Research Article

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Singularities of spherical surface in R4

https://doi.org/10.1515/math-2024-0033
received October 10, 2023; accepted June 26, 2024

Abstract: In this article, we mainly study the geometric properties of spherical surface of a curve on a hypersurface Σ in four-dimensional Euclidean space. We define a family of tangent height functions of a curve on Σ as the main tool for research and combine the relevant knowledge of singularity theory. It is shown that there are three types of singularities of spherical surface, that is, in the local sense, the spherical surface is respectively diffeomorphic to the cuspidal edge, the swallowtail, and the cuspidal beaks. In addition, we give two examples of the spherical surface.

Keywords: spherical surface, hypersurface, cuspidal beaks

MSC 2020: 58K05, 53D10, 53B20

1 Introduction

Singularity theory is a subject with strong application, which runs through the fields of differential geometry and differential topology, and is also one of the flourishing fields in modern mathematics. The classification of singular points of curves has always been the focus of research in singularity theory. For surfaces, we can also study their differential geometric properties from the viewpoint of singularity theory. In other words, when studying an unknown surface, we hope to make its local diffeomorphic to a familiar surface, thereby obtaining the properties of the unknown surface.

The application of singularity theory has achieved significant results in different spaces. Most of the research focuses on the classification of singular points of sub-manifolds [1–12]. On the other hand, the study of singularities on hypersurface has also received extensive attention from scholars. In [13], Sun and Pei introduced in detail the geometric property of Lorentzian hypersurfaces on pseudo n-spheres and the one parameter Gauss indicatrices on Lorentzian hypersurfaces. Moreover, they used the Legendrian singularity theory to complete the singularity analysis of the one parameter Gauss indicatrices of Lorentzian hypersurfaces on pseudo n-spheres. In [14], Izumiya et al. classified singularities of lightlike hypersurfaces in Minkowski 4-space. As a generalization of the study on lightlike hypersurface in Minkowski space, Pei et al. studied the singularities of lightlike hypersurface and Lorentzian surface in semi-Euclidean 4-space with index 2 in [15]. The aforementioned research makes the obtained results more systematic, which is what scholars are willing to see. In [16], Izumiya et al. defined the hyperbolic surface and de Sitter surface of a curve in a spacelike hypersurface in Minkowski 4-space and techniques from singularity theory were applied to obtain the generic shape of such surface and their singular value sets. There are also many studies on spherical surfaces. Not only in the field of mathematics but also in fields such as chemistry and physics [17–19]. However, the classification of singular points on spherical surfaces has not been resolved yet. This is also our main research motivation.

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Inspired by the aforementioned research, we chose the four-dimensional Euclidean space $\mathbb{R}^4$ as the outer space. Then we considered a embedding $\Pi : U \to \mathbb{R}^4$, from an open subset $U \subset \mathbb{R}^3$ and identify hypersurface Σ and U through the embedding Π. For a curve whose curvature does not disappear $\gamma : I \to \Sigma$, we defined a spherical surface in S^3 associated with curve γ. We used the classical deformation theory of singularity theory to study the generic differential geometry of spherical surfaces and their singular sets. The conclusion reached is that spherical surface is respectively diffeomorphic to the cuspidal edge, the swallowtail, and the cuspidal beaks.

The article is organized as follows: In Section 2, we introduce the definition of A_k-singularities and discriminant sets. Moreover, we build a moving frame along γ and calculate the Frenet-Serret type formulae. In Section 3, we define the tangential height functions that measure the contact of curve t with special hyperplanes and whose differentiation yields invariants related to each surface. In Section 4, the spherical surface of γ is described as the discriminant set of the family of tangential height functions. By using the theory of deformations, we get a classification and a characterisation of the diffeomorphim type of such surfaces. Finally, we provide some examples of spherical surfaces in Section 5.

2 Preliminaries

The four-dimensional Euclidean space is

$$\mathbb{R}^4 = \{(a_0, a_1, a_2, a_3) | a_i \in \mathbb{R}(i = 0, 1, 2, 3)\}$$

with scalar product

$$\langle a, b\rangle = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3,$$

for any vectors $a = (a_0, a_1, a_2, a_3)$, and $b = (b_0, b_1, b_2, b_3)$ in $\mathbb{R}^4$. We define the vector product of a, b and $z = (z_0, z_1, z_2, z_3)$ as follows:

$$a \wedge b \wedge z = \begin{vmatrix} e_0 & e_1 & e_2 & e_3 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ z_0 & z_1 & z_2 & z_3 \end{vmatrix},$$

where $a, b, z \in \mathbb{R}^4$, and $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4$, $e_0 = (1, 0, 0, 0)$. The norm of a nonzero vector $a \in \mathbb{R}^4$ is defined by $||a|| = \sqrt{\langle a, a\rangle}$, and when $||a|| = 1$, we call a a unit vector.

For a non-zero vector $v \in \mathbb{R}^4$ and a real number c, we define a hyperplane with pseudo-normal v by

$$\mathrm{HP}(v, c) = \{a \in \mathbb{R}^4 | \langle a, v\rangle = c\}.$$

The sphere in $\mathbb{R}^4$ is defined by

$$S^3 = \{a \in \mathbb{R}^4 | \langle a, a\rangle = 1\}.$$

We consider an embedding $\Pi : U \to \mathbb{R}^4$, where U is an open subset in $\mathbb{R}^3$. We write $\Sigma = \Pi(U)$ and identify Σ and U through the embedding Π. Let $\bar{\gamma} : I \to U$ be a regular curve. Then we have a curve $\gamma : I \to \Sigma \subset \mathbb{R}^4$ defined by $\gamma(s) = \Pi(\bar{\gamma}(s))$. We say that γ is a curve in the hypersurface Σ. To facilitate calculation, we reparametrize γ by the arc length s. So we have the unit tangent vector $t(s) = \dot{\gamma}(s)$ with $||t(s)|| = 1$. In this case, we call γ a unit speed curve. Then, we have a unit normal vector field n along $\Sigma = \Pi(U)$ defined by

$$n(p) = \frac{\Pi_{u_1}(u) \wedge \Pi_{u_2}(u) \wedge \Pi_{u_3}(u)}{||\Pi_{u_1}(u) \wedge \Pi_{u_2}(u) \wedge \Pi_{u_3}(u)||},$$

for $p = \Pi(u)$, where $\Pi_{u_i} = \partial\Pi/\partial u_i, i = 1, 2, 3$. A unit normal vector field n_γ along γ is defined by $n_\gamma(s) = n \circ \gamma(s)$.

