Research Article

Dong Yeol Oh*

Noetherian rings of composite generalized power series

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Abstract: Let $A \subseteq B$ be an extension of commutative rings with identity, $(S, \leq)$ a nonzero strictly ordered monoid, and $S^* = S \backslash \{0\}$. Let $A + [\![B^{S^*,\leq}]\!] = \{f \in [\![B^{S,\leq}]\!] \mid f(0) \in A\}$. In this study, we determine when the ring $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring. We prove that when S is a strict monoid, if $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring, then A is a Noetherian ring, B is a finitely generated A-module, and S is finitely generated. We also show that if B is a finitely generated A-module over a Noetherian ring A and $(S, \leq)$ is a positive strictly ordered monoid, which is finitely generated, then $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring.

Keywords: generalized power series ring, Noetherian ring, strictly ordered monoid

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1 Introduction

Throughout this study, a *monoid* is a nonzero commutative semigroup with identity element. The operation is written additively, and the identity element is denoted by 0, unless otherwise stated.

Let $A \subseteq B$ be an extension of commutative rings with identity, X an indeterminate over B, S a monoid, and $S^* = S \backslash \{0\}$. The set $\{f \in B[X] | f(0) \in A\}$ is a ring denoted by $A + XB[X]$. The famous Hilbert's basis theorem [1, Theorem 69] says that the polynomial ring $B[X]$ over a Noetherian ring B is a Noetherian ring. Hizem [2, Proposition 2.1] extended this to obtain that $A + XB[X]$ is a Noetherian ring if and only if A is a Noetherian ring and B is a finitely generated A-module.

As is well known, the polynomial ring over B is isomorphic to a semigroup ring over B. Gilmer [3, Theorem 7.7] generalized Hilbert's basis theorem to the semigroup ring as follows: for a monoid S, the semigroup ring $B[S]$ is a Noetherian ring if and only if B is a Noetherian ring and S is finitely generated. Lim and Oh [4, Theorem 2.1] showed that for a torsion-free cancellative monoid S with $S \cap -S = \{0\}$, $A + B[S^*] := \{f \in B[S] | f(0) \in A\}$ is a Noetherian ring if and only if A is a Noetherian ring, B is a finitely generated A-module, and S is finitely generated.

The power series ring analog for Hilbert's basis theorem is that the formal power series ring $B[\![X]\!]$ over a Noetherian ring B is a Noetherian ring. Hizem and Benhissi [5, Theorem 4] showed that $A + XB[\![X]\!] := \{f \in B[\![X]\!] |$ the constant term of $f \in A\}$ is a Noetherian ring if and only if A is a Noetherian ring and B is a finitely generated A-module.

In [6–9] and in [10] rings of power series with exponents in a partially ordered commutative monoid were studied extensively. Let $(S, \leq)$ be a strictly ordered additive monoid, and let $[\![B^{S,\leq}]\!]$ be the set of all maps $f : S \to B$ such that $\text{supp}(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow (Definitions 2.2 and 2.3).

*** Corresponding author: Dong Yeol Oh**, Department of Mathematics Education, Chosun University, Gwangju 61452, Republic of Korea, e-mail: dyoh@chosun.ac.kr, dongyeol70@gmail.com

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Then, for $f, g \in [\![B^{S,\le}]\!]$ and $s \in S$, the set $X_s(f,g) := \{(t,u) \in S \times S | f(t) \neq 0,\ g(u) \neq 0, \text{ and } t+u=s\}$ is finite [8, (1.16)]. On $[\![B^{S,\le}]\!]$, the addition and the multiplication are defined as follows: for $f, g \in [\![B^{S,\le}]\!]$ and $s \in S$,

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (fg)(s) = \sum_{(t,u)\in X_s(f,g)} f(t)g(u).$$

Then, $[\![B^{S,\le}]\!]$ becomes a commutative ring with identity e, namely, $e(0) = 1$ and $e(s) = 0$ for all $s \in S^*$, which is called the *generalized power series ring with coefficients in B and exponents in S*. The element $f \in [\![B^{S,\le}]\!]$ with $a_s = f(s)$ for each $s \in S$ is written as $f = \sum_{s\in S} a_s X^s$. The set $A + [\![B^{S^*,\le}]\!] := \{f \in [\![B^{S,\le}]\!] | f(0) \in A\}$ is a subring of $[\![B^{S,\le}]\!]$, which is called the *composite generalized power series ring of A and B*.

