and Ricci-semisymmetric submanifolds obeying Chen's inequality in real space forms [4–6]. As a result, we discovered the following problems posed by Chen [4]:

**Problem 1.1.** Suppose that $\mathcal{B}_1 \times_f \mathcal{B}_2$ is an arbitrary warped product isometrically immersed in $\mathbb{E}^m$ (or in $\mathbb{Q}^m(c)$) as Riemannian submanifold. What are the connections between the warping function and the extrinsic structures $\mathcal{B}_1 \times_f \mathcal{B}_2$?

**Problem 1.2.** Given a warped product $\mathcal{B}_1 \times_f \mathcal{B}_2$, what are the necessary conditions for the warped product to admit minimal isometric immersion in a Euclidean $m$-space $\mathbb{E}^m$ (or in $\mathbb{Q}^m(c)$)?

Aside from Chen's optimal inequality for CR-warped products in complex space [7], there have not been many studies on $\delta$-invilliant for warped product structure. It has recently been reported that Mustafa et al. [8] and other authors in [9–11] have constructed the first Chen invariant for warped product submanifolds on real space forms and documented the minimality conditions for them. Conversely, product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ have been intensively researched in recent years [12–16]. For example, $n = 2$, they are two of the eight models proposed by Thurston in his geometrization conjecture. A generalized Hopf differential was presented by Abresch and Rosenberg in [17] and used to categorize constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. Since then, numerous studies have been published on this topic about the geometry of submanifolds in those product spaces, the majority of them are focused on submanifolds with constant mean curvature or minimal submanifolds. The product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ demonstrate greater submanifold geometrically characteristics richer than the space forms $\mathbb{S}^n$, $\mathbb{R}^n$, and $\mathbb{H}^n$. In this context, the Ricci and Codazzi and Gauss equations are essential tools in the study of submanifolds. The Codazzi and Gauss equations are defined intrinsically on submanifolds when the ambient space is a space form. However, the vertical vector $\frac{\partial}{\partial t}$ concerning the $\mathbb{R}$ direction is implicated in the preceding product spaces. We shall provide the basic equations for an $n$-dimensional manifold $\mathcal{B}^n$ isometrically embedded into $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$ in Section 2 such that $\mathbb{Q}_\varepsilon^m$ stands for a real space forms with constant sectional curvature $\varepsilon$. The basic equations in such a scenario consist of only the Ricci and Codazzi and Gauss equations but also two additional equations generated by $\frac{\partial}{\partial t}$'s parallelism. Let $\mathcal{T}$ represent $\frac{\partial}{\partial t}$'s projection on $\mathcal{B}^n$. Because $\mathcal{T}$ appears in both the basic equations and the fundamental theorems for isometric immersions in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$, the first fundamental form is a symmetric 2-tensor that measures the scalar curvature of the submanifold, while the second fundamental form is the normal curvature tensor that measures the curvature of the submanifold in the normal direction. This invariant $\mathcal{T}$ is important because it is used to calculate the length of curves on the submanifold and to define the submanifold's area.

Here, we present a new approach for constructing $\delta$-invariant curvature inequalities for warped product submanifolds immersed in Riemannian product manifolds of the type $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$, which has been discussed in [14]. Several inequalities for Euclidean spaces and hyperbolic spaces and spheres were also generalized based on the main results.

# 2 Preliminaries

Suppose that $(\mathcal{B}, g)$ represents a Riemannian manifold of dimension $m$. In Riemannian space form, a Riemannian manifold is described by the following relation for Riemannian curvature tensor $\mathcal{R}$:

$$\mathcal{R}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) = \varepsilon\{g(\mathcal{Z}_1, \mathcal{Z}_4)g(\mathcal{Z}_2, \mathcal{Z}_3) - g(\mathcal{Z}_2, \mathcal{Z}_4)g(\mathcal{Z}_1, \mathcal{Z}_3)\}, \tag{2.1}$$

for any $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4 \in \mathfrak{X}(\mathcal{B})$ and it is denoted by $\mathcal{B}^m(\varepsilon)$.

If $\nabla$ and $\nabla^\perp$ are generated connections on the tangent bundle $T\mathcal{B}^n$ and normal bundle $T^\perp\mathcal{B}^n$ of $\mathcal{B}^n$ $n$-dimensional Riemannian submanifold of an $m$-dimensional Riemannian manifold $\widetilde{\mathcal{B}}^m$ with generated metric $g$, respectively. Following are the Gauss and Weingarten formulas

$$\text{(i)} \quad \widetilde{\nabla}_{\mathcal{Z}_1}\mathcal{Z}_2 = \nabla_{\mathcal{Z}_1}\mathcal{Z}_2 + \Pi(\mathcal{Z}_1, \mathcal{Z}_2), \qquad \text{(ii)} \quad \widetilde{\nabla}_{\mathcal{Z}_1}N = -A_N\mathcal{Z}_1 + \nabla^\perp_{\mathcal{Z}_1}N, \tag{2.2}$$

for each $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathfrak{X}(\mathcal{B}^n)$ and $N \in \mathfrak{X}^\perp(\mathcal{B}^n)$, where $\widetilde{\nabla}$ is the connection on $\widetilde{\mathcal{B}}^m$, and $\Pi$ and $A_N$ are the second fundamental form and shape operator (corresponding to the normal vector field $N$), respectively. Their relationship is as follows:

$$g(\Pi(\mathcal{Z}_1, \mathcal{Z}_2), N) = g(A_N\mathcal{Z}_1, \mathcal{Z}_2). \tag{2.3}$$

As well, Gauss' and Codazzi's equations are as follows:

$$R(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) = \widetilde{R}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) + g(\Pi(\mathcal{Z}_1, \mathcal{Z}_4), \Pi(\mathcal{Z}_2, \mathcal{Z}_3)) - g(\Pi(\mathcal{Z}_1, \mathcal{Z}_3), \Pi(\mathcal{Z}_2, \mathcal{Z}_4)). \tag{2.4}$$

$$(\widetilde{R}(\mathcal{Z}_1, \mathcal{Z}_2)\mathcal{Z}_3)^\perp = (\widetilde{\nabla}_{\mathcal{Z}_1}\Pi)(\mathcal{Z}_2, \mathcal{Z}_3) - (\widetilde{\nabla}_{\mathcal{Z}_2}\Pi)(\mathcal{Z}_1, \mathcal{Z}_3). \tag{2.5}$$

for all $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4 \in \mathfrak{X}(\widetilde{\mathcal{B}}^m)$, where $\widetilde{\mathcal{R}}$ and $\mathcal{R}$ are the curvature tensor of $\widetilde{\mathcal{B}}^m$ and $\mathcal{B}^n$, respectively. This is the mean curvature of Riemannian submanifolds $\mathcal{H}$

$$\mathcal{H} = \frac{1}{n}\text{trace}(\Pi). \tag{2.6}$$

A submanifold $\mathcal{B}^n$ of Riemannian manifold $\widetilde{\mathcal{B}}^m$ is said to be *totally umbilical* and *totally geodesic* if $\Pi(\mathcal{Z}_1, \mathcal{Z}_2) = g(\mathcal{Z}_1, \mathcal{Z}_1)\mathcal{H}$ and $\Pi(\mathcal{Z}_1, \mathcal{Z}_2) = 0$, for any $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathfrak{X}(\mathcal{B}^n)$, respectively. Moreover, if $\mathcal{H} = 0$, then $\mathcal{B}^n$ is called *minimal* in $\widetilde{\mathcal{B}}^m$. Furthermore, the kernel or the null space of the second fundamental form is defined as follows:

$$\mathcal{B}_x = \{\mathcal{Z} \in T_x\mathcal{B}^n : \Pi(\mathcal{Z}, \mathcal{Z}_1) = 0, \text{ for all } \mathcal{Z}_1 \in T_x(\mathcal{B}^n)\}. \tag{2.7}$$

In this context, if we have $\{e_1, ..., e_m\}$ be an orthonormal basis of the tangent space $T_x\widetilde{\mathcal{B}}^m$ at some $x$ in $\widetilde{\mathcal{B}}^m$, we give the definition of another important Riemannian intrinsic invariant called the scalar curvature of $\widetilde{\mathcal{B}}^m$, and denoted at $\widetilde{\tau}(x)$, which is given by

$$\widetilde{\tau}(x) = \sum_{1 \le i < j \le m} \widetilde{\mathcal{K}}_{ij}, \tag{2.8}$$

$\widetilde{\mathcal{K}}_{ij} = \widetilde{\mathcal{K}}(e_i \wedge e_j)$ stand for the sectional curvature of space form $\widetilde{\mathcal{B}}^m(c)$. In a later proof, it will be frequently used to illustrate that first equality (2.8) is congruent with the following equation:

$$2\widetilde{\tau}(x) = \sum_{1 \le i \ne j \le m} \widetilde{\mathcal{K}}_{ij}, \tag{2.9}$$

and the scalar curvature $\widetilde{\tau}(L)$ of $r$-dimensional subspace of $T_x\widetilde{\mathcal{B}}^m$ is as follows:

$$\widetilde{\tau}(L) = \sum_{1 \le i < j \le r} \widetilde{\mathcal{K}}_{ij}. \tag{2.10}$$

Suppose that $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space $T_x\mathcal{B}^n$ and if $e_r \in \{e_{n+1}, ..., e_m\}$ an orthonormal basis of the normal space $T^\perp\mathcal{B}^n$, then we obtain

$$\Pi^r_{ij} = g(\Pi(e_i, e_j), e_r) \quad \text{and} \quad ||\Pi||^2 = \sum_{r=n+1}^{m} \sum_{i,j=1}^{n} (\Pi^r_{ij})^2. \tag{2.11}$$