Under the assumption that $||t'(s) - \langle t'(s), n_\gamma(s)\rangle n_\gamma(s)|| \neq 0$, we can construct

$$n_1(s) = \frac{t'(s) - \langle t'(s), n_\gamma(s)\rangle n_\gamma(s)}{||t'(s) - \langle t'(s), n_\gamma(s)\rangle n_\gamma(s)||}.$$

It follows that $\langle t, n_1\rangle = 0$ and $\langle n_\gamma, n_1\rangle = 0$. Moreover, we have a unit vector defined by $n_2(s) = t(s) \wedge n_\gamma(s) \wedge n_1(s)$. Then, we have a orthonormal frame $\{t(s), n_\gamma(s), n_1(s), n_2(s)\}$. By standard arguments, we have the Frenet-Serret type formulae for the aforementioned frame as follows:

$$\begin{cases} t'(s) = k_n(s)n_\gamma(s) + k_g(s)n_1(s) \\ n_\gamma'(s) = -k_n(s)t(s) + \tau_1(s)n_1(s) + \tau_2(s)n_2(s) \\ n_1'(s) = -\tau_1(s)n_\gamma(s) - k_g(s)t(s) + \tau_g(s)n_2(s)' \\ n_2'(s) = -\tau_2(s)n_\gamma(s) - \tau_g(s)n_1(s) \end{cases}$$

where $k_n(s) = \langle n_\gamma(s), t'(s)\rangle$, $\tau_1(s) = \langle n_\gamma'(s), n_1(s)\rangle$, $\tau_2(s) = \langle n_\gamma'(s), n_2(s)\rangle$, $k_g(s) = ||t'(s) - \langle t'(s), n_\gamma(s)\rangle n_\gamma(s)|| = ||t'(s) - k_n(s)n_\gamma(s)||$, and $\tau_g(s) = \langle n_1'(s), n_2(s)\rangle$. The invariant k_n is called a normal curvature, τ_1 is a first normal torsion, τ_2 is a second normal torsion, k_g is a geodesic curvature, and τ_g is a geodesic torsion. Under the assumption $k_g(s) = ||t'(s) - \langle t'(s), n_\gamma(s)\rangle n_\gamma(s)|| \neq 0$, we have $k_g > 0$.

Definition 2.1. Let $X : \mathbb{R}^4 \to \mathbb{R}$ be a submersion and $\gamma : I \to \Sigma$ be a regular curve. We say that γ and $X^{-1}(0)$ have contact of order k at s_0, if the function $g(s) = X \circ \gamma(s)$ satisfies $g(s_0) = g'(s_0) = \dots = g^{(k)}(s_0) = 0$ and $g^{(k+1)}(s_0) \neq 0$, i.e., g has an A_k-singularity at s_0.

Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, x_0) \to \mathbb{R}$ be a family of germs of functions. We call G an r-parameter deformation of f if $f(s) = G_{x_0}(s)$. We assume that f has an A_k-singularity ($k \geq 1$) at s_0, we can write

$$j^{k-1}\left(\frac{\partial G}{\partial x_i}(s, x_0)\right)(s_0) = \sum_{j=0}^{k-1} a_{ji}(s - s_0)^j,$$

for $i = 1, \dots, r$. Then G is a versal deformation if the $k \times r$ matrix of coefficients (a_{ji}) has rank $k (k \leq r)$.

The discriminant set of G is given by

$$\mathcal{D}_G = \left\{x \in (\mathbb{R}^r, x_0) | G = \frac{\partial G}{\partial s} = 0 \quad \text{at } (s, x) \text{ for some } s \in (\mathbb{R}, s_0)\right\},$$

and the bifurcation set of G is given by

$$\mathcal{B}_G = \left\{x \in (\mathbb{R}^r, x_0) | \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = 0 \quad \text{at } (s, x) \text{ for some } s \in (\mathbb{R}, s_0)\right\}.$$

The next result is from [20].

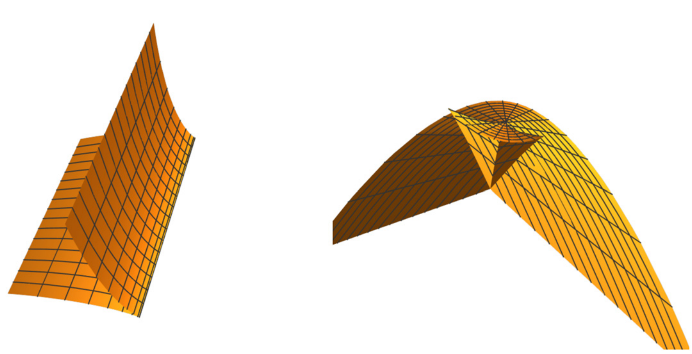

Figure 1: Cuspidal edge (left) and swallowtail (right).

Theorem 2.2. *Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, x_0) \to \mathbb{R}$ be an r-parameter deformation of f such that f has an A_k-singularity at s_0. We assume that G is a versal deformation, then $\mathcal{D}_G$ is locally diffeomorphic to*
(1) $C \times \mathbb{R}^{r-2}$ *if* $k = 2$,
(2) $\mathrm{SW} \times \mathbb{R}^{r-3}$ *if* $k = 3$,

where $C \times \mathbb{R} = \{(x_1, x_2, x_3) | x_1^2 = x_2^3\} \times \mathbb{R}$ *is the cuspidal edge and* $\mathrm{SW} = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *is the swallowtail surface* (Figure 1).