We give some simple examples of generalized power series rings. If S is a monoid with trivial order (i.e., $s \le t$ implies $s = t$ for all $s, t \in S$), then the artinian and narrow subsets of S are any finite subsets of S. So $[\![B^{S,\le}]\!] = B[S]$, the semigroup ring of S with coefficients in B. If $S = \mathbb{N}$ is the set of nonnegative integers with the usual order $\le$, then the artinian and narrow subsets of $\mathbb{N}$ are any subsets of $\mathbb{N}$; thus, $[\![B^{\mathbb{N},\le}]\!] \cong B[\![X]\!]$. If $\mathbb{N}$ is with the usual order and $S = \mathbb{N}^n = \mathbb{N} \times \ldots \times \mathbb{N}$ (n times) with the order $\le$, where $\le$ is the product order, then $[\![B^{\mathbb{N}^n,\le}]\!] \cong B[\![X_1, \ldots, X_n]\!]$, ring of power series over B with n indeterminates [7, Examples 1, 2, 3].

Let R be a commutative ring with identity and $(S, \le)$ be a strictly ordered monoid. Ribenboim [7, 5.2, 5.5] determined when $[\![R^{S,\le}]\!]$ is a Noetherian ring. Brookfield [11, Theorem 4.3] determined an equivalent condition for $[\![R^{S,\le}]\!]$ to be Noetherian as follows: for a positive strictly ordered monoid $(S, \le)$, $[\![R^{S,\le}]\!]$ is a Noetherian ring if and only if R is a Noetherian ring and S is finitely generated.

Lim and Oh gave necessary conditions for $A + [\![B^{S^*,\le}]\!]$ to be Noetherian as follows: if $A + [\![B^{S^*,\le}]\!]$ is a Noetherian ring and $(S, \le)$ is a positive strictly ordered monoid, then (1) A is Noetherian; (2) if S is cancellative, then B is finitely generated A-module and S is finitely generated; and (3) $(S, \le)$ is narrow [12, Theorem 2.1]. They also proved the converse as follows: when $(S, \le)$ is a positive totally ordered monoid, if B is a finitely generated A-module over a Noetherian ring A and S is finitely generated, then $A + [\![B^{S^*,\le}]\!]$ is Noetherian [12, Theorem 2.10].

In this study, by modifying concepts and results in [11], we extend the results of Lim and Oh [12, Theorems 2.1 and 2.10] to determine when $A + [\![B^{S^*,\le}]\!]$ is a Noetherian ring. More precisely, in Theorem 3.1, we show that when $(S, \le)$ is a strictly ordered monoid, if $A + [\![B^{S^*,\le}]\!]$ is a Noetherian ring and S is a strict monoid, then A is Noetherian, B is a finitely generated A-module, and S is finitely generated. In Theorem 3.7, we also show that the converse of Theorem 3.1 holds when $(S, \le)$ is positive strictly ordered. As a corollary, if $(S, \le)$ is positive strictly ordered, then $A + [\![B^{S^*,\le}]\!]$ is a Noetherian ring if and only if A is a Noetherian ring, B is a finitely generated A-module, and S is finitely generated.

2 Preliminaries

We start with recalling concepts and known results, which will be needed in our proofs.

Definition 2.1. [3] A monoid S is said to be
(1) *cancellative* if $s + t = s + u$ implies $t = u$ for all $s, t, u \in S$,
(2) *torsion-free* if for any $s, t \in S$ and a positive integer n, $ns = nt$ implies $s = t$,
(3) *finitely generated* if there exists a finite subset $\{s_1, \ldots, s_n\}$ of S such that $S = \langle s_1, \ldots, s_n\rangle$, where $\langle s_1, \ldots, s_n\rangle$ is the set of all elements $\sum_{i=1}^n k_i s_i$ with nonnegative integers k_i.

Definition 2.2. ([9]) A partially ordered set $(S, \le)$ is said to be
(1) *artinian* if every strictly descending sequence of elements of S is finite,
(2) *narrow* if every subset of pairwise order-incomparable elements of S is finite,
(3) *an ordered monoid* if it is an additive monoid and the order $\le$ is compatible with the operation, i.e., for any $s, t, u \in S$, $s \le t$ implies $s + u \le t + u$.

Definition 2.3. [9] An ordered monoid $(S, \leq)$ is said to be
(1) *strictly ordered* if $s < s'$ implies $s + t < s' + t$ for all $s, s', t \in S$,
(2) *positively ordered* if $0 \leq s$ for all $s \in S$,
(3) *positive strictly ordered* if it is both positively ordered and strictly ordered.

For a partially ordered set $(S, \leq)$, a *lower* set of S is a subset I of S such that for all $x, y \in S$, $x \leq y, y \in I$ implies $x \in I$. We write $\Downarrow(S, \leq)$ for the set of lower sets of S ordered by inclusion. Using the result of Higman [13], Brookfield [11, Lemma 2.2] determined when $\Downarrow(S, \leq)$ is artinian without proof. We include a proof for the reader.