Suppose that $\mathcal{K}_{ij}$ denotes the sectional curvatures of the plane section spanned by $\{e_i, e_j\}$ in the submanifold $\mathcal{B}^n$. Therefore, $\widetilde{\mathcal{K}}_{ij}$ and $\mathcal{K}_{ij}$ are the extrinsic and intrinsic sectional curvature of the span $\{e_i, e_j\}$ at $x$; therefore, from the Gauss equation (2.4), we obtain

$$\mathcal{K}_{ij} = \widetilde{\mathcal{K}}_{ij} + \sum_{r=n+1}^{m} (\Pi_{ii}^r \Pi_{jj}^r - (\Pi_{ij}^r)^2). \tag{2.12}$$

A second invariant is known as the Chen first invariant, and the definition is given as $x \in \widetilde{\mathcal{B}}^m$:

$$\delta_{\widetilde{\mathcal{B}}^m}(x) = \tilde{\tau}(x) - \inf\{\widetilde{\mathcal{K}}(\pi) : \pi \subset T_x \widetilde{\mathcal{B}}^m, \dim \pi = 2\}. \tag{2.13}$$

Assume that $(\mathcal{B}_1^{d_1}, g_1)$ and $(\mathcal{B}_2^{d_2}, g_2)$ are two Riemannian manifolds. Suppose that $f$ is a smooth function defined on $\mathcal{B}_1^{d_1}$. Then, the warped product manifold $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ of the manifold $\mathcal{B}_1^{d_1} \times \mathcal{B}_2^{d_2}$ is furnished by the Riemannian metric $g = g_1 + f^2 g_2$. Then, for any $\mathcal{Z}_1 \in \mathfrak{X}(\mathcal{B}^{d_1})$ and $\mathcal{Z}_2 \in \mathfrak{X}(\mathcal{B}^{d_2})$, we obtain

$$\nabla_{\mathcal{Z}_2} \mathcal{Z}_1 = \nabla_{\mathcal{Z}_1} \mathcal{Z}_2 = (\mathcal{Z}_1 \ln f) \mathcal{Z}_2. \tag{2.14}$$

According to units, vector fields $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are tangent to $\mathcal{B}_1^{d_1}$ and $\mathcal{B}_2^{d_2}$, respectively, and thus derive

$$\begin{aligned}
\mathcal{K}(\mathcal{Z}_1 \wedge \mathcal{Z}_2) &= g(R(\mathcal{Z}_1, \mathcal{Z}_2)\mathcal{Z}_1, \mathcal{Z}_2) \\
&= (\nabla_{\mathcal{Z}_1}\mathcal{Z}_1)\ln f g(\mathcal{Z}_2, \mathcal{Z}_2) - g(\nabla_{\mathcal{Z}_1}((\mathcal{Z}_1 \ln f)\mathcal{Z}_2), \mathcal{Z}_2) \\
&= (\nabla_{\mathcal{Z}_1}\mathcal{Z}_1)\ln f g(\mathcal{Z}_2, \mathcal{Z}_2) - g(\nabla_{\mathcal{Z}_1}(\mathcal{Z}_1 \ln f)\mathcal{Z}_2 + (\mathcal{Z}_1 \ln f)\nabla_{\mathcal{Z}_1}\mathcal{Z}_2, \mathcal{Z}_2) \\
&= (\nabla_{\mathcal{Z}_1}\mathcal{Z}_1)\ln f - (\mathcal{Z}_1 \ln f)^2 - \mathcal{Z}_1(\mathcal{Z}_1 \ln f).
\end{aligned} \tag{2.15}$$

Suppose that $\{e_1, ..., e_n\}$ is an orthonormal frame for $\mathcal{B}^n$, then summing up over the vector fields such that

$$\sum_{j=d_1+1}^{n} \sum_{i=1}^{d_1} \mathcal{K}(e_i \wedge e_j) = \sum_{j=d_1+1}^{n} \sum_{i=1}^{d_1} ((\nabla_{e_i} e_i)\ln f - e_i(e_i \ln f) - (e_i \ln f)^2),$$

which implies that

$$\sum_{j=d_1+1}^{n} \sum_{i=1}^{d_1} \mathcal{K}(e_i \wedge e_j) = d_2(\Delta(\ln f) - \|\nabla(\ln f)\|^2), \quad n = d_1 + d_2. \tag{2.16}$$

Nevertheless, for arbitrary warped product submanifolds, it has been shown in [4] that

$$\sum_{j=d_1+1}^{n} \sum_{i=1}^{d_1} \mathcal{K}(e_i \wedge e_j) = \frac{d_2 \Delta f}{f}. \tag{2.17}$$

Thus, from (2.16) and (2.17), we obtain

$$\frac{\Delta f}{f} = \Delta(\ln f) - \|\nabla(\ln f)\|^2. \tag{2.18}$$

Warped product submanifolds tend to result in the following remarks:

**Remark 2.1.** A warped product manifold $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is said to be *trivial* if the warping function $f$ is constant or simply a Riemannian product manifold.

**Remark 2.2.** If $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is a warped product manifold, then $\mathcal{B}_1^{d_1}$ is totally geodesic and $\mathcal{B}_2^{d_2}$ is totally umbilical submanifold of $\mathcal{B}^n$.

A key algebraic figure can be found in the following lemma:

**Lemma 2.1.** *Let $l_1, l_2, ..., l_n, s$ be $(n + 1)(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} (t_i)\right)^2 = (d_1)\left(\sum_{i=1}^{n} t_i^2 + s\right). \tag{2.19}$$

*Then, $2l_1 l_2 \geq s$, with equality holds if and only if $l_1 + l_2 = l_3 = ... = l_n$.*

Let $\mathcal{B}^n$ be an $n$-dimensional submanifold of $\mathbb{Q}_{\varepsilon}^m \times \mathbb{R}$ with codimension $p$ that $m + 1 = n + p$. Assume that the $\mathcal{R}$ stands for curvature operator for $\mathcal{B}^n$ and $\widetilde{\mathcal{R}}$ for $\mathbb{Q}_{\varepsilon}^m \times \mathbb{R}$, respectively. Let $\mathcal{A}$ is the second fundamental form of $\mathcal{B}^n$ such that $\Pi : \mathfrak{X}(\mathcal{B}^n) \odot \mathfrak{X}(\mathcal{B}^n) \to \mathfrak{X}^{\perp}(\mathcal{B}^n)$ as $g(\mathcal{A}_{\mu}\mathcal{Z}_1, \mathcal{Z}_2) = g(\Pi(\mathcal{Z}_1, \mathcal{Z}_2), \mu)$, for all $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathfrak{X}(\mathcal{B}^n)$ and $\mu \in \mathfrak{X}^{\perp}(\mathcal{B}^n)$. If the squared norm of $\mathcal{A}$ and $\Pi$ are equal, then $\|A\|^2 = \|\Pi\|^2$, where $\Pi$ is also a second fundamental form of $\mathcal{B}^n$. If $\{e_1, e_2, ..., e_n\}$ be a local orthonormal frame of tangent bundle and $\{\xi_1, \xi_2, ..., \xi_p\}$ be an orthonormal frame for normal bundle, then we have

$$\frac{\partial}{\partial t} = \mathcal{T} + \sum_{\alpha} v_{\alpha}\xi_{\alpha} = \sum_{i} \mathcal{T}_i e_i + \sum_{\alpha} v_{\alpha}\xi_{\alpha}, \tag{2.20}$$

where $\mathcal{T}_i$ and $v_{\alpha}$ stand for smooth functions for all $1 \le i \le n$ and $1 \le \alpha \le p$. The following formulas are given as fundamental equations for submanifold $\mathcal{B}^n$ isometrically immersed into $\mathbb{Q}_{\varepsilon}^m \times \mathbb{R}$ [14]:

$$\begin{aligned} g(\mathcal{R}(\mathcal{Z}_1, \mathcal{Z}_2)\mathcal{Z}_3, \mathcal{Z}_4) = \varepsilon\{&g(\mathcal{Z}_2, \mathcal{Z}_3)g(\mathcal{Z}_1, \mathcal{Z}_4) + g(\mathcal{Z}_1, \mathcal{T})g(\mathcal{Z}_1, \mathcal{Z}_3)g(\mathcal{Z}_4, \mathcal{T}) \\ &+ g(\mathcal{Z}_1, \mathcal{T})g(\mathcal{Z}_3, \mathcal{T})g(\mathcal{Z}_2, \mathcal{Z}_4) - g(\mathcal{Z}_1, \mathcal{Z}_3)g(\mathcal{Z}_2, \mathcal{Z}_4) \\ &- g(\mathcal{Z}_1, \mathcal{T})g(\mathcal{Z}_2, \mathcal{Z}_3)g(\mathcal{Z}_4, \mathcal{T}) - g(\mathcal{Z}_2, \mathcal{T})g(\mathcal{Z}_3, \mathcal{T})g(\mathcal{Z}_1, \mathcal{Z}_4)\} \\ &+ g(\Pi(\mathcal{Z}_2, \mathcal{Z}_3), \Pi(\mathcal{Z}_1, \mathcal{Z}_4)) - g(\Pi(\mathcal{Z}_1, \mathcal{Z}_3), \Pi(\mathcal{Z}_2, \mathcal{Z}_4)), \end{aligned} \tag{2.21}$$

$$g((\nabla_{\mathcal{Z}_1}\Pi)(\mathcal{Z}_2, \mathcal{Z}_3), \xi_{\alpha}) - g((\nabla_{\mathcal{Z}_2}\Pi)(\mathcal{Z}_1, \mathcal{Z}_3), \xi_{\alpha}) = \varepsilon v_{\alpha}(g(\mathcal{Z}_1, \mathcal{T})\mathcal{Z}_2 - g(\mathcal{T}, \mathcal{Z}_2)\mathcal{Z}_1), \tag{2.22}$$

$$g(\mathcal{R}^{\perp}(\mathcal{Z}_1, \mathcal{Z}_2)\xi, \eta) = g([\mathcal{A}_{\xi}, \mathcal{A}_{\eta}]\mathcal{Z}_1, \mathcal{Z}_2), \tag{2.23}$$

$$\nabla_{\mathcal{Z}_1}\mathcal{T} = \sum_{j} v_j \mathcal{A}_j(\mathcal{Z}_1), \tag{2.24}$$

$$dv_{\alpha}(\mathcal{Z}_1) = \sum_{j} v_{\alpha}\omega_{\alpha}^{j}(\mathcal{Z}_1) - g(\mathcal{A}_{\alpha}(\mathcal{Z}_1), \mathcal{T}), \tag{2.25}$$

for all $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4 \in \mathfrak{X}(\mathcal{B}^n)$, and $\xi, \eta \in \mathfrak{X}^{\perp}(\mathcal{B}^n)$. As a result of a straightforward calculation, we obtain the Gauss equation as follows:

$$\mathcal{R}_{ijkl} = \varepsilon((\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \mathcal{T}_j\mathcal{T}_k\delta_{il} + \mathcal{T}_i\mathcal{T}_l\delta_{jk} - \mathcal{T}_i\mathcal{T}_k\delta_{jl} - \mathcal{T}_j\mathcal{T}_l\delta_{ik}) + \sum_{r}(\Pi_{ik}^{r}\Pi_{jl}^{r} - \Pi_{il}^{r}\Pi_{kk}^{r}). \tag{2.26}$$

We are giving an example of a minimal surface in product spaces.