3 Tangential height functions

In this section, we define a family of functions on a curve in a hypersurface Σ.

Let $\gamma : I \to \Sigma \subset \mathbb{R}^4$, we give the following definitions:

$$H : I \times S^3 \to \mathbb{R};\ (s, v) \mapsto \langle t(s), v \rangle.$$

The functions H are called a family of tangential height functions of γ. The meaning of H is that it measures the contact of the curve t with hyperplanes in $\mathbb{R}^4$. Generically, this contact can be of order k, $k = 1, 2, 3$. For any fixed $v \in S^3$, we denote $h_v(s) = H(s, v)$.

In the following proposition, we find the conditions for characterizing the A_k-singularity, $k = 1, 2, 3$.

Proposition 3.1. *Let $\gamma : I \to \Sigma$ be a unit speed curve, we assume that $k_n \neq 0, k_g \neq 0$, so $k_g > 0$, and $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$. Thus, we obtain the following:*
(1) $h_v(s) = 0$ *if and only if there exists* $\delta, \xi, \varphi \in \mathbb{R}$ *such that* $\delta^2 + \xi^2 + \varphi^2 = 1$ *and*

$$v = \delta n_\gamma(s) + \xi n_1(s) + \varphi n_2(s).$$

(2) $h_v(s) = h_v'(s) = 0$ *if and only if there exists* $\theta \in \mathbb{R}$ *such that*

$$v = \frac{\cos\theta}{\sqrt{k_g^2(s) + k_n^2(s)}}(-k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin\theta n_2(s).$$

(3) $h_v(s) = h_v'(s) = h_v''(s) = 0$ *if and only if*

$$v = \frac{\cos\theta}{\sqrt{k_g^2(s) + k_n^2(s)}}(-k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin\theta n_2(s)$$

and $\tan\theta = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s)$.

(4) $h_v(s) = h_v'(s) = h_v''(s) = h_v'''(s) = 0$ *if and only if*

$$v = \frac{\cos\theta}{\sqrt{k_g^2(s) + k_n^2(s)}}(-k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin\theta n_2(s),$$

$\tan\theta = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s)$ *and* $\chi(s) = 0$, *where*

$$\begin{aligned}\chi(s) = ((&-k_n''k_g + 2k_gk_g'\tau_1 + k_g^2\tau_1' + k_gk_n\tau_2^2 + k_g^2\tau_g\tau_2 + 2k_nk_n'\tau_1 + k_n^2\tau_1' + k_nk_g'' \\ &- k_n^2\tau_2\tau_g - k_nk_g\tau_g^2)(k_n\tau_2 + k_g\tau_g) + (2k_n'\tau_2 - k_g\tau_2\tau_1 + k_n\tau_1\tau_g + 2k_g'\tau_g \\ &+ k_n\tau_2' + k_g\tau_g')(k_n'k_g - k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g'))(s).\end{aligned}$$

(5) $h_v(s) = h_v'(s) = h_v''(s) = h_v'''(s) = h_v^{(4)}(s) = 0$ *if and only if*

$$v = \frac{\cos\theta}{\sqrt{k_g^2(s) + k_n^2(s)}}(-k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin\theta n_2(s),$$

$\tan\theta = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s)$ *and* $\chi(s) = \chi'(s) = 0$.

Proof. (1) We can know $v \in S^3$, there are $\lambda, \delta, \xi, \varphi \in \mathbb{R}$ with $\lambda^2 + \delta^2 + \xi^2 + \varphi^2 = 1$ such that $v = \lambda t(s) + \delta n_\gamma(s) + \xi n_1(s) + \varphi n_2(s)$. According to $h_v(s) = \langle t(s), v\rangle = 0$, we have $\lambda = 0$. So $\delta^2 + \xi^2 + \varphi^2 = 1$ and $v = \delta n_\gamma(s) + \xi n_1(s) + \varphi n_2(s)$. Thus, (1) holds.

(2) Because $h_v(s) = h_v'(s) = 0$, so we have

$$\begin{aligned}\langle t(s), v\rangle = \langle t'(s), v\rangle &= \langle k_n(s)n_\gamma(s) + k_g(s)n_1(s), \delta n_\gamma(s) + \xi n_1(s) + \varphi n_2(s)\rangle \\ &= \delta k_n(s) + \xi k_g(s) = 0.\end{aligned}$$

we calculate that $\delta = -\xi\frac{k_g(s)}{k_n(s)}$, and $\left(\frac{k_g^2(s)+k_n^2(s)}{k_n^2(s)}\right)\xi^2 + \varphi^2 = 1$. It follows that $\delta = -\frac{\cos\theta k_g(s)}{\sqrt{k_g^2(s)+k_n^2(s)}}$, $\xi = \frac{\cos\theta k_n(s)}{\sqrt{k_g^2(s)+k_n^2(s)}}$, $\varphi = \sin\theta$. Thus, (2) holds.

(3) When $h_v(s) = h_v'(s) = h_v''(s) = 0$, we have $\langle t(s), v\rangle = \langle t'(s), v\rangle = \langle t''(s), v\rangle = 0$. Where $v = \frac{\cos\theta}{\sqrt{k_g^2(s)+k_n^2(s)}}(-k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin\theta n_2(s)$, and we calculate that

$$\begin{aligned}t''(s) = &(-k_n^2(s) - k_g^2(s))t(s) + (k_n'(s) - k_g(s)\tau_1(s))n_\gamma(s) \\ &+ (k_n(s)\tau_1(s) + k_g'(s))n_1(s) + (k_n(s)\tau_2(s) + k_g(s)\tau_g(s))n_2(s).\end{aligned}$$

Hence, there exists θ such that $\tan\theta = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2+k_n^2}(k_n\tau_2 + k_g\tau_g)}(s)$. Thus, (3) holds.