Lemma 2.4. [11, Lemma 2.2] *Let* $(S, \leq)$ *be a partially ordered set. The following conditions are equivalent*:
(1) $\Downarrow(S, \leq)$ *is artinian.*
(2) *For every infinite sequence* (s_n) *of S, there exist* $i < j$ *such that* $s_i \leq s_j$.
(3) $(S, \leq)$ *is artinian and narrow.*

Proof. (2) $\Leftrightarrow$ (3) It follows from [13, Theorem 2.1].

(1) $\Rightarrow$ (2) Let $(s_n)_{n\geq 1}$ be an infinite sequence of S. Put $S_i := \{s_j | j \geq i\}$ for each $i \geq 1$. For each i, put $I_i := \{x \in S | x \leq y \text{ for } y \in S_i\}$. Note that I_i is a lower set of S. Since $S_1 \supseteq S_2 \supseteq \ldots$, we have $I_1 \supseteq I_2 \supseteq \ldots$. Note that $s_i \in I_i$ for each i. Since $\Downarrow(S, \leq)$ is artinian, there exist $i < j$ such that $I_i = I_j$. Thus, $s_i \in I_j$, which means that $s_i \leq s_k$ for some $k \geq j > i$.

(2) $\Rightarrow$ (1) Suppose that $(I_n)_{n\geq 1}$ is an infinite strictly descending chain of lower sets of S. Consider an infinite sequence $(s_n)_{n\geq 1}$ such that $s_i \in I_i \backslash I_{i+1}$. By assumption, there exist $i < j$ such that $s_i \leq s_j$. Since I_j is a lower set of S and $s_j \in I_j$, we have $s_i \in I_j$, which is a contradiction. □

Let S be a monoid. The *algebraic preorder* (or *natural preorder*) on S is the relation $\preccurlyeq$ defined as follows: for every $s, t \in S$,

$$s \preccurlyeq t, \quad \text{if and only if } s + u = t, \quad \text{for some } u \in S.$$

In general, $s \preccurlyeq t \preccurlyeq s$ does not imply $s = t$; so $\preccurlyeq$ is not always a partial order on S. We collect some known results on a monoid S with the algebraic preorder $\preccurlyeq$ in [11]. A monoid S is said to be *strict* if $s + t + u = s$ implies $t = u = 0$ for every $s, t, u \in S$.

Lemma 2.5. *Let S be a strict monoid and let* $s_1, \ldots, s_n$ *be nonzero elements of S. If* $\sum_{i=1}^{n} k_i s_i = 0$ *for some nonnegative integers* k_i, *then* $k_i = 0$ *for all i.*

Proof. Let $s_1, \ldots, s_n$ be nonzero elements of S such that $\sum_{i=1}^{n} k_i s_i = 0$ for each $k_i \geq 0$. Assume that $k_i \neq 0$ for some $1 \leq i \leq n$. Without loss of generality, we may assume that $k_1 \neq 0$. Then, $0 = 0 + s_1 + ((k_1 - 1)s_1 + \sum_{i=2}^{n} k_i s_i)$. Since S is strict, we have $s_1 = 0$, which is a contradiction. □

For a monoid S, we denote by $G(S)$ the largest subgroup of S, *i.e.*, $G(S) := \{s \in S | s + t = 0 \text{ for some } t \in S\}$.

Lemma 2.6. *Let S be a nonzero monoid. Then, the following statements hold*:
(1) *If S is strict, then* $G(S) = \{0\}$.
(2) *If S is cancellative with* $G(S) = \{0\}$, *then S is strict.*
(3) [11, Lemma 3.1] *If* $(S, \leq)$ *is a positive strictly ordered monoid, then S is strict.*
(4) [11, Lemma 3.2] *S is strict if and only if* $(S, \preccurlyeq)$ *is a strictly ordered monoid.*
(5) [11, Lemma 3.3] *Let S be a strict monoid. Then, S is finitely generated if and only if* $\Downarrow(S, \preccurlyeq)$ *is artinian.*