**Example 2.1.** [18] The horizontal sphere $\mathbb{S}^2 \times \mathbb{R}$ and vertical cylinders $\mathbb{S}^1 \times \mathbb{R}$ are examples of product spaces and helicoids most famous examples of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$. Let us assume that $\mathcal{M}_{\beta}$ is a helicoid such that $\beta \ne 0$. We define conformal immersion as

$$\phi(u_1, v_1) = \begin{pmatrix} \sin\phi(u_1)\cos\beta v_1 \\ \sin\phi(u_1)\sin\beta v_1 \\ \cos\phi(u_1) \\ v_1 \end{pmatrix}$$

such that $\phi$ satisfied the relation $\phi'(u_1)^2 = 1 + \beta^2 \sin^2\phi(u_1)$, $\phi''(u_1)^2 = \beta^2 \sin\phi(u_1)\cos\phi(u_1)$, if we consider that $\phi(0) = 0$ and $\phi'(u_1) > 0$. We can classify that $\mathcal{M}_{\beta}$ is a right helicoid if $\beta > 0$ and left helicoid if $\beta < 0$. The normal to $\mathbb{S}^2 \times \mathbb{R}$ in $\mathbb{R}^4$ is given by

$$\widetilde{N}(u_1, v_2) = \begin{pmatrix} \sin\phi(u_1)\cos\beta v_1 \\ \sin\phi(u_1)\sin\beta v_1 \\ \cos\phi(u_1) \\ 0 \end{pmatrix}.$$

The normal to $\mathcal{M}_\beta$ in $\mathbb{S}^2 \times \mathbb{R}$ is given by

$$\widetilde{N}(u_1, v_2) = \frac{1}{\phi'(u_1)}\begin{pmatrix} \sin\beta v_1 \\ \cos\beta v_1 \\ 0 \\ \beta\sin\phi(u_1) \end{pmatrix}.$$

Now, we calculate that as

$$g\left(\frac{\partial^2 x}{\partial u_1^2}, N\right) = g\left(\frac{\partial^2 x}{\partial v_1^2}, N\right) = 0, \qquad g\left(\frac{\partial^2 x}{\partial u_1 \partial v_1}, N\right) = -\beta\cos\phi(u_1).$$

Therefore, $g(S\mathcal{Z}_1, \mathcal{Z}_2) = g(\mathrm{d}\mathcal{Z}_2(\mathcal{Z}_1), \mathcal{N})$, we derive the matrix of $S$ in the frame $\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$ is $-\frac{\beta\cos\phi(u_1)}{\phi'(u_1)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For umbilical point such that $\cos\phi(u_1) = 0$. We find that $\mathcal{T} = -\frac{1}{\phi'(u_1)^2}\frac{\partial}{\partial v_1}$ and $\nu = \frac{\beta\sin\phi(u_1)}{\phi'(u_1)}$. If $\beta = 0$, the formula defines a vertical cylinder $\mathbb{S}^2 \times \mathbb{R}$ and if $\beta \to \infty$, the surface reduces to horizontal sphere $\mathbb{S}^2 \times \{t\}$.

# 3 Main results

The results presented in this section provide solutions to Problems 1.1 and 1.2.

**Theorem 3.1.** *Assuming that $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is an isometric immersion into a product manifold $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. Then, for each point $x \in \mathcal{B}^n$ and each plane section $\pi_i \subset T_x\mathcal{B}_i^{d_i}$, for $i = 1, 2$, we obtain*
*(1) Let $\pi_1 \subset T_x\mathcal{B}_1^{d_1}$, then*

$$\delta_{\mathcal{B}_1^{d_1}}(x) \le \frac{n^2}{2}\|\mathcal{H}\|^2 + d_2\|\nabla(\ln f)\|^2 - d_2\Delta(\ln f) + \left\{\frac{d_1}{2}(d_1 + 2d_2 - 1) - (d_1 - 1)\|\mathcal{T}\|^2 - 1\right\}\varepsilon. \tag{3.1}$$

*It is only possible to compute the equality of the above inequality at $x \in \mathcal{B}^n$ if and only if there exists an orthonormal basis $\{e_1, ..., e_n\}$ of $T_x\mathcal{B}^n$ and an orthonormal basis $\{e_{n+1}, ..., e_m\}$ of $T_x^\perp\mathcal{B}^n$ such that*
*(a) $\pi_1 = \mathrm{Span}\{e_1, e_2\}$ and*
*(b) Shape operators can be expressed as follows:*

$$(i) \quad A_{e_{n+1}} = \left[\begin{array}{cccccc|cccc} \mu_1 & \Pi_{12}^{n+1} & 0 & \cdots & & 0_{1d_1} & 0_{1d_1+1} & \cdots & & 0_{1n} \\ \Pi_{12}^{n+1} & \mu_2 & 0 & \cdots & & \vdots & \vdots & \cdots & & \vdots \\ 0 & 0 & \mu & \cdots & & \vdots & \vdots & \cdots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \cdots & & \vdots \\ 0_{d_11} & 0 & 0 & \cdots & \mu & 0_{d_1d_1+1} & \cdots & & 0_{d_1n} \\ \hline 0_{d_1+11} & \cdots & \cdots & \cdots & & 0_{d_1+1d_1} & \Pi_{d_1+1d_1+1}^{n+1} & \cdots & & \Pi_{d_1+1n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & & 0_{nd_1} & \Pi_{nd_1+1}^{n+1} & \cdots & & \Pi_{nn}^{n+1} \end{array}\right],$$

*where $\mu = \mu_1 + \mu_2$. If $r \in \{n + 2, ..., m\}$, then it calculates the matrix*

$$(ii) \quad A_{e_r} = \left[\begin{array}{cccccc|cccc} \Pi_{11}^r & \Pi_{12}^r & 0 & \cdots & & 0_{1d_1} & 0_{1d_1+1} & \cdots & & 0_{1n} \\ \Pi_{21}^r & -\Pi_{11}^r & 0 & \cdots & & \vdots & \vdots & \cdots & & \vdots \\ 0 & 0 & 0_{33} & \cdots & & \vdots & \vdots & \cdots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \cdots & & \vdots \\ 0_{d_11} & 0 & 0 & \cdots & & 0_{d_1d_1} & 0_{d_1d_1+1} & \cdots & & 0_{d_1n} \\ \hline 0_{d_1+11} & \cdots & \cdots & \cdots & & 0_{d_1+1d_1} & \Pi_{d_1+1d_1+1}^r & \cdots & & \Pi_{d_1+1n}^r \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & & 0_{nd_1} & \Pi_{nd_1+1}^r & \cdots & & \Pi_{nn}^r \end{array}\right].$$

(2) *If $\pi_2 \subset T_x\mathcal{B}_2^{d_2}$, then*

$$\delta_{\mathcal{B}_2^{d_2}}(x) \le \frac{n^2}{2}||\mathcal{H}||^2 + d_2||\nabla(\ln f)||^2 - d_2\Delta(\ln f) + \left\{\frac{d_2}{2}(d_2 + 2d_1 - 1) - (d_2 - 1)||\mathcal{T}||^2 - 1\right\}\varepsilon. \tag{3.2}$$

It follows that the above equation has equalities if and only if there exists an orthonormal basis $\{e_1, ..., e_n\}$ of $T_x\mathcal{B}^n$ and an orthonormal basis $\{e_{n+1}, ..., e_m\}$ of $T_x^\perp\mathcal{B}^n$ such that

(a) $\pi_2 = \text{Span}\{e_{d_1+1}, e_{d_1+2}\}$ and

(b) the Shape operators are represented as follows:

$$(iii) \quad A_{e_{n+1}} = \begin{bmatrix} \Pi_{11}^{n+1} & \cdots & \cdots & \Pi_{1d_1}^{n+1} & 0_{1d_1+1} & \cdots & & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \Pi_{d_11}^{n+1} & \cdots & \cdots & \Pi_{d_1d_1}^{n+1} & 0_{d_1d_1+1} & & \cdots & & \cdots & 0_{d_1n} \\ 0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \mu_1 & \Pi_{d_1+1d_1+2}^{n+1} & 0 & \cdots & & 0_{d_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \Pi_{d_1+2d_1+1}^{n+1} & \mu_2 & 0 & \cdots & & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & & 0 & \mu \end{bmatrix},$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n + 2, ...,m\}$, thus we have

$$(iv) \quad A_{e_r} = \begin{bmatrix} \Pi_{11}^{r} & \cdots & \cdots & \Pi_{1d_1}^{r} & 0_{1d_1+1} & \cdots & & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \Pi_{d_11}^{r} & \cdots & \cdots & \Pi_{d_1d_1}^{r} & 0_{d_1d_1+1} & & \cdots & & \cdots & 0_{d_1n} \\ 0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \Pi_{d_1+1d_1+1}^{r} & \Pi_{d_1+1d_1+2}^{r} & 0 & \cdots & & 0_{d_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \Pi_{d_1+2d_1+1}^{r} & -\Pi_{d_1+1d_1+1}^{r} & 0 & \cdots & & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & & 0 & 0 \end{bmatrix}.$$