(4) Based on (3) and $h_v'''(s) = 0$, we have $\langle t'''(s), v\rangle = 0$. Then we calculate that

$$\begin{aligned}t'''(s) = &(-3k_n(s)k_n'(s) - 3k_g(s)k_g'(s))t(s) + (-k_n^3(s) - k_g^2(s)k_n(s) + k_n''(s) \\ &- 2k_g'(s)\tau_1(s) - k_g(s)\tau_1'(s) - k_n(s)\tau_1^2(s) - k_n(s)\tau_2^2(s) - k_g(s)\tau_g(s)\tau_2(s))n_\gamma(s) \\ &+ (-k_g^3(s) - k_n^2(s)k_g(s) + k_g''(s) + 2k_n'(s)\tau_1(s) + k_n(s)\tau_1'(s) - k_g(s)\tau_1^2(s) \\ &- k_g(s)\tau_g^2(s) - k_n(s)\tau_g(s)\tau_2(s))n_1(s) + (2k_n'(s)\tau_2(s) - k_g(s)\tau_1(s)\tau_2(s) \\ &+ k_n(s)\tau_g(s)\tau_1(s) + 2k_g'(s)\tau_g(s) + k_n(s)\tau_2'(s) + k_g(s)\tau_g'(s))n_2(s).\end{aligned}$$

Thus, $h_v(s) = h_v'(s) = h_v''(s) = h_v'''(s) = 0$ if and only if

$$v = \frac{\cos\theta}{\sqrt{k_g^2(s) + k_n^2(s)}}(-k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin\theta n_2(s),$$

$$\tan\theta = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s),$$

and

$$\begin{aligned}\chi(s) = &((-k_n''k_g + 2k_gk_g'\tau_1 + k_g^2\tau_1' + k_gk_n\tau_2^2 + k_g^2\tau_g\tau_2 + 2k_nk_n'\tau_1 + k_n^2\tau_1' + k_nk_g'' \\ &- k_n^2\tau_2\tau_g - k_nk_g\tau_g^2)(k_n\tau_2 + k_g\tau_g) + (2k_n'\tau_2 - k_g\tau_2\tau_1 + k_n\tau_1\tau_g + 2k_g'\tau_g \\ &+ k_n\tau_2' + k_g\tau_g')(k_n'k_g - k_g^2\tau_1 - k_n^2\tau_1 - k_nk_g'))(s) = 0.\end{aligned}$$

(5) Based on (4), we know $\chi(s) = \langle t'''(s), v\rangle = 0$, so $h_v^{(4)}(s) = \langle t^{(4)}(s), v\rangle = \chi'(s) = 0$. Thus, (5) holds. Proof completed. □

In the following proposition, we find that the family of tangential height functions on a curve in Σ is a versal deformation of an A_k-singularity, $k = 2, 3$.

Proposition 3.2. *Let $\gamma : I \to \Sigma$ be a unit speed curve with $k_g \neq 0$, so $k_g > 0$, and $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$. Thus, we have*
(1) *If h_{v_0} has an A_2-singularity at s_0, then H is a versal deformation of h_{v_0}.*
(2) *If h_{v_0} has an A_3-singularity at s_0, then H is a versal deformation of h_{v_0}.*

Proof. The family of tangential height functions is given by

$$H(s,v) = \langle t(s), v\rangle = v_0x_0'(s) + v_1x_1'(s) + v_2x_2'(s) + v_3x_3'(s),$$

where $v = (v_0, v_1, v_2, v_3)$, $t(s) = (x_0'(s), x_1'(s), x_2'(s), x_3'(s))$, and $v_1 = \sqrt{1 - v_0^2 - v_2^2 - v_3^2}$. In order not to lose generality, we assume that $v_1 \neq 0$. So we have

$$\frac{\partial H}{\partial v_0}(s,v) = x_0'(s) - \frac{v_0}{v_1}x_1'(s), \quad \frac{\partial^2 H}{\partial s\partial v_0}(s,v) = x_0''(s) - \frac{v_0}{v_1}x_1''(s),$$
$$\frac{\partial^3 H}{\partial^2 s\partial v_0}(s,v) = x_0'''(s) - \frac{v_0}{v_1}x_1'''(s), \quad \frac{\partial H}{\partial v_i}(s,v) = x_i'(s) - \frac{v_i}{v_1}x_1'(s),$$
$$\frac{\partial^2 H}{\partial s\partial v_i}(s,v) = x_i''(s) - \frac{v_i}{v_1}x_1''(s), \quad \frac{\partial^3 H}{\partial^2 s\partial v_i}(s,v) = x_i'''(s) - \frac{v_i}{v_1}x_1'''(s), \quad (i = 2,3).$$

Therefore, the 1-jet of $\frac{\partial H}{\partial v_i}(s,v)$ at s_0 is given by

$$x_i'(s_0) - \frac{v_i}{v_1}x_1'(s_0) + (x_i''(s_0) - \frac{v_i}{v_1}x_1''(s_0))(s - s_0),$$

and the 2-jet of $\frac{\partial H}{\partial v_i}(s,v)$ at s_0 is given by

$$x_i'(s_0) - \frac{v_i}{v_1}x_1'(s_0) + (x_i''(s_0) - \frac{v_i}{v_1}x_1''(s_0))(s - s_0) + \frac{1}{2}(x_i'''(s_0) - \frac{v_i}{v_1}x_1'''(s_0))(s - s_0)^2,$$

where $i = 0, 2, 3$.