Remark 2.7.
(1) For two strictly ordered monoids $(S, \leq)$ and $(S, \preccurlyeq)$, the fact that $(S, \preccurlyeq)$ is narrow may not imply that $(S, \leq)$ is narrow. For an example, let $S = \mathbb{N}$ be an additive monoid. Then, $\mathbb{N}$ is a finitely generated strict monoid and $(\mathbb{N}, \preccurlyeq)$ is narrow. But if $\leq$ is the trivial order on $\mathbb{N}$, then $(\mathbb{N}, \leq)$ is not narrow.
(2) Let $\leq'$ and $\leq$ be partial orders on a monoid S such that $\leq'$ is finer than $\leq$ (i.e., $s \leq t$ implies $s \leq' t$ for all $s, t \in S$). It follows from Lemma 2.4 that if $(S, \leq)$ is artinian and narrow, then $(S, \leq')$ is artinian and narrow. The converse does not hold; $\preccurlyeq$ is finer than the trivial order $\leq$ on $\mathbb{N}$, $(\mathbb{N}, \preccurlyeq)$ is artinian and narrow, but $(\mathbb{N}, \leq)$ is not narrow.
(3) Let $(S, \leq)$ be a positive strictly ordered monoid. Then, S is strict by Lemma 2.6. So $(S, \preccurlyeq)$ is a strictly ordered monoid and $\leq$ is finer than $\preccurlyeq$.

From Remark 2.7, we have the following.

Lemma 2.8. *Let $(S, \leq)$ be a positive strictly ordered monoid. If S is finitely generated, then $(S, \leq)$ is artinain and narrow.*

We note that if either S is cancellative with $G(S) = \{0\}$ or $(S, \leq)$ is a positive strictly ordered monoid, then S is strict. We now give an example that is a non-cancellative monoid S such that $(S, \leq)$ is positive strictly ordered.

Example 2.9. Let $S = (\mathbb{N} \times \{0, 1\})\backslash\{(0, 0)\} = \{(n, a) | n = 0, 1, \ldots, a = 0, 1\}\backslash\{(0, 0)\}$. Define an addition on S as follows: for $x = (n, a), y = (m, b) \in S$,

$$x + y = (n + m, ab).$$

Then, $(S, +)$ is a monoid with identity $(0, 1)$. Since $(2, 0) + (1, 0) = (3, 0) = (2, 0) + (1, 1)$, S is not cancellative. We now define an order $\leq$ on S as follows: for $x = (n, a), y = (m, b) \in S$,

$$x < y, \quad \text{if } n < m, \quad \text{and} \quad x = y, \quad \text{if } n = m \text{ and } a = b.$$

Then $(S, \leq)$ is a positive strictly ordered monoid.

3 Composite generalized power series ring of the form $A + [\![B^{S^*,\leq}]\!]$

Let $A \subseteq B$ be an extension of commutative rings with identity, $(S, \leq)$ a strictly ordered monoid, and $S^* = S\backslash\{0\}$. In this section, we determine when a composite generalized power series ring $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring. We first give necessary conditions for the ring $A + [\![B^{S^*,\leq}]\!]$ to be Noetherian. Lim and Oh [12, Theorem 2.1] provided necessary conditions for $A + [\![B^{S^*,\leq}]\!]$ to be Noetherian when $(S, \leq)$ is a positive strictly ordered monoid that is cancellative. We extend [12, Theorem 2.1] to the case when S is a strict monoid.

Theorem 3.1. *Let $A \subseteq B$ be an extension of commutative rings with identity and $(S, \leq)$ a strictly ordered monoid. Suppose that S is a strict monoid. If $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring, then the following statements hold:*
(1) *A is a Noetherian ring and B is a finitely generated A-module.*
(2) *S is finitely generated.*
(3) *If $(S, \leq)$ is positively ordered, then $(S, \leq)$ is narrow.*

Proof. Let $R := A + [\![B^{S^*,\leq}]\!]$. We recall that the element f of R with $a_s = f(s)$ for each $s \in S$ is written as $f = \sum_{s\in S} a_s X^s$.

(1) It is clear that $[\![B^{S^*,\leq}]\!]$ is an ideal of R. So $R/[\![B^{S^*,\leq}]\!] \cong A$ is a Noetherian ring. Let $s \in S^*$. Consider the ideal $X^s[\![B^{S,\leq}]\!]$ of $[\![B^{S,\leq}]\!]$ generated by X^s:

$$X^s[\![B^{S,\leq}]\!] = \{X^s g | g \in [\![B^{S,\leq}]\!]\}.$$

Note that $G(S) = \{0\}$ since S is strict. Since $s \in S^*$, $X^s g \in R$ for every $g \in [\![B^{S,\leq}]\!]$. Thus $X^s[\![B^{S,\leq}]\!]$ is an ideal of R. Since R is Noetherian, there exist $g_1, \ldots, g_n \in X^s[\![B^{S,\leq}]\!]$ such that $X^s[\![B^{S,\leq}]\!] = (g_1, \ldots, g_n)R$. Since each $g_i = X^s f_i$ for some $f_i \in [\![B^{S,\leq}]\!]$, the ideal $X^s[\![B^{S,\leq}]\!]$ can be written as follows:

$$X^s[\![B^{S,\leq}]\!] = X^s f_1 R + \ldots + X^s f_n R.$$

For any $b \in B$, there are $h_i \in R$ such that

$$bX^s = \sum_{i=1}^{n} X^s f_i h_i = X^s\left(\sum_{i=1}^{n} f_i h_i\right).$$

Since S is strict, $G(S) = \{0\}$ by Lemma 2.6 (i), and $s + t = s$ (for $s, t \in S$) implies $t = 0$. Recall that for $f, g \in [\![B^{S,\leq}]\!]$ and $s \in S$, $X_s(f, g)$ is the set $\{(t, u) \in S \times S | f(t) \neq 0,\ g(u) \neq 0,\ \text{and } t + u = s\}$. Hence, we have the following:

$$\begin{aligned} X_s\left(X^s, \sum_{i=1}^{n} f_i h_i\right) &= \left\{(u, v) \in S \times S | X^s(u) \neq 0, \left(\sum_{i=1}^{n} f_i h_i\right)(v) \neq 0,\ \text{and } u + v = s\right\} \\ &= \left\{(s, 0) \in S \times S \middle| \left(\sum_{i=1}^{n} f_i h_i\right)(0) \neq 0\right\}, \\ X_0(f_i, h_i) &= \{(u, v) \in S \times S | f_i(u) \neq 0, h_i(v) \neq 0,\ \text{and } u + v = 0\} \\ &= \{(0, 0) \in S \times S | f_i(0) \neq 0 \text{ and } h_i(0) \neq 0\}. \end{aligned}$$

Therefore, for the element $s \in S$, we have the following:

$$b = (bX^s)(s) = \sum_{u+v=s} X^s(u)\left(\sum_{i=1}^{n} f_i h_i\right)(v) = \sum_{s+v=s}\sum_{i=1}^{n} (f_i h_i)(v) = \sum_{i=1}^{n} f_i(0) h_i(0).$$

Consequently, $b \in (f_1(0), \ldots, f_n(0))A$, and B is a finitely generated A-module.

(2) Let (s_n) be an infinite sequence in S. Without loss of generality, we may assume that $s_n \neq 0$ for all $n \geq 1$. Let I_n be the ideal of R generated by $X^{s_1}, \ldots, X^{s_n}$:

$$I_n = (X^{s_1}, \ldots, X^{s_n})R = X^{s_1}R + \ldots + X^{s_n}R.$$

Since R is Noetherian, there exists $N \geq 1$ such that $X^{s_j} \in I_N$ for all $j > N$. Hence, for $j > N$, $X^{s_j} = X^{s_1} f_1 + \ldots + X^{s_N} f_N$, where each $f_i \in R$. Therefore,

$$s_j \in \operatorname{supp}\left(\sum_{i=1}^{N} X^{s_i} f_i\right) \subseteq \bigcup_{i=1}^{N} \operatorname{supp}(X^{s_i} f_i) = \bigcup_{i=1}^{N} (s_i + \operatorname{supp}(f_i)).$$

So $s_j = s_i + t_i$ for some $t_i \in \operatorname{supp}(f_i)$, where $i < j$. Hence, $s_i \preccurlyeq s_j$ for $i < j$. Note that since S is strict, $(S, \preccurlyeq)$ is a strictly ordered monoid. It follows from Lemmas 2.4 and 2.6 that S is finitely generated.

(3) It follows from Lemma 2.8. □

Note that if a strictly ordered monoid $(S, \leq)$ is either positive or cancellative with $G(S) = \{0\}$, then S is strict by Lemma 2.6. By applying Theorem 3.1 to these cases, we have the following.

Corollary 3.2. [12, Theorem 2.1] *Let $A \subseteq B$ be an extension of commutative rings with identity and $(S, \leq)$ a positive strictly ordered monoid. If $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring, then A is Noetherian, B is a finitely generated A-module, S is finitely generated, and $(S, \leq)$ is narrow.*

Corollary 3.3. *Let $A \subseteq B$ be an extension of commutative rings with identity. Let $(S, \leq)$ be strictly ordered, where S is cancellative with $G(S) = \{0\}$. If $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring, then A is Noetherian, B is a finitely generated A-module, and S is finitely generated.*

Let S and S' be monoids. A map σ from S to S' is a *monoid homomorphism* if $\sigma(0) = 0$ and $\sigma(s + t) = \sigma(s) + \sigma(t)$ for all $s, t \in S$. Given two ordered monoids $(S, \leq)$ and $(S', \leq')$, a map σ from $(S, \leq)$ to $(S', \leq')$ is a homomorphism of ordered monoids if σ is a monoid homomorphism, and $s \leq t$ implies $\sigma(s) \leq \sigma(t)$ for all $s, t \in S$, and a homomorphism σ of ordered monoids is *strict* if $s < t$ implies $\sigma(s) < \sigma(t)$ for all $s, t \in S$.