(v) *In the case of equality in (1) or (2), the warped product $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_1}$ is mixed totally geodesic in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. Moreover, $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is both $\mathcal{B}_1^{d_1}$-minimal and $\mathcal{B}_2^{d_2}$-minimal. Thus, $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is a minimal warped product submanifold in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$.*

**Proof.** Let $\pi_1 \subset T_x\mathcal{B}_1^{d_1}$ be a 2-plane for $x \in \mathcal{B}^n$, then we consider the orthonormal basis $\{e_1, ..., e_{d_1}, e_{d_1+1}, ..., e_n\}$ of $T_x\mathcal{B}^n$ such that $\{e_1, ..., e_{d_1}\}$ is an orthonormal basis for $T_x\mathcal{B}_1^{d_1}$ and $\{e_{d_1+1}, ..., e_n\}$ is for $T_x\mathcal{B}_2^{d_2}$. Similarly, $\{e_{n+1}, ..., e_m\}$ is an orthonormal basis for $T_x^\perp\mathcal{B}^n$. Given $\pi = \text{Span}\{e_1, e_2\}$, the normal vector $e_{n+1}$ lies in the direction of the mean curvature vector $H$, thus from (2.1) and (2.26), we obtain

$$n^2||H||^2 + n(n - 1)\left(1 - \frac{2}{n}||\mathcal{T}||^2\right)\varepsilon = 2\tau(x) + ||\Pi||^2, \tag{3.3}$$

which implies that

$$\left(\sum_{i=1}^{d_1}\Pi_{ii}^{n+1}\right)^2 = 2\tau(x) + ||\Pi||^2 - n(n-1)\left(1 - \frac{2}{n}||\mathcal{T}||^2\right)\varepsilon - \left(\sum_{j=d_1+1}^{n}\Pi_{jj}^{n+1}\right)^2 - 2\sum_{A=1}^{d_1}\sum_{B=d_1+1}^{n}\Pi_{AA}^{n+1}\Pi_{BB}^{n+1}. \tag{3.4}$$

Taking these factors into account,

$$\Omega = 2\tau(x) - n(n-1)\left(1 - \frac{2}{n}||\mathcal{T}||^2\right)\varepsilon - \frac{(d_1 - 2)}{(d_1 - 1)}\left(\sum_{i=1}^{d_1}\Pi_{ii}^{n+1}\right)^2 - \left(\sum_{j=d_1+1}^{n}\Pi_{jj}^{n+1}\right)^2 - 2\sum_{A=1}^{d_1}\sum_{B=d_1+1}^{n}\Pi_{AA}^{n+1}\Pi_{BB}^{n+1}. \tag{3.5}$$

The following is the result of looking at (3.4) and (3.5):

$$\left(\sum_{i=1}^{d_1} \Pi_{ii}^{n+1}\right)^2 = (d_1 - 1)(\Omega + ||\Pi||^2). \tag{3.6}$$

The above equation can be expressed as

$$\left(\sum_{i=1}^{d_1} \Pi_{ii}^{n+1}\right)^2 = (d_1 - 1)\left[\Omega + \sum_{i=1}^{d_1}(\Pi_{ii}^{n+1})^2 + \sum_{j=d_1+1}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{i\neq j=1}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(\Pi_{ij}^{r})^2\right]. \tag{3.7}$$

Thus, if Lemma 2.1 is applied to the above equation, it becomes, i.e.,

$$t_\alpha = \Pi_{\alpha\alpha}^{n+1}, \quad \forall \alpha \in \{1, ..., d_1\}$$

and

$$s = \Omega + \sum_{j=d_1+1}^{n}(\Pi_{jj}^{n+1})^2 + \sum_{i\neq j=1}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(\Pi_{ij}^{r})^2.$$

Thus, we obtain that

$$\Pi_{11}^{n+1}\Pi_{22}^{n+1} \geq \frac{1}{2}\left\{\Omega + \sum_{j=d_1+1}^{n}(\Pi_{jj}^{n+1})^2 + \sum_{i\neq j=1}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(\Pi_{ij}^{r})^2\right\}. \tag{3.8}$$

We derive the following values from (2.1) and (2.12):

$$K(\pi_1) = (1 - ||\mathcal{T}||^2)\varepsilon + \sum_{r=n+1}^{m}(\Pi_{11}^{r}\Pi_{22}^{r} - (\Pi_{12}^{r})^2). \tag{3.9}$$

Equations (3.8) and (3.9) are combined to give

$$K(\pi_1) \geq (1 - ||\mathcal{T}||^2)\varepsilon + \frac{1}{2}\Omega + \frac{1}{2}\sum_{j=d_1+1}^{n}(\Pi_{jj}^{n+1})^2$$
$$+ \sum_{r=n+1}^{m}(\Pi_{11}^{r}\Pi_{22}^{r} - (\Pi_{12}^{r})^2) + \frac{1}{2}\sum_{i\neq j=1}^{n}(\Pi_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(\Pi_{ij}^{r})^2. \tag{3.10}$$

Taking the last two terms from the equation above, we can obtain the following result:

$$\sum_{\substack{i,j=1\\i\neq j}}^{n}(\Pi_{ij}^{n+1})^2 = \sum_{\substack{i,j=3\\i\neq j}}^{n}(\Pi_{ij}^{n+1})^2 + 2\sum_{j=3}^{n}(\Pi_{1j}^{n+1})^2 + 2(\Pi_{12}^{n+1})^2 + 2\sum_{j=3}^{n}(\Pi_{2j}^{n+1})^2. \tag{3.11}$$

Moreover, for the last term, we obtain

$$\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(\Pi_{ij}^{r})^2 = \sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(\Pi_{ij}^{r})^2 + 2\sum_{r=n+2}^{m}\sum_{j=3}^{n}(\Pi_{1j}^{r})^2 + 2\sum_{r=n+2}^{m}\sum_{j=3}^{n}(\Pi_{2j}^{r})^2 + 2(\Pi_{12}^{r})^2 + \sum_{r=n+2}^{m}((\Pi_{11}^{r})^2 + (\Pi_{22}^{r})^2). \tag{3.12}$$

Furthermore, we have

$$\sum_{r=n+2}^{m}\Pi_{11}^{r}\Pi_{22}^{r} + \frac{1}{2}\sum_{r=n+2}^{m}((\Pi_{11}^{r})^2 + (\Pi_{22}^{r})^2) = \frac{1}{2}\sum_{r=n+2}^{m}(\Pi_{11}^{r} + \Pi_{22}^{r})^2, \tag{3.13}$$

$$\sum_{j=3}^{n}((\Pi_{1j}^{n+1})^2 + (\Pi_{2j}^{n+1})^2) + \sum_{r=n+2}^{m}\sum_{j=3}^{n}(\Pi_{1j}^{r})^2 + \sum_{r=n+2}^{m}\sum_{j=3}^{n}(\Pi_{2j}^{r})^2 = \sum_{r=n+1}^{m}\sum_{j=3}^{n}\{(\Pi_{1j}^{r})^2 + (\Pi_{2j}^{r})^2\}. \tag{3.14}$$

After adding (3.11) and (3.12), then using (3.13) and (3.14), and taking account that $(\Pi_{12}^{n+1})^2 + \sum_{r=n+2}^{m}(\Pi_{12}^{n+1})^2 = \sum_{r=n+1}^{m}(\Pi_{12}^{n+1})^2$, we obtain

$$
\begin{aligned}
&\sum_{\substack{i,j=1 \\ i\neq j}}^{n} (\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(\Pi_{ij}^{r})^2 \\
&= 2\sum_{r=n+1}^{m}\sum_{j=3}^{n}\{(\Pi_{1j}^{r})^2 + (\Pi_{2j}^{r})^2\} + \sum_{\substack{i,j=3 \\ i\neq j}}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(\Pi_{jj}^{r})^2 - 2\sum_{r=n+2}^{m}\{\Pi_{11}^{r}\Pi_{22}^{r} - (\Pi_{12}^{r})^2\} \\
&\quad + \sum_{r=n+2}^{m}(\Pi_{11}^{r} + \Pi_{22}^{r})^2.
\end{aligned}
\tag{3.15}
$$

It follows from (3.10) and (3.15), one derives

$$
\begin{aligned}
K(\pi_1) \geq (1 - \|\mathcal{T}\|^2)\varepsilon + \frac{1}{2}\Omega + \frac{1}{2}\sum_{\beta=d_1+1}^{n}(\Pi_{\beta\beta}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{j=3}^{n}\{(\Pi_{2j}^{r})^2 + (\Pi_{2j}^{r})^2\} \\
+ \frac{1}{2}\left[\sum_{\substack{i,j=3 \\ i\neq j}}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(\Pi_{jj}^{r})^2\right] + \frac{1}{2}\sum_{r=n+2}^{m}(\Pi_{11}^{r} + \Pi_{22}^{r})^2,
\end{aligned}
$$

which implies that

$$
K(\pi_1) \geq (1 - \|\mathcal{T}\|^2)\varepsilon + \frac{1}{2}\left[\Omega + \sum_{\substack{i,j=3 \\ i\neq j}}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(\Pi_{jj}^{r})^2 + \sum_{\beta=d_1+1}^{n}(\Pi_{\beta\beta}^{n+1})^2\right].
$$

From (3.5), we arrive at

$$
\begin{aligned}
K(\pi_1) \geq (1 - \|\mathcal{T}\|^2)\varepsilon + \tau(x) + \frac{1}{2(d_1-1)}\left(\sum_{a=1}^{n}\Pi_{aa}^{n+1}\right)^2 - \frac{n^2}{2}\|\mathcal{H}\|^2 - \frac{n(n-1)}{2}\left(1 - \frac{2}{n}\|\mathcal{T}\|^2\right)\varepsilon \\
+ \frac{1}{2}\left[\sum_{\substack{i,j=3 \\ i\neq j}}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(\Pi_{jj}^{r})^2 + \sum_{\beta=d_1+1}^{n}(\Pi_{\beta\beta}^{n+1})^2\right].
\end{aligned}
\tag{3.16}
$$