(1) If h_v has an A_2-singularity at $s = s_0$. Let us consider the following matrix:

$$B = \begin{pmatrix} x_0'(s_0) - \frac{v_0}{v_1}x_1'(s_0) & x_2'(s_0) - \frac{v_2}{v_1}x_1'(s_0) & x_3'(s_0) - \frac{v_3}{v_1}x_1'(s_0) \\ x_0''(s_0) - \frac{v_0}{v_1}x_1''(s_0) & x_2''(s_0) - \frac{v_2}{v_1}x_1''(s_0) & x_3''(s_0) - \frac{v_3}{v_1}x_1''(s_0) \end{pmatrix}.$$

We calculate the Gram-Schmidt matrix of $\widetilde{B} = v_1B$. We denote the lines of $\widetilde{B}$ by

$$F = (x_0'(s_0)v_1 - x_1'(s_0)v_0, x_2'(s_0)v_1 - x_1'(s_0)v_2, x_3'(s_0)v_1 - x_1'(s_0)v_3),$$
$$G = (x_0''(s_0)v_1 - x_1''(s_0)v_0, x_2''(s_0)v_1 - x_1''(s_0)v_2, x_3''(s_0)v_1 - x_1''(s_0)v_3).$$

Since $\langle v,v\rangle = 1$, $\langle t(s), t(s)\rangle = 1$, $\langle t(s), v\rangle = 0$, $\langle t'(s), v\rangle = 0$, and $\langle t'(s), t'(s)\rangle = \langle (k_nn_\gamma + k_gn_1)(s), (k_nn_\gamma + k_gn_1)(s)\rangle = k_g^2(s) + k_n^2(s)$, we have the following Euclidean inner product

$$F\cdot F = v_1^2 + (x_1')^2, F\cdot G = x_1'x_1'', G\cdot G = (k_g^2(s) + k_n^2(s))v_1^2 + (x_1'')^2.$$

Therefore, the Gram-Schmidt matrix of $\widetilde{B}$ is given by

$$G_{\widetilde{B}} = \begin{pmatrix} v_1^2 + (x_1')^2 & x_1'x_1'' \\ x_1'x_1'' & v_1^2(k_g^2(s) + k_n^2(s)) + (x_1'')^2 \end{pmatrix}.$$

We assume that $n_\gamma(s_0) = (0,1,0,0)$. In this case, we have $x_1'(s_0) = 0$, $x_1''(s_0) = k_n(s_0)$ and $v_1 = -\frac{k_g(s_0)\cos\theta_0}{\sqrt{k_g^2(s) + k_n^2(s)}}$. Thus, the determinant of $G_{\widetilde{B}}$ is

$$((x_1')^2 + v_1^2)[(k_g^2(s_0) + k_n^2(s_0))v_1^2 + (x_1'')^2] - (x_1'x_1'')^2 = \frac{k_g^2(s_0)\cos^2\theta_0}{k_g^2(s_0) + k_n^2(s_0)}(k_g^2(s_0)\cos^2\theta_0 + k_n^2(s_0))$$

that is different from zero. Thus, the rank of the matrix B is equal to two and so the assertion (1) follows.

(2) We now assume that h_v has an A_3-singularity at $s = s_0$. In this case, we show that the determinant of the matrix

$$A = \begin{pmatrix} x_0'(s_0) - \frac{v_0}{v_1}x_1'(s_0) & x_2'(s_0) - \frac{v_2}{v_1}x_1'(s_0) & x_3'(s_0) - \frac{v_3}{v_1}x_1'(s_0) \\ x_0''(s_0) - \frac{v_0}{v_1}x_1''(s_0) & x_2''(s_0) - \frac{v_2}{v_1}x_1''(s_0) & x_3''(s_0) - \frac{v_3}{v_1}x_1''(s_0) \\ x_0'''(s_0) - \frac{v_0}{v_1}x_1'''(s_0) & x_2'''(s_0) - \frac{v_2}{v_1}x_1'''(s_0) & x_3'''(s_0) - \frac{v_3}{v_1}x_1'''(s_0) \end{pmatrix}$$

is nonzero. Denote

$$a_i = \begin{pmatrix} x_i'(s_0) \\ x_i''(s_0) \\ x_i'''(s_0) \end{pmatrix} \quad (i = 0, 1, 2, 3).$$

By a simple calculation, we have

$$\det A = -\frac{v_0}{v_1}\det(a_1, a_2, a_3) + \frac{v_1}{v_1}\det(a_0, a_2, a_3) - \frac{v_2}{v_1}\det(a_0, a_1, a_3) + \frac{v_3}{v_1}\det(a_0, a_1, a_2).$$

On the other hand,

$$\gamma'(s_0) \wedge \gamma''(s_0) \wedge \gamma'''(s_0) = (\det(a_1, a_2, a_3), -\det(a_0, a_2, a_3), \det(a_0, a_1, a_3), -\det(a_0, a_1, a_2)).$$

Therefore, $\det A = -\left\langle \left(\frac{v_0}{v_1}, \frac{v_1}{v_1}, \frac{v_2}{v_1}, \frac{v_3}{v_1}\right), (\gamma' \wedge \gamma'' \wedge \gamma''')(s_0) \right\rangle$. We calculate that

$$\gamma'(s_0) \wedge \gamma''(s_0) \wedge \gamma'''(s_0) = -k_g(k_n\tau_2 + k_g\tau_g)n_\gamma + k_n(k_n\tau_2 + k_g\tau_g)n_1 + (k_g(k_n' - k_g\tau_1) + k_n(k_n\tau_1 + k_g'))n_2(s_0).$$