It is known that if $S = \langle s_1, \ldots, s_n\rangle$ is a finitely generated monoid, then there is a surjective homomorphism $\sigma : \mathbb{N}^n \to S$, which is defined by $\sigma(k) = k_1s_1 + \ldots + k_ns_n$ for $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ [14, Theorem 1.2]. If S is, moreover, strict, then we have the following.

Lemma 3.4. *Let S be a strict monoid that is finitely generated. Then, there is a surjective strict homomorphism from $(\mathbb{N}^n, \preccurlyeq)$ to $(S, \preccurlyeq)$ for some n.*

Proof. Let $S = \langle s_1, \ldots, s_n\rangle$ be a finitely generated strict monoid. Note that since $\mathbb{N}^n$ and S are strict, it follows from Lemma 2.6 that $(\mathbb{N}^n, \preccurlyeq)$ and $(S, \preccurlyeq)$ are strictly ordered monoids. We define a map σ from $(\mathbb{N}^n, \preccurlyeq)$ to $(S, \preccurlyeq)$ as follows:

$$\sigma : \mathbb{N}^n \to S, \quad \sigma(k) = k_1s_1 + \ldots + k_ns_n, \quad \text{for } k = (k_1, \ldots, k_n) \in \mathbb{N}^n.$$

Then, σ is a surjective monoid homomorphism by [14, Theorem 1.2]. It follows from Lemma 2.5 that if $0 \neq x \in \mathbb{N}^n$, then $0 \neq \sigma(x) \in S$. If $a, b \in \mathbb{N}^n$ with $a \prec b$, then $b = a + c$ for some $0 \neq c \in \mathbb{N}^n$. Since σ is a monoid homomorphism, $\sigma(b) = \sigma(a) + \sigma(c)$ and $0 \neq \sigma(c) \in S$. Hence, $\sigma(a) \prec \sigma(b)$ in S. Therefore, σ is strict. □

Let R be a commutative ring with identity, and let $(S, \leq)$ and $(S', \leq')$ be strictly ordered monoids. We recall that the element $f \in [\![R^{S,\leq}]\!]$ with $a_s = f(s)$ for each $s \in S$ is written as $f = \sum_{s\in S} a_sX^s$. If $\sigma : (S, \leq) \to (S', \leq')$ is a strict homomorphism of ordered monoids, then σ induces a ring homomorphism $\overline{\sigma} : [\![R^{S,\leq}]\!] \to [\![R^{S',\leq'}]\!]$, which is defined by $\overline{\sigma}(f) = \sum_{s\in S} a_sX^{\sigma(s)}$ for $f = \sum_{s\in S} a_sX^s \in [\![R^{S,\leq}]\!]$ (see [8, p. 78]). It is shown in [8, 1.17] that if σ is surjective and there exists a homomorphism of ordered monoids $\tau : (S', \leq') \to (S, \leq)$ such that $\sigma \circ \tau$ is the identity map, then $\overline{\sigma}$ is surjective.

The following is a special case of [8, 1.17].

Lemma 3.5. *Let R be a commutative ring with identity, and let $\sigma : (S, \leq) \to (S', \leq')$ be a strict homomorphism of strictly ordered monoids. If $(S, \leq)$ is narrow and σ is surjective, then the induced ring homomorphism $\overline{\sigma} : [\![R^{S,\leq}]\!] \to [\![R^{S',\leq'}]\!]$ is surjective.*

Proof. It follows from [8, p. 78] that the induced map $\overline{\sigma} : [\![R^{S,\leq}]\!] \to [\![R^{S',\leq'}]\!]$ is a ring homomorphism. Let $f' = \sum_{s'\in S'} a_{s'}X^{s'} \in [\![R^{S',\leq'}]\!]$. Then, $\text{supp}(f')$ is artinian and narrow in S'. Since σ is surjective, there is a map τ from S' to S such that $\sigma \circ \tau$ is the identity map on S'. Since σ is strict and $(S, \leq)$ is narrow, $\tau(\text{supp}(f'))$ is artinian and narrow in S. Put $f \coloneqq \sum_{s\in\tau(\text{supp}(f'))} a_{\sigma(s)}X^s$. Then, $f \in [\![R^{S,\leq}]\!]$ and $\overline{\sigma}(f) = f'$. Hence, $\overline{\sigma}$ is surjective. □

Remark 3.6. (1) If $(S, \leq)$ and $(S, \leq')$ are strictly ordered monoids, and $\leq'$ is finer than $\leq$, then $[\![R^{S,\leq}]\!]$ is a subring of $[\![R^{S,\leq'}]\!]$. In particular, if $(S, \leq)$ is narrow, then $[\![R^{S,\leq}]\!] = [\![R^{S,\leq'}]\!]$ (see also [8, 1.18]).