We can obtain the following equation using (2.9) and (2.17) in (3.16):

$$
\begin{aligned}
K(\pi_1) \geq \tau_1(x) + \tau_2(x) + \frac{d_2\nabla f}{f} - \frac{n^2}{2}\|\mathcal{H}\|^2 + \left(1 - \frac{n(n-1)}{2} + (n-2)\|\mathcal{T}\|^2\right)\varepsilon \\
+ \frac{1}{2}\left[\sum_{\substack{i,j=3 \\ i\neq j}}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(\Pi_{ij}^{r})^2 + \sum_{\beta=d_1+1}^{n}(\Pi_{\beta\beta}^{n+1})^2\right],
\end{aligned}
$$

where $\tau_i(x)$, $i = 1, 2$, is the scalar curvature of $\mathcal{B}_i^{d_i}$. This implies that

$$
\begin{aligned}
\tau_1(x) - K(\pi_1) \leq \frac{n^2}{2}\|\mathcal{H}\|^2 - \tau_2(x) - \frac{d_2\nabla f}{f} + \left(\frac{n(n-1)}{2} - (n-2)\|\mathcal{T}\|^2 - 1\right)\varepsilon \\
- \frac{1}{2}\left[\sum_{\substack{i,j=3 \\ i\neq j}}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(\Pi_{ij}^{r})^2 + \sum_{\beta=d_1+1}^{n}(\Pi_{\beta\beta}^{n+1})^2\right].
\end{aligned}
\tag{3.17}
$$

The Gauss equation (2.4) for $\tau_2(x)$ gives us

$$\tau_2(x) = \frac{d_2(d_2-1)}{2}\left[1 - \frac{2}{d_2}\|\mathcal{T}\|^2\right]\varepsilon - \frac{1}{2}\sum_{r=n+1}^{m}\sum_{A,B=d_1+1}^{n}(\Pi_{AB}^r)^2 - \frac{1}{2}\sum_{r=n+1}^{m}(\Pi_{d_1+1d_1+1}^r + \ldots + \Pi_{nn}^r). \tag{3.18}$$

According to equations (3.17) and (3.18), we can state the following.

$$\tau_1(x) - K(\pi_1) \le \frac{n^2}{2}\|\mathcal{H}\|^2 - \frac{d_2(d_2-1)}{2}\left[1 - \frac{2}{d_2}\|\mathcal{T}\|^2\right]\varepsilon - \frac{1}{2}\left[\sum_{\substack{i,j=3\\i\ne j}}^{n}(\Pi_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(\Pi_{ij}^r)^2\right.$$

$$\left. + \sum_{\beta=d_1+1}^{n}(\Pi_{\beta\beta}^{n+1})^2 - \sum_{r=n+1}^{m}\sum_{A,B=d_1+1}^{n}(\Pi_{AB}^r)^2\right] + \left[\frac{n(n-1)}{2} - (n-2)\|\mathcal{T}\|^2 - 1\right]\varepsilon - \frac{d_2\nabla f}{f}. \tag{3.19}$$

Then, the last relation turns into

$$\tau_1(x) - K(\pi_1) \le \frac{n^2}{2}\|\mathcal{H}\|^2 - \frac{d_2(d_2-1)}{2}\left[1 - \frac{2}{d_2}\|\mathcal{T}\|^2\right]\varepsilon + \left[\frac{n(n-1)}{2} - (n-2)\|\mathcal{T}\|^2 - 1\right]\varepsilon$$

$$- \frac{1}{2}\left[\sum_{\substack{k,l=3\\k\ne l}}^{d_1}(\Pi_{kl}^{n+1})^2 + 2\sum_{k=3}^{m}\sum_{l=d_1+1}^{n}(\Pi_{kl}^{n+1})^2 + \sum_{\substack{A,B=1\\A\ne B}}^{d_1}(\Pi_{kl}^{n+1})^2\right.$$

$$+ \sum_{r=n+2}^{m}\sum_{k,l=3}^{d_1}(\Pi_{kl}^r)^2 + 2\sum_{r=n+2}^{m}\sum_{k=3}^{d_1}\sum_{A=d_1+1}^{n}(\Pi_{kl}^r)^2$$

$$\left. + \sum_{r=n+2}^{m}\sum_{A,B=d_1+1}^{n}(\Pi_{AB}^r)^2 + \sum_{\beta=d_1+1}^{n}(\Pi_{\beta\beta}^{n+1})^2 - \sum_{r=n+1}^{m}\sum_{A,B=d_1+1}^{n}(\Pi_{AB}^r)^2\right] - \frac{d_2\nabla f}{f}. \tag{3.20}$$

In order to solve the previous equation, we can use the following two relations:

$$\sum_{A=d_1+1}^{n}(\Pi_{AA}^{n+1})^2 + \sum_{\substack{A,B=d_1+1\\A\ne B}}^{n}(\Pi_{AB}^{n+1})^2 = \sum_{A,B=d_1+1}^{n}(\Pi_{AB}^{n+1})^2$$

and

$$\sum_{A,B=d_1+1}^{n}(\Pi_{AB}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{A,B=d_1+1}^{n}(\Pi_{AB}^r)^2 = \sum_{r=n+1}^{m}\sum_{A,B=d_1+1}^{n}(\Pi_{AB}^r)^2.$$

Assertion (3.20) follows as

$$\tau_1(x) - K(\pi_1) \le \left[\frac{d_1}{2}(d_1 + 2d_2 - 1) - (d_1-1)\|\mathcal{T}\|^2 - 1\right]\varepsilon$$

$$- \frac{1}{2}\left[\sum_{\substack{k,l=3\\k\ne l}}^{d_1}(\Pi_{kl}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{k,l=3}^{d_1}(\Pi_{kl}^r)^2 + 2\sum_{a=3}^{d_1}\sum_{\beta=d_1+1}^{n}(\Pi_{a\beta}^{n+1})^2 + 2\sum_{r=n+1}^{m}\sum_{A=3}^{d_1}\sum_{B=d_1+1}^{n}(\Pi_{AB}^r)^2\right]$$

$$+ \frac{n^2}{2}\|\mathcal{H}\|^2 - \frac{d_2\nabla f}{f}. \tag{3.21}$$

The first inequality of Theorem 3.1 holds from the above equation and (2.13). For the second case if $\pi_2 \subset T_x\mathcal{B}_2^{d_2}$, we consider $\pi_2 = \mathrm{Span}\{e_{d_1+1}, e_{d_1+2}\}$. Similar to the first case, the following results can be obtained:

$$\left(\sum_{a=d_1+1}^{n}\Pi_{aa}^{n+1}\right)^2 = 2\tau(x) + \|\Pi\|^2 - n(n-1)\left[1 - \frac{2}{n}\|\mathcal{T}\|^2\right]\varepsilon - \left(\sum_{\beta=1}^{d_1}\Pi_{\beta\beta}^{n+1}\right)^2 - 2\sum_{a=1}^{d_1}\sum_{\beta=d_1+1}^{n}\Pi_{aa}^{n+1}\Pi_{\beta\beta}^{n+1}.$$

Next, we will consider

$$\Psi = 2\tau(x) - n(n-1)\left(1 - \frac{2}{n}\|\mathcal{T}\|^2\right)\varepsilon - \frac{(d_1-2)}{(d_1-1)}\left(\sum_{\alpha=d_1+1}^{n}\Pi_{\alpha\alpha}^{n+1}\right)^2 - \left(\sum_{\beta=d_1+1}^{n}\Pi_{\beta\beta}^{n+1}\right)^2 - 2\sum_{\alpha=1}^{d_1}\sum_{\beta=d_1+1}^{n}\Pi_{\alpha\alpha}^{n+1}\Pi_{\beta\beta}^{n+1}.$$

The last two equation implies that

$$\left(\sum_{\alpha=d_1+1}^{n}\Pi_{\alpha\alpha}^{n+1}\right)^2 = (d_2-1)(\Psi + \|\Pi\|^2),$$

which implies that

$$\left(\sum_{\alpha=d_1+1}^{n}\Pi_{\alpha\alpha}^{n+1}\right)^2 = (d_2-1)\left\{\Psi + \left(\sum_{\alpha=1}^{d_1}\Pi_{\alpha\alpha}^{n+1}\right)^2 + \left(\sum_{\beta=d_1+1}^{n}\Pi_{\beta\beta}^{n+1}\right)^2 + \sum_{\substack{\alpha,\beta=1\\\alpha\neq\beta}}^{n}(\Pi_{\alpha\beta}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{\alpha=\beta=1}^{n}(\Pi_{\alpha\beta}^{r})^2\right\}. \tag{3.22}$$

Similarly, we obtain the following equation when we apply Lemma 2.1:

$$\Pi_{d_1+1d_1+1}^{n+1}\Pi_{d_1+2d_1+2}^{n+1} \geq \frac{1}{2}\left\{\Psi + \left(\sum_{\alpha=1}^{d_1}\Pi_{\alpha\alpha}^{n+1}\right)^2 + \sum_{\substack{\alpha,\beta=1\\\alpha\neq\beta}}^{n}(\Pi_{\alpha\beta}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{\alpha=\beta=1}^{n}(\Pi_{\alpha\beta}^{r})^2\right\}. \tag{3.23}$$

From (2.1) and (2.12), we find that

$$K(\pi_2) = (1 - \|\mathcal{T}\|^2)\varepsilon + \sum_{r=n+1}^{m}(\Pi_{d_1+1d_1+1}^{r}\Pi_{d_1+2d_1+2}^{r} - (\Pi_{d_1+1d_1+2}^{r})^2). \tag{3.24}$$