If h_v has an A_3-singularity at $s = s_0$, thus, using $v = \frac{\cos\theta}{\sqrt{k_g^2(s) + k_n^2(s)}}(-k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin\theta n_2(s)$, and $\tan\theta = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s)$, we have

$$\begin{aligned} \det A &= -\left\langle \left(\frac{v_0}{v_1}, \frac{v_1}{v_1}, \frac{v_2}{v_1}, \frac{v_3}{v_1}\right), (\gamma' \wedge \gamma'' \wedge \gamma''')(s_0) \right\rangle \\ &= -\left\langle -\frac{k_g\cos\theta_0}{\sqrt{k_g^2 + k_n^2}}n_\gamma + \frac{k_n\cos\theta_0}{\sqrt{k_g^2 + k_n^2}}n_1 + \sin\theta_0 n_2, -k_g(k_n\tau_2 + k_g\tau_g)n_\gamma \right. \\ &\quad \left. + k_n(k_n\tau_2 + k_g\tau_g)n_1 + (k_g(k_n' - k_g\tau_1) + k_n(k_n\tau_1 + k_g'))n_2 \right\rangle(s_0) = \frac{\sqrt{k_g^2 + k_n^2}}{k_g\cos\theta_0}(s_0). \end{aligned}$$

Therefore, if h_v has an A_3-singularity at s_0, then $\det A \neq 0$ and H is a versal deformation of h_{v_0}. This completes the proof. □

We now define a deformation $\widetilde{H} : I \times S^3 \times \mathbb{R} \to \mathbb{R}$ by $\widetilde{H}(s, v, u) = H(s, v) + u(s - s_0)^2 = \langle t(s), v\rangle + u(s - s_0)^2$. The germ at $(s_0, v_0, 0)$ represented by $\widetilde{H}$ is considered.

Proposition 3.3. *If h_{v_0} has an A_3-singularity at s_0, then $\widetilde{H}$ is a versal deformation of h_{v_0}.*

Proof. We have

$$\widetilde{H}(s, v, u) = H(s, v) + u(s - s_0)^2 = v_0x_0' + v_1x_1' + v_2x_2' + v_3x_3' + u(s - s_0)^2,$$

where $v = (v_0, v_1, v_2, v_3)$, $t(s) = (x_0'(s), x_1'(s), x_2'(s), x_3'(s))$ and $v_1 = \sqrt{1 - v_0^2 - v_2^2 - v_3^2}$. Thus,

$$\frac{\partial\widetilde{H}}{\partial v_i}(s, v, 0) = x_i'(s) - \frac{v_i}{v_1}x_1'(s),$$

for $i = 0, 2, 3$. The 2-jet of $\frac{\partial \widetilde{H}}{\partial v_i}(s, v, 0)$ at s_0 is given by

$$x_i'(s_0) - \frac{v_i}{v_1}x_1'(s_0) + (x_i''(s_0) - \frac{v_i}{v_1}x_1''(s_0))(s - s_0) + \frac{1}{2}(x_i'''(s_0) - \frac{v_i}{v_1}x_1'''(s_0))(s - s_0)^2,$$

and the 2-jet of $\frac{\partial \widetilde{H}}{\partial u}(s, v, 0)$ at s_0 is $(s - s_0)^2$. We assume that h_v has an A_3-singularity at $s = s_0$. Then we can show that

$$\operatorname{rank}\begin{pmatrix} x_0'(s_0) - \frac{v_0}{v_1}x_1'(s_0) & x_2'(s_0) - \frac{v_2}{v_1}x_1'(s_0) & x_3'(s_0) - \frac{v_3}{v_1}x_1'(s_0) & 0 \\ x_0''(s_0) - \frac{v_0}{v_1}x_1''(s_0) & x_2''(s_0) - \frac{v_2}{v_1}x_1''(s_0) & x_3''(s_0) - \frac{v_3}{v_1}x_1''(s_0) & 0 \\ x_0'''(s_0) - \frac{v_0}{v_1}x_1'''(s_0) & x_2'''(s_0) - \frac{v_2}{v_1}x_1'''(s_0) & x_3'''(s_0) - \frac{v_3}{v_1}x_1'''(s_0) & 1 \end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ x_0'(s_0) - \frac{v_0}{v_1}x_1'(s_0) & x_0''(s_0) - \frac{v_0}{v_1}x_1''(s_0) & 0 \\ x_2'(s_0) - \frac{v_2}{v_1}x_1'(s_0) & x_2''(s_0) - \frac{v_2}{v_1}x_1''(s_0) & 0 \\ x_3'(s_0) - \frac{v_3}{v_1}x_1'(s_0) & x_3''(s_0) - \frac{v_3}{v_1}x_1''(s_0) & 0 \end{pmatrix} = 3.$$

The rank of the last matrix has the same value as the rank of

$$\begin{pmatrix} 1 & 0 & 1 \\ x_0'(s_0) - \frac{v_0}{v_1}x_1'(s_0) & x_0''(s_0) - \frac{v_0}{v_1}x_1''(s_0) & 0 \\ x_2'(s_0) - \frac{v_2}{v_1}x_1'(s_0) & x_2''(s_0) - \frac{v_2}{v_1}x_1''(s_0) & 0 \\ x_3'(s_0) - \frac{v_3}{v_1}x_1'(s_0) & x_3''(s_0) - \frac{v_3}{v_1}x_1''(s_0) & 0 \end{pmatrix}.$$

Consider

$$l_1(s_0) = \left(1, x_0'(s_0) - \frac{v_0}{v_1}x_1'(s_0), x_2'(s_0) - \frac{v_2}{v_1}x_1'(s_0), x_3'(s_0) - \frac{v_3}{v_1}x_1'(s_0)\right),$$

$$l_2(s_0) = \left(0, x_0''(s_0) - \frac{v_0}{v_1}x_1''(s_0), x_2''(s_0) - \frac{v_2}{v_1}x_1''(s_0), x_3''(s_0) - \frac{v_3}{v_1}x_1''(s_0)\right),$$

and $l_3(s_0) = (1, 0, 0, 0)$. It is enough to show that $l_1(s_0)$, $l_2(s_0)$, and $l_3(s_0)$ are linearly independent. Because, if $l_1(s_0), l_2(s_0), l_3(s_0)$ are linearly dependent, then we have $x_0'(s_0) = \frac{v_0}{v_1}x_1'(s_0)$, $x_2'(s_0) = \frac{v_2}{v_1}x_1'(s_0)$, and $x_3'(s_0) = \frac{v_3}{v_1}x_1'(s_0)$. That is, $t(s_0)$ and v are parallel, and so we have a contradiction because $\langle t(s_0), v\rangle = 0$. □

4 Spherical surface

In this section, we give the definition of a spherical surface. In addition, we study the classification of singular points on spherical surfaces.