(2) In Lemma 3.5, the condition that $(S, \leq)$ is narrow is necessary. We give an example such that the induced map $\overline{\sigma}$ is not surjective when the map σ is surjective. Let K be a field. Let $\leq$ and $\leq'$ be the trivial order and usual order on $\mathbb{N}$, respectively. The identity map $\sigma : \mathbb{N} \to \mathbb{N}$ is a surjective strict homomorphism of strictly ordered monoids from $(\mathbb{N}, \leq)$ to $(\mathbb{N}, \leq')$. Note that $[\![K^{\mathbb{N},\leq}]\!] = K[X]$ and $[\![K^{\mathbb{N},\leq'}]\!] = K[\![X]\!]$. Then, there does not exist surjective ring homomorphism from $K[X]$ to $K[\![X]\!]$. Hence, the induced map $\overline{\sigma}$ is not surjective.

We now prove the converse of Theorem 3.1 when $(S, \leq)$ is a positive strictly ordered monoid. Using [11, Lemma 4.2], Brookfield [11, Theorem 4.3] showed that if R is a Noetherian ring and S is finitely generated, then $[\![R^{S,\leq}]\!]$ is a Noetherian ring. However, [11, Lemma 4.2] requires an additional assumption that $(S, \leq)$ is narrow (Remark 3.6 (2)). Therefore, we provide Lemma 3.5, which corrects [11, Lemma 4.2]. Using Lemma 3.5 and following the arguments as in the proof of [11, Theorem 4.3], we establish the following theorem.

Theorem 3.7. *If B is a finitely generated A-module over a Noetherian ring A and $(S, \leq)$ is a positive strictly ordered monoid that is finitely generated, then $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring.*

Proof. Since $(S, \leq)$ is a positive strictly ordered monoid and S is finitely generated, it follows from Lemmas 2.6 and 3.4 that there is a surjective strict monoid homomorphism $\tau : (\mathbb{N}^n, \preccurlyeq) \to (S, \preccurlyeq)$ for some n. Since $(S, \leq)$ is positive strictly ordered, we have $s \preccurlyeq t \Rightarrow s \leq t$ for all $s, t \in S$. Thus, the identity map $i : (S, \preccurlyeq) \to (S, \leq)$ is a surjective strict monoid homomorphism. Hence, the composite map $\sigma = i \circ \tau : (\mathbb{N}^n, \preccurlyeq) \to (S, \leq)$ is a surjective strict monoid homomorphism. Note that $(\mathbb{N}^n, \preccurlyeq)$ is narrow by Lemmas 2.4 and 2.6. It follows from Lemma 3.5 that σ induces the surjective ring homomorphism $\overline{\sigma} : [\![B^{\mathbb{N}^n,\preccurlyeq}]\!] \to [\![B^{S,\leq}]\!]$, which is defined as follows: for each $f = \sum_{u\in\mathbb{N}^n} a_u X^u \in [\![B^{\mathbb{N}^n,\preccurlyeq}]\!]$,

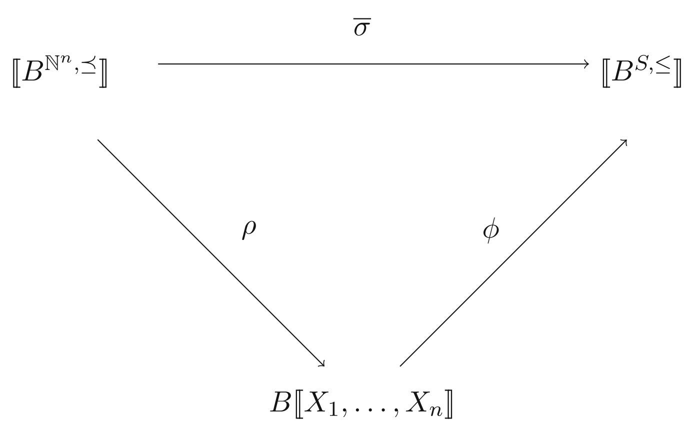
$$\overline{\sigma}(f) = \sum_{u\in\mathbb{N}^n} a_u X^{\tau(u)}.$$