Using equations (3.23) and, respectively, (3.24), we obtain

$$K(\pi_2) \geq (1 - \|\mathcal{T}\|^2)\varepsilon + \sum_{r=n+1}^{m}(\Pi_{d_1+1d_1+1}^{r}\Pi_{d_1+2d_1+2}^{r} - (\Pi_{d_1+1d_1+2}^{r})^2)$$

$$\frac{1}{2}\left\{\Psi + \left(\sum_{\alpha=1}^{d_1}\Pi_{\alpha\alpha}^{n+1}\right)^2 + \sum_{\substack{\alpha,\beta=1\\\alpha\neq\beta}}^{n}(\Pi_{\alpha\beta}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{\alpha=\beta=1}^{n}(\Pi_{\alpha\beta}^{r})^2\right\}. \tag{3.25}$$

The second inequality of Theorem 3.1 can be found using the same method as (3.5) and (3.21). Considering the case, $\pi_1 \subset T_x\mathcal{B}_1^{d_1}$, then the equality holds if and only if in (3.8), (3.10), (3.17), (3.18), and (3.21), equalities are preserved. Based on this, we obtain the following result:

$$\Pi_{11}^{n+1} + \Pi_{22}^{n+1} = \Pi_{33}^{n+1} = \ldots = \Pi_{d_1d_1}^{n+1}, \tag{3.26}$$

$$\sum_{r=n+2}^{m}\sum_{j=3}^{n}((\Pi_{2j}^{r})^2 + (\Pi_{2j}^{r})^2) + \sum_{r=n+2}^{m}(\Pi_{11}^{r} + \Pi_{22}^{r})^2 = 0, \tag{3.27}$$

$$\sum_{r=n+1}^{m}(\Pi_{d_1+1d_1+1}^{r} + \ldots + \Pi_{nn}^{r}) = \left(\sum_{\alpha=1}^{d_1}\Pi_{\alpha\alpha}^{n+1}\right)^2 = 0, \tag{3.28}$$

$$\sum_{\substack{k,l=3\\k\neq l}}^{d_1}(\Pi_{kl}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{k,l=3}^{d_1}(\Pi_{kl}^{r})^2 + \sum_{\alpha=3}^{d_1}\sum_{\beta=d_1+1}^{n}(\Pi_{\alpha\beta}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{A=3}^{d_1}\sum_{B=d_1+1}^{n}(\Pi_{AB}^{r})^2 = 0. \tag{3.29}$$

Equation (3.28) confirms that the warped product $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_1}$ is both $\mathcal{B}_1^{d_1}$-minimal and $\mathcal{B}_2^{d_2}$-minimal warped product submanifold in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. Based on our results, we conclude that the warped product submanifold

$\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_1}$ is minimal in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. The other case will be classified into two ways based on vector fields $e_r$. Based on the assumption that $r = n + 1$, here is what we define

$$\Pi_{11}^{n+1} + \Pi_{22}^{n+1} = \Pi_{33}^{n+1} = \dots = \Pi_{d_1 d_1}^{n+1}$$

and

$$\sum_{j=3}^{n} (\Pi_{1j}^{n+1})^2 = \sum_{j=3}^{n} (\Pi_{2j}^{n+1})^2 = \sum_{\substack{k,l=3 \\ k \neq l}}^{d_1} (\Pi_{kl}^{n+1})^2 = \sum_{\alpha=3}^{d_1} \sum_{\beta=d_1+1}^{n} (\Pi_{\alpha\beta}^{n+1})^2 = 0.$$

This condition can be represented by the following matrices:

$$\text{(i)} \quad A_{e_{n+1}} = \begin{bmatrix} \mu_1 & \Pi_{12}^{n+1} & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\ \Pi_{12}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{d_11} & 0 & 0 & \cdots & \mu & 0_{d_1d_1+1} & \cdots & 0_{d_1n} \\ 0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \Pi_{d_1+1d_1+1}^{n+1} & \cdots & \Pi_{d_1+1n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \Pi_{nd_1+1}^{n+1} & \cdots & \Pi_{nn}^{n+1} \end{bmatrix}$$

where $\mu = \mu_1 + \mu_2$ gives (i) of Theorem 3.1. As well, if $r \in \{n + 2, \dots, m\}$, then it must follow that

$$\Pi_{11}^{r} + \Pi_{22}^{r} = \sum_{j=3}^{n} (\Pi_{1j}^{r})^2 = \sum_{j=3}^{n} (\Pi_{2j}^{r})^2 = \sum_{\substack{k,l=3 \\ k \neq l}}^{d_1} (\Pi_{kl}^{r})^2 = \sum_{\alpha=3}^{d_1} \sum_{\beta=d_1+1}^{n} (\Pi_{\alpha\beta}^{r})^2 = 0.$$

That is equivalent to the second metric:

$$\text{(ii)} \quad A_{e_r} = \begin{bmatrix} \Pi_{11}^{r} & \Pi_{12}^{r} & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\ \Pi_{21}^{r} & -\Pi_{11}^{r} & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{d_11} & 0 & 0 & \cdots & 0_{d_1d_1} & 0_{d_1d_1+1} & \cdots & 0_{d_1n} \\ 0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \Pi_{d_1+1d_1+1}^{r} & \cdots & \Pi_{d_1+1n}^{r} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \Pi_{nd_1+1}^{r} & \cdots & \Pi_{nn}^{r} \end{bmatrix}.$$

It is evident from the first two conditions that $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_1}$ is mixed totally geodesic submanifold in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$.

It is also necessary for the equality sign in (ii) to hold if the following two matrices are satisfied:

$$\text{(iii)} \quad A_{e_{n+1}} = \begin{bmatrix} \Pi_{11}^{n+1} & \cdots & \cdots & \Pi_{1d_1}^{n+1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \Pi_{d_11}^{n+1} & \cdots & \cdots & \Pi_{d_1d_1}^{n+1} & 0_{d_1d_1+1} & \cdots & \cdots & \cdots & 0_{d_1n} \\ 0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \mu_1 & \Pi_{d_1+1d_1+2}^{n+1} & 0 & \cdots & 0_{d_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \Pi_{d_1+2d_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & \mu \end{bmatrix},$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n + 2, ...,m\}$, thus we have

$$
\text{(iv)} \quad A_{e_r} = \begin{bmatrix}
\Pi^r_{11} & \cdots & \cdots & \Pi^r_{1d_1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\Pi^r_{d_1 1} & \cdots & \cdots & \Pi^r_{d_1 d_1} & 0_{d_1 d_1+1} & \cdots & \cdots & \cdots & 0_{d_1 n} \\
0_{d_1+1 1} & \cdots & \cdots & 0_{d_1+1 d_1} & \Pi^r_{d_1+1 d_1+1} & \Pi^r_{d_1+1 d_1+2} & 0 & \cdots & 0_{d_1+1 n} \\
\vdots & \ddots & \ddots & \vdots & \Pi^r_{d_1+2 d_1+1} & -\Pi^r_{d_1+1 d_1+1} & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & 0
\end{bmatrix}.
$$

It is also determined by the above that $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_1}$ is both $\mathcal{B}_1^{d_1}$-minimal and $\mathcal{B}_2^{d_2}$-minimal warped product sub-manifold in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$, which suggest that the warped product submanifold $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_1}$ is minimal in $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$.   □

A wide range of distinct geometrical properties can be obtained from immersion in warped product manifolds. It is now possible to find the inequalities for the Riemannian manifolds with constant sectional curvature $\in \{1, 0, -1\}$ that can be presented as a product manifold $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. The following result is obtained, the result is relevant to the application of the warped product submanifold in $\mathbb{S}^n \times \mathbb{R}$ with $\varepsilon = 1$.

**Theorem 3.2.** *Assume that $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is an isometric immersion into a Euclidean sphere $\mathbb{S}^n \times \mathbb{R}$. Then, for each point $x \in \mathcal{B}^n$ and each plane section $\pi_i \subset T_x \mathcal{B}_i^{d_i}$, for $i = 1, 2$, we obtain the following for*
*(a) $\pi_1 \subset T_x \mathcal{B}_1^{d_1}$*

$$
\delta_{\mathcal{B}_1^{d_1}}(x) \leq \frac{n^2}{2}\|\mathcal{H}\|^2 + d_2\|\nabla(\ln f)\|^2 - d_2\Delta(\ln f) + \left\{\frac{d_1}{2}(d_1 + 2d_2 - 1) - (d_1 - 1)\|\mathcal{T}\|^2 - 1\right\}.
$$

*(b) For $\pi_2 \subset T_x \mathcal{B}_2^{d_2}$*

$$
\delta_{\mathcal{B}_2^{d_2}}(x) \leq \frac{n^2}{2}\|\mathcal{H}\|^2 + d_2\|\nabla(\ln f)\|^2 - d_2\Delta(\ln f) + \left\{\frac{d_2}{2}(d_2 + 2d_1 - 1) - (d_2 - 1)\|\mathcal{T}\|^2 - 1\right\}.
$$

*According to Theorem* 3.1, *the above inequality implies equalities.*

**Proof.** Now, we consider the constant sectional curvature $\varepsilon = 1$ and $\mathbb{Q}_\varepsilon^m = \mathbb{S}^m$ for the product manifold $\mathbb{S}^m \times \mathbb{R}$. Then, inserting the proceeding value in (3.1) and (3.2), we obtain the result.   □

Here, we present the application of the warped product submanifold in $\mathbb{H}^m \times \mathbb{R}$ with $\varepsilon = -1$.