Let $\gamma : I \to \Sigma$ be a unit speed curve with $k_g(s) \neq 0$ and $(k_n\tau_2 + k_g\tau_g) \neq 0$, and a surface $S_\gamma : I \times J \to S^3$ is given by

$$S_\gamma(s, \theta) = \frac{\cos\theta}{\sqrt{k_g^2(s) + k_n^2(s)}}(-k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin\theta n_2(s),$$

where $J = [0, 2\pi]$. We call S_γ a spherical surface of γ.

Corollary 4.1. *The spherical surface of γ is the discriminant set D_H of the family of tangential height functions H.*

The cuspidal beaks are defined to be a germ of surface diffeomorphic to $\mathrm{CBK} = \{(x_1, x_2, x_3) | x_1 = v, x_2 = -2u^3 + v^2u, x_3 = 3u^4 - v^2u^2\}$. The cuspidal lips are defined to be a germ of surface diffeomorphic to $CLK = \{(x_1, x_2, x_3) | x_1 = v, x_2 = 2u^3 + v^2u, x_3 = 3u^4 + v^2u^2\}$ (Figure 2).

By using Theorem 2.2 and Propositions 3.2 and 3.3, we can obtain the diffeomorphism type of the spherical surface in the following theorem.

Theorem 4.2. *Let $\gamma : I \to \Sigma$ be a unit speed curve with $k_g \neq 0$ and $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$, and S_γ is the spherical surface of γ. We obtain the following*:

(1) *S_γ is singular at (s_0, θ_0) if and only if*

$$\tan\theta_0 = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s_0).$$

(2) *The germ of S_γ* at (s_0, θ_0) *is diffeomorphic to a cuspidal edge if*

$$\tan\theta_0 = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s_0) \quad \text{and} \quad \chi(s_0) \neq 0.$$

(3) *The germ of S_γ* at (s_0, θ_0) *is diffeomorphic to a swallowtail if*

$$\tan\theta_0 = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s_0), \chi(s_0) = 0 \quad \text{and} \quad \chi'(s_0) \neq 0.$$

(4) *The germ of S_γ at (s_0, θ_0) is diffeomorphic to a cuspidal beaks if*

$$\tan\theta_0 = \frac{k_n'k_g - k_g^2\tau_1 - k_g'k_n - k_n^2\tau_1}{\sqrt{k_g^2 + k_n^2}(k_n\tau_2 + k_g\tau_g)}(s_0), \lambda_1 \neq 0, \chi(s_0) = 0 \quad \text{and} \quad \chi'(s_0) \neq 0,$$

where $\lambda_1(s_0) = (k_n'\tau_2 + k_n\tau_2' + k_g'\tau_g + k_g\tau_g')(s_0)$.

(5) *A cuspidal lip does not appear.*

Figure 2: Cuspidal beaks (left) and cuspidal lips (right).

Proof. (1) By using the definition of the spherical surface, we have

$$S_\gamma(s,\theta)=\frac{\cos\theta}{\sqrt{k_g^2(s)+k_n^2(s)}}(-k_g(s)n_\gamma(s)+k_n(s)n_1(s))+\sin\theta n_2(s).$$

Taking the partial derivative of s, we obtain

$$\begin{aligned}\frac{\partial S_\gamma}{\partial s}(s,\theta)=\Bigg(&\frac{\cos\theta(-k_n^2k_g'+k_gk_nk_n'-k_g^2k_n\tau_1-k_n^3\tau_1)-\sin\theta\tau_2(k_g^2+k_n^2)\sqrt{k_g^2+k_n^2}(k_g^2+k_n^2)}{\sqrt{k_g^2+k_n^2}(k_g^2+k_n^2)}n_\gamma\\&+\frac{\cos\theta(k_g^2k_n'-k_gk_nk_g'-k_n^2k_g\tau_1-k_g^3\tau_1)-\sin\theta\tau_g(k_g^2+k_n^2)\sqrt{k_g^2+k_n^2}(k_g^2+k_n^2)}{\sqrt{k_g^2+k_n^2}(k_g^2+k_n^2)}n_1\\&+\frac{\cos\theta(k_n\tau_g-k_g\tau_2)}{\sqrt{k_g^2+k_n^2}}n_2\Bigg)(s).\end{aligned}$$

By taking the partial derivative of θ, we obtain

$$\frac{\partial S_\gamma}{\partial\theta}(s,\theta)=\frac{\sin\theta k_g(s)}{\sqrt{k_g^2(s)+k_n^2(s)}}n_\gamma(s)-\frac{\sin\theta k_n(s)}{\sqrt{k_g^2(s)+k_n^2(s)}}n_1(s)+\cos\theta n_2(s).$$

Therefore, when (s_0,θ_0) is a singularity if and only if the vectors $\{\frac{\partial S_\gamma}{\partial s}(s_0,\theta_0),\frac{\partial S_\gamma}{\partial\theta}(s_0,\theta_0)\}$ are linearly dependent. That is, the corresponding coefficients are proportional. Through calculation, we obtain if and only if

$$\tan\theta_0=\frac{k_gk_n'-k_g^2\tau_1-k_n^2\tau_1-k_nk_g'}{\sqrt{k_g^2+k_n^2}(k_g\tau_g+k_n\tau_2)}(s_0).$$

Thus, (1) holds.

(2) It follows from assertions (3) and (4) of Proposition 3.1 that h_v has an A_2-singularity at $s=s_0$ if and only if

$$\tan\theta_0=\frac{k_gk_n'-k_g^2\tau_1-k_n^2\tau_1-k_nk_g'}{\sqrt{k_g^2+k_n^2}(k_g\tau_g+k_n\tau_2)}(s_0)\quad\text{and}\quad\chi(s_0)\neq0.$$

Therefore, by (1) of Proposition 3.2 and Theorem 2.2, we have assertions (2).