Since σ is a surjective strict monoid homomorphism and $(\mathbb{N}^n, \preccurlyeq)$ is a positive strictly ordered monoid, we have that $\sigma(0) = 0$ and $\sigma(\mathbb{N}^{n*}) = S^*$. For each $\overline{f} = \sum_{s\in S} a_s X^s \in [\![B^{S^*,\leq}]\!]$, $\mathrm{supp}(f)$ is artinian and narrow. Since $(\mathbb{N}^n, \preccurlyeq)$ is artinian and narrow, $\sigma^{-1}(\mathrm{supp}(f))$ is an artinian and narrow subset of $\mathbb{N}^n$. For each $s \in \mathrm{supp}(f)$, choose $u_s \in \mathbb{N}^n$ such that $\sigma(u_s) = s$. Put $X \coloneqq \{u_s \in \mathbb{N}^n | s \in \mathrm{supp}(f)\} \subseteq \mathbb{N}^{n*}$. Note that X is artinian and narrow. Consider $f = \sum_{u_s\in X} a_s X^{u_s}$. Then, $f \in [\![B^{\mathbb{N}^n,\preccurlyeq}]\!]$ and $\overline{\sigma}(f) = \overline{f}$. Hence, we have that $\overline{\sigma}(a) = a$ for $a \in B$ and $\overline{\sigma}([\![B^{\mathbb{N}^{n*},\preccurlyeq}]\!]) = [\![B^{S^*,\leq}]\!]$.

We also note that there is a ring isomorphism $\rho : [\![B^{\mathbb{N}^n,\preccurlyeq}]\!] \to B[\![X_1, \dots, X_n]\!]$, which is defined as follows: for each $f = \sum_{u=(k_1,\dots,k_n)\in\mathbb{N}^n} a_u X^u \in [\![B^{\mathbb{N}^n,\preccurlyeq}]\!]$,

$$\rho(f) = \sum_{u\in\mathbb{N}^n} a_u X_1^{k_1} \dots X_n^{k_n}.$$

We note that $\rho(a) = a$ for $a \in B$ and $\rho([\![B^{\mathbb{N}^{n*},\preccurlyeq}]\!]) = (X_1, \dots, X_n)B[\![X_1, \dots, X_n]\!]$. Now, we have the following (commutative) diagram:



Here, the ring homomorphism $\phi : B[\![X_1, \dots, X_n]\!] \to [\![B^{S,\leq}]\!]$ is surjective, which is defined as follows: for each $f' \in B[\![X_1, \dots, X_n]\!]$,

$$\phi(f') \coloneqq \overline{\sigma}(f), \quad \text{where } f \in [\![B^{\mathbb{N}^n,\preccurlyeq}]\!], \text{ with } \rho(f) = f'.$$

Note that $A + [\![B^{\mathbb{N}^{n*},\preccurlyeq}]\!]$ is a subring of $[\![B^{\mathbb{N}^n,\preccurlyeq}]\!]$. Hence, we have the following:

$$\overline{\sigma}(A + [\![B^{\mathbb{N}^{n*},\preccurlyeq}]\!]) = A + [\![B^{S^*,\leq}]\!],$$
$$\rho(A + [\![B^{\mathbb{N}^{n*},\preccurlyeq}]\!]) = A + (X_1, \dots, X_n)B[\![X_1, \dots, X_n]\!].$$

It follows from [2, Proposition 2.1] that $A + (X_1, \dots, X_n)B[\![X_1, \dots, X_n]\!]$ is a Noetherian ring. Since ρ is an isomorphism, $A + [\![B^{\mathbb{N}^{n*},\preccurlyeq}]\!]$ is a Noetherian ring. Hence, $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring. □

Note that if $(S, \leq)$ is a positive totally ordered monoid that is finitely generated, Theorem 3.7 is exactly same as [12, Theorem 2.10]. By combining Theorems 3.1 and 3.7 for the cases when $(S, \leq)$ is a positive strictly ordered monoid, we have the following.

Theorem 3.8. *Let $A \subseteq B$ be an extension of commutative rings with identity and $(S, \leq)$ be a positive strictly ordered monoid. Then, $A + [\![B^{S^*,\leq}]\!]$ is a Noetherian ring if and only if A is a Noetherian ring, B is a finitely generated A-module, and S is finitely generated.*

As a corollary, we recover [11, Theorem 4.3].

Corollary 3.9. [11, Theorem 4.3] *Let A be a commutative ring with identity and $(S, \leq)$ be a positive strictly ordered monoid. Then, $[\![A^{S,\leq}]\!]$ is a Noetherian ring if and only if A is a Noetherian ring and S is finitely generated.*

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