**Theorem 3.3.** *Assume that $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is an isometric immersion into a Hyperbolic spaces $\mathbb{H}^m \times \mathbb{R}$. Then, for each point $x \in \mathcal{B}^n$ and each plane section $\pi_i \subset T_x \mathcal{B}_i^{n_i}$, for $i = 1, 2$, we obtain the following for*
*(a) $\pi_1 \subset T_x \mathcal{B}_1^{d_1}$ or $\pi_2 \subset T_x \mathcal{B}_2^{d_2}$*

$$
\delta_{\mathcal{B}_1^{d_1}}(x) \leq \frac{n^2}{2}\|H\|^2 + d_2\|\nabla(\ln f)\|^2 - d_2\Delta(\ln f) - \left\{\frac{d_1}{2}(d_1 + 2d_2 - 1) - (d_1 - 1)\|\mathcal{T}\|^2 - 1\right\}.
$$

*(b) For $\pi_2 \subset T_x \mathcal{B}_2^{d_2}$*

$$
\delta_{\mathcal{B}_2^{d_2}}(x) \leq \frac{n^2}{2}\|H\|^2 + d_2\|\nabla(\ln f)\|^2 - d_2\Delta(\ln f) - \left\{\frac{d_2}{2}(d_2 + 2d_1 - 1) - (d_2 - 1)\|\mathcal{T}\|^2 - 1\right\}.
$$

*According to Theorem* 3.1, *the above inequality implies equalities.*

**Proof.** Now we assume that $\mathbb{Q}_\varepsilon^m = \mathbb{H}^m$ and constant sectional curvature $\varepsilon = -1$ for the product manifold $\mathbb{H}^m \times \mathbb{R}$. Then using these values in (3.1) and (3.2), we obtain required result.   □

# 4 Generalized the results for space form $\mathbb{Q}_\varepsilon^m$

Our final result is obtained by concluding that $\mathbb{R} = 0$, which implies that $\mathcal{T} = 0$, given Theorem 3.1.

**Theorem 4.1.** *Assume that* $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ *is an isometric immersion into a space form* $\mathbb{Q}_\varepsilon^m$. *Then, for each point* $x \in \mathcal{B}^n$ *and each plane section* $\pi_i \subset T_x \mathcal{B}_i^{d_i}$, *for* $i = 1, 2$, *we obtain*
(1) *Let* $\pi_1 \subset T_x \mathcal{B}_1^{d_1}$, *then*

$$\delta_{\mathcal{B}_1^{d_1}}(x) \leq \frac{n^2}{2}\|\mathcal{H}\|^2 + d_2\|\nabla(\ln f)\|^2 + \frac{1}{2}d_1(d_1 + 2d_2 - 1)\varepsilon - \varepsilon - d_2\Delta(\ln f). \tag{4.1}$$

*It is only possible to compute the equality of the above inequality at* $x \in \mathcal{B}^n$ *if and only if there exists an orthonormal basis* $\{e_1, ..., e_n\}$ *of* $T_x\mathcal{B}^n$ *and an orthonormal basis* $\{e_{n+1}, ..., e_m\}$ *of* $T_x^\perp\mathcal{B}^n$ *such that*
(a) $\pi_1 = \text{Span}\{e_1, e_2\}$ *and*
(b) *Shape operators can be expressed as follows:*

$$
\text{(i)} \quad A_{e_{n+1}} = \left[
\begin{array}{ccccc|ccc}
\mu_1 & \Pi_{12}^{n+1} & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\
\Pi_{12}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0_{d_11} & 0 & 0 & \cdots & \mu & 0_{d_1d_1+1} & \cdots & 0_{d_1n} \\
\hline
0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \Pi_{d_1+1d_1+1}^{n+1} & \cdots & \Pi_{d_1+1n}^{n+1} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \Pi_{nd_1+1}^{n+1} & \cdots & \Pi_{nn}^{n+1}
\end{array}
\right],
$$

*where* $\mu = \mu_1 + \mu_2$. *If* $r \in \{n + 2, ..., m\}$, *then we have the matrix*

$$
\text{(ii)} \quad A_{e_r} = \left[
\begin{array}{ccccc|ccc}
\Pi_{11}^r & \Pi_{12}^r & 0 & \cdots & 0_{1d_1} & 0_{1d_1+1} & \cdots & 0_{1n} \\
\Pi_{21}^r & -\Pi_{11}^r & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0_{d_11} & 0 & 0 & \cdots & 0_{d_1d_1} & 0_{d_1d_1+1} & \cdots & 0_{d_1n} \\
\hline
0_{d_1+11} & \cdots & \cdots & \cdots & 0_{d_1+1d_1} & \Pi_{d_1+1d_1+1}^r & \cdots & \Pi_{d_1+1n}^r \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0_{n1} & \cdots & \cdots & \cdots & 0_{nd_1} & \Pi_{nd_1+1}^r & \cdots & \Pi_{nn}^r
\end{array}
\right].
$$

(2) *If* $\pi_2 \subset T_x \mathcal{B}_2^{d_2}$, *then*

$$\delta_{\mathcal{B}_2^{d_2}}(x) \leq \frac{n^2}{2}\|\mathcal{H}\|^2 + d_2\|\nabla(\ln f)\|^2 + \frac{1}{2}d_2(d_2 + 2d_1 - 1)\varepsilon - \varepsilon - d_2\Delta(\ln f). \tag{4.2}$$

*It follows that the above equation has equalities if and only if there exists an orthonormal basis* $\{e_1, ..., e_n\}$ *of* $T_x\mathcal{B}^n$ *and an orthonormal basis* $\{e_{n+1}, ..., e_m\}$ *of* $T_x^\perp\mathcal{B}^n$ *such that*
(a) $\pi_2 = \text{Span}\{e_{d_1+1}, e_{d_1+2}\}$ *and*
(b) *The shape operators are represented as follows:*

$$
\text{(iii)} \quad A_{e_{n+1}} = \left[
\begin{array}{cccc|ccccc}
\Pi_{11}^{n+1} & \cdots & \cdots & \Pi_{1d_1}^{n+1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\Pi_{d_11}^{n+1} & \cdots & \cdots & \Pi_{d_1d_1}^{n+1} & 0_{d_1d_1+1} & \cdots & \cdots & \cdots & 0_{d_1n} \\
\hline
0_{d_1+11} & \cdots & \cdots & 0_{d_1+1d_1} & \mu_1 & \Pi_{d_1+1d_1+2}^{n+1} & 0 & \cdots & 0_{d_1+1n} \\
\vdots & \ddots & \ddots & \vdots & \Pi_{d_1+2d_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & \mu
\end{array}
\right],
$$

*where $\mu = \mu_1 + \mu_2$. If $r \in \{n + 2, \ldots, m\}$, thus we have*

$$
\text{(iv)} \quad A_{e_r} =
\begin{bmatrix}
\Pi^r_{11} & \cdots & \cdots & \Pi^r_{1d_1} & 0_{1d_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\Pi^r_{d_1 1} & \cdots & \cdots & \Pi^r_{d_1 d_1} & 0_{d_1 d_1+1} & \cdots & \cdots & \cdots & 0_{d_1 n} \\
0_{d_1+1 1} & \cdots & \cdots & 0_{d_1+1 d_1} & \Pi^r_{d_1+1 d_1+1} & \Pi^r_{d_1+1 d_1+2} & 0 & \cdots & 0_{d_1+1 n} \\
\vdots & \ddots & \ddots & \vdots & \Pi^1_{d_1+2 d_1+1} & -\Pi^r_{d_1+1 d_1+1} & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{nd_1} & 0_{nd_1+1} & 0 & \cdots & 0 & 0
\end{bmatrix},
$$

(v) *If the equality holds in* (1) *or* (2), *then* $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ *is mixed totally geodesic in space form* $\mathbb{Q}_\varepsilon^m$. *Moreover,* $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ *is both* $\mathcal{B}_1^{d_1}$-*minimal and* $\mathcal{B}_2^{d_2}$-*minimal. Thus,* $\mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ *is a minimal warped product submanifold in space form* $\mathbb{Q}_\varepsilon^m$.

# 5 Several applications of the obtained Dirichlet eigenvalue inequalities

Determining the upper bound of the eigenvalue of the Laplacian on a specific manifold is an essential aspect of Riemannian geometry. This pursuit aims to examine eigenvalues, which arise as solutions to the Dirichlet boundary value problems for curvature functions. Understanding these eigenvalues is crucial, and one key objective is to establish their upper bound. This approach is particularly valuable when considering the diverse range of boundary conditions on a manifold and focusing on the Dirichlet boundary condition. An appropriate Laplacian limit on the given manifold can be defined by determining the upper bound of the eigenvalue. As an example, when we have the compact domain $\Sigma$ in a complete noncompact Riemannian manifold, then we have the Dirichlet boundary condition whose first eigenvalue is $\upsilon_1(\Sigma) > 0$

$$\Delta\sigma + \upsilon\sigma = 0 \text{ on } \Sigma \quad \text{and} \quad \sigma = 0 \text{ on } \partial\Sigma, \tag{5.1}$$

This equation corresponds to $\Delta$ where $\sigma$ is a nonzero function defined on $\mathcal{B}^n$. Then, $\upsilon_1(\mathcal{B}^n)$ is expressed as $\inf_\Sigma \upsilon_1(\Sigma)$.

If Dirichlet boundary conditions are considered, the Dirichlet eigenvalues are identical to the Laplace eigenvalues. The eigenvalues of differential equations are of significant interest to many branches of mathematics, including number theory, mathematical physics, and number theory. Among their most notable results is their ability to characterize a domain's geometry. For instance, the first Dirichlet eigenvalue of a domain relates to its diameter, while the higher eigenvalues reflect the domain's curvature and its embedding within Euclidean space. Additionally, the Dirichlet eigenvalues play a crucial role in solving the heat equation on a domain, where they determine both the eigenfunctions and the rate of decay of the solution. Based on the assumption that $\sigma$ is the non-constant warping function on the compact warped product submanifold $\mathcal{B}^n$. In this case, the minimum principle on $\upsilon_1$ leads to (see, e.g., [19])

$$\int_{\mathcal{B}^n} ||\nabla\sigma||^2 dV \geq \upsilon_1 \int_{\mathcal{B}^n} (\sigma)^2 dV. \tag{5.2}$$

As a result, equality can only be achieved if and only if

$$\Delta\sigma = \upsilon_1\sigma. \tag{5.3}$$

We know that the boundary for compact support is covered by $\mathcal{B}_1 \times \{d_2\}$. Implementation of the integration along the base manifold $\mathcal{B}^{d_1}$ in Eqs (3.1) and (3.2), we obtain the following result.

**Theorem 5.1.** *Assume that $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is a compact warped product submanifold into a product manifold $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. Then, we have*

$$
\int_{\mathcal{B}_1^{d_1} \times \{d_2\}} \delta_{\mathcal{B}_1^{d_1}}(x)\mathrm{d}V \leq \frac{n^2}{2} \int_{\mathcal{B}_1^{d_1} \times \{d_2\}} \|H\|^2 \mathrm{d}V + v_1 d_2 \int_{\mathcal{B}_1^{d_1} \times \{d_2\}} (\ln f)^2 \,\mathrm{d}V
$$
$$
+ \int_{\mathcal{B}_1^{d_1} \times \{d_2\}} \left[ \left( \frac{d_1}{2}(d_1 + 2d_2 - 1) - (d_1 - 1)\|\mathcal{T}\|^2 - 1 \right)\varepsilon \right] \mathrm{d}V, \tag{5.4}
$$

*for $\pi_1 \subset T\mathcal{B}_1^{d_1}$. Moreover, we have*

$$
\int_{\{d_1\} \times \mathcal{B}_2^{d_2}} \delta_{\mathcal{B}_2^{d_2}}(x)\mathrm{d}V
$$
$$
\leq \frac{n^2}{2} \int_{\{d_1\} \times \mathcal{B}_2^{d_2}} \|H\|^2 \mathrm{d}V + v_1 d_2 \int_{\{d_1\} \times \mathcal{B}_2^{d_2}} (\ln f)^2 \,\mathrm{d}V + \int_{\{d_1\} \times \mathcal{B}_2^{d_2}} \left[ \left( \frac{d_2}{2}(d_2 + 2d_1 - 1) - (d_2 - 1)\|\mathcal{T}\|^2 - 1 \right)\varepsilon \right] \mathrm{d}V, \tag{5.5}
$$

*for $\pi_2 \subset T\mathcal{B}_2^{d_2}$.*

**Proof.** We obtain the result easily by replacing $\sigma = \ln f$ in (3.1) and (3.2) with the Stokes theorem condition $\int \Delta \sigma \mathrm{d}V = 0$ for compact support.      □

In the following, we present the applications of Brochler formulas:

**Theorem 5.2.** *Assuming that $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ is a compact warped product submanifold into a product manifold $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. Then, we have*

$$
\int_{\mathcal{B}_1^{d_1} \times \{d_2\}} \mathrm{Ric}(\nabla \ln f, \nabla \ln f)\mathrm{d}V
$$
$$
\geq \frac{v_1}{d_2} \int_{\mathcal{B}_1^{d_1} \times \{d_2\}} \delta_{\mathcal{B}_1^{d_1}}(x)\mathrm{d}V - \frac{n^2 v_1}{2d_2} \int_{\mathcal{B}_1^{d_1} \times \{d_2\}} \|H\|^2 \mathrm{d}V
$$
$$
+ \frac{v_1}{d_2} \int_{\mathcal{B}_1^{d_1} \times \{d_2\}} \left[ 1 + (d_1 - 1)\|\mathcal{T}\|^2 - \left( \frac{d_1}{2}(d_1 + 2d_2 - 1) \right)\varepsilon \right] \mathrm{d}V - \int_{\mathcal{B}_1^{d_1} \times \{d_2\}} \|\nabla^2 \ln f\|^2 \mathrm{d}V, \tag{5.6}
$$

*for $\pi_1 \subset T\mathcal{B}_1^{d_1}$. Moreover, we have*

$$
\int_{\{d_1\} \times \mathcal{B}_2^{d_2}} \mathrm{Ric}(\nabla \ln f, \nabla \ln f)\mathrm{d}V
$$
$$
\geq \frac{v_1}{d_2} \int_{\{d_1\} \times \mathcal{B}_2^{d_2}} \delta_{\mathcal{B}_2^{d_2}}(x)\mathrm{d}V(x)\mathrm{d}V - \frac{n^2 v_1}{2d_2} \int_{\{d_1\} \times \mathcal{B}_2^{d_2}} \|H\|^2 \mathrm{d}V
$$
$$
+ \frac{v_1}{d_2} \int_{\{d_1\} \times \mathcal{B}_2^{d_2}} \left[ 1 + (d_2 - 1)\|\mathcal{T}\|^2 - \left( \frac{d_2}{2}(d_2 + 2d_1 - 1) \right)\varepsilon \right] \mathrm{d}V - \int_{\{d_1\} \times \mathcal{B}_2^{d_2}} \|\nabla^2 \ln f\|^2 \mathrm{d}V, \tag{5.7}
$$

*for $\pi_2 \subset T\mathcal{B}_2^{d_2}$.*

**Proof.** A Laplacian whose first eigenfunction is $\sigma$ is equal to $\mathrm{div}(\nabla \sigma)$ for $\mathcal{B}^n$ connected to the first non zero eigenvalue $v_1$, so that, $\Delta \sigma = -v_1 \sigma$. As a result, recalling Bochner's formula (see [1]), the differentiable function $\sigma$ at the Riemannian manifold has the following relation:

$$
\frac{1}{2}\Delta\|\nabla \sigma\|^2 = \|\nabla^2 \sigma\|^2 + \mathrm{Ric}(\nabla \sigma, \nabla \sigma) + g(\nabla \sigma, \nabla(\Delta \sigma)).
$$

The Stokes theorem tells us that we have to integrate the previous equation to obtain the following result:

$$\int_{\mathcal{B}_1 \times \{d_2\}} \|\nabla^2 \sigma\|^2 dV + \int_{\mathcal{B}_1 \times \{d_2\}} \text{Ric}(\nabla\sigma, \nabla\sigma) dV + \int_{\mathcal{B}_1 \times \{d_2\}} g(\nabla\sigma, \nabla(\Delta\sigma)) dV = 0. \tag{5.8}$$

Now, using $\Delta\sigma = -\upsilon_1 \sigma$ and making some rearrangement in Eq. (5.8), we derive

$$\int_{\mathcal{B}_1 \times \{d_2\}} \|\nabla\sigma\|^2 dV = \frac{1}{\upsilon_1} \left( \int_{\mathcal{B}_1 \times \{d_2\}} \|\nabla^2 \sigma\|^2 dV + \int_{\mathcal{B}_1 \times \{d_2\}} \text{Ric}(\nabla\sigma, \nabla\sigma) dV \right). \tag{5.9}$$

Integrating in (3.1) and (3.2) and using the above equation, we obtain the desired results. □

# 6 Chern's problem: Finding the conditions under which warped products must be minimal

Our solution to the Chern problem [20] in this section is to provide a partial solution to why a warped submanifold must be minimal in a product manifold $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$.

**Corollary 6.1.** *Let* $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ *be an isometric immersion of a warped product submanifold into a product manifold* $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. *Then, for each point* $x \in \mathcal{B}^n$ *and each* $\pi_1 \subset T_x \mathcal{B}_1^{d_1}$, *we have*

$$\delta_{\mathcal{B}_1^{d_1}}(x) + d_2 \Delta(\ln f) \leq d_2 \|\nabla(\ln f)\|^2 + \left\{ \frac{d_1}{2}(d_1 + 2d_2 - 1) - (d_1 - 1)\|\mathcal{T}\|^2 - 1 \right\} \varepsilon,$$

*and if the equality satisfies, then* $\mathcal{B}^n$ *is minimal.*

The second result is:

**Corollary 6.2.** *Let* $\mathcal{B}^n = \mathcal{B}_1^{d_1} \times_f \mathcal{B}_2^{d_2}$ *be an isometric immersion of a warped product submanifold into a product manifold* $\mathbb{Q}_\varepsilon^m \times \mathbb{R}$. *Then, for each point* $x \in \mathcal{B}^n$ *and each* $\pi_2 \subset T_x \mathcal{B}_2^{d_2}$, *we have*

$$\delta_{\mathcal{B}_2^{d_2}}(x) + d_1 \Delta(\ln f) \leq d_2 \|\nabla(\ln f)\|^2 + \left\{ \frac{d_2}{2}(d_2 + 2d_1 - 1) - (d_2 - 1)\|\mathcal{T}\|^2 - 1 \right\} \varepsilon,$$

*and if the equality satisfies, then* $\mathcal{B}^n$ *is minimal.*

**Remark 6.1.** Finally, we noticed that Theorem 3.1 is the solution of Problem 1.1. Furthermore, Corollaries 6.1 and 6.2 are the solution of Problem 6.2.

# 7 Conclusion remarks

Riemannian submanifolds have intrinsic and extrinsic invariants. Creating connections between these invariants is one of the fundamental problems of submanifold theory. This pursuit is motivated by Nash's renowned theory of isometric immersion, which suggests that viewing each Riemannian manifold as a submanifold in a Euclidean space is significant and influential [21]. In this context, the squared mean curvature is the primary extrinsic invariant, while the Ricci curvature and the scalar curvature act as the primary intrinsic invariants [5]. Furthermore, the Chen delta invariant, one of the numerical invariants in algebraic topology, assumes significance in measuring the degree to which a loop in space does not represent the boundary of a surface. Specifically, if a loop functions as a boundary, the Chen delta invariant attains zero value. Otherwise, it

quantifies how much the loop deviates from being a boundary. The applications of the delta invariant span a broad spectrum within mathematics, including differential geometry, differential topology, and algebraic geometry and algebraic topology. For instance, researchers have employed it to investigate the topology and geometry of moduli spaces of algebraic curves, examine the geometry of the Kähler-Einstein metric on a complex manifold, and explore the topology and geometry of configuration spaces of particles in a Euclidean space. Moreover, the delta invariant finds practical applications in physics, particularly in topological field theories. Several applications have also been made to physics, including studies of topological field theories [2,22,23].

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