(3) It also follows from assertions (4) and (5) of Proposition 3.1 that h_v has an A_3-singularity at $s=s_0$ if and only if

$$\tan\theta_0=\frac{k_gk_n'-k_g^2\tau_1-k_n^2\tau_1-k_nk_g'}{\sqrt{k_g^2+k_n^2}(k_g\tau_g+k_n\tau_2)}(s_0),\quad\chi(s_0)=0\quad\text{and}\quad\chi'(s_0)\neq0.$$

Therefore, by Proposition 3.2 and Theorem 2.2, we have assertions (3).

(4) By using Proposition 7.5 in [21] and Proposition 3.3, we can obtain that H is a Morse family of hypersurfaces. We now calculate $\sigma=(\partial^2H/\partial s^2)|\mathcal{D}_H$. Then, we have

$$\begin{aligned}\frac{\partial^2H}{\partial s^2}(s,\theta)&=\left\langle t''(s),\frac{\cos\theta}{\sqrt{k_g^2(s)+k_n^2(s)}}(-k_g(s)n_\gamma(s)+k_n(s)n_1(s))+\sin\theta n_2(s)\right\rangle\\&=\frac{-\cos\theta}{\sqrt{k_g^2(s)+k_n^2(s)}}(k_gk_n'-k_g^2\tau_1-k_n^2\tau_1-k_nk_g')(s)+\sin\theta(k_n\tau_2+k_g\tau_g)(s).\end{aligned}$$

The Hessian matrix of

$$\sigma(s,\theta)=\frac{-\cos\theta}{\sqrt{k_g^2(s)+k_n^2(s)}}(k_gk_n'-k_g^2\tau_1-k_n^2\tau_1-k_nk_g')(s)+\sin\theta(k_n\tau_2+k_g\tau_g)(s)$$

is

$$\mathrm{Hess}(\sigma)(s_0,0)=\begin{pmatrix}\frac{\partial^2\sigma}{\partial s^2}(s_0,0) & \lambda_1(s_0)\\ \lambda_1(s_0) & 0\end{pmatrix}.$$

So when $\lambda_1(s_0)\neq 0$, we have $\det \mathrm{Hess}(\sigma)(s_0,0)\neq 0$. By Lemma 7.7 in [21], H is P-$\mathcal{K}$-equivalent to $t^4 \pm v_1^2t^2 + v_2t + v^3$. The singular set of $\mathcal{D}_H$ is given by $\sigma(s,\theta)=0$. Therefore, it consists of two curves that transversally intersect at $(s_0,0)$. So the normal form is $t^4 - v_1^2t^2 + v_2t + v^3$, and the surface is diffeomorphic to the cuspidal beaks. Thus, we obtain assertions (4) and (5). □

5 Examples

In this section, we give two examples of spherical surfaces.

Example 5.1. We suppose $\Sigma = \mathbb{R}^3 = \{x=(x_0,x_1,x_2,x_3)\in\mathbb{R}^4 | x_0 = 0\}$. For $\gamma: I\to\mathbb{R}^3$, we have $n_\gamma = e_0$, $t(s)=\gamma'(s)$, $n_1(s)=n(s)$ and $n_2(s)=b(s)$. Here, $\{t,n,b\}$ is the ordinary Frenet frame. In this case, $k_n=\tau_1=\tau_2=0$, $k_g=k$ and $\tau_g=\tau$. The Frenet-Serret type formulae are the original Frenet-Serret formulae:

$$\begin{cases} e_0'(s)=0,\\ t'(s)=k(s)n(s),\\ n'(s)=-k(s)t(s)+\tau(s)b(s),\\ b'(s)=-\tau(s)n(s).\end{cases}$$

The spherical surface of γ is given by

$$S_\gamma(s,\theta) = -\cos\theta e_0(s) + \sin\theta b(s).$$

Let $\gamma: I\to\mathbb{R}^3$ be a curve defined by

$$\gamma(s)=\left(0,\cos\left(\frac{\sqrt{3}}{3}s\right),-\sin\left(\frac{\sqrt{3}}{3}s\right),\frac{\sqrt{6}}{3}s\right),$$

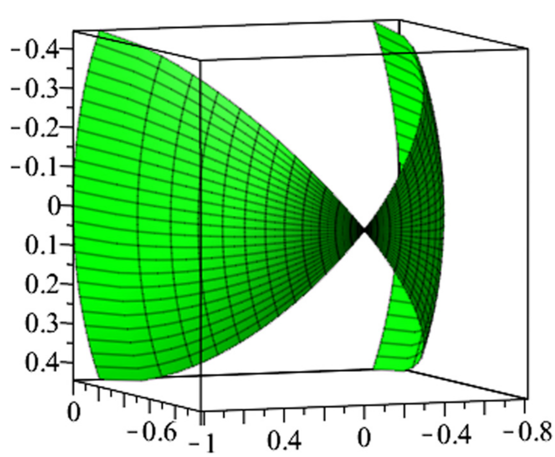

Figure 3: Projection of the image of the spherical surface on $x_1x_2x_3$-space.

we have

$$t(s)=\gamma'(s)=\left(0,-\frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{3}s\right),-\frac{\sqrt{3}}{3}\cos\left(\frac{\sqrt{3}}{3}s\right),\frac{\sqrt{6}}{3}\right),$$

$$n(s)=\left(0,-\cos\left(\frac{\sqrt{3}}{3}s\right),\sin\left(\frac{\sqrt{3}}{3}s\right),0\right),$$

$$b(s)=\left(0,-\frac{\sqrt{6}}{3}\sin\left(\frac{\sqrt{3}}{3}s\right),-\frac{\sqrt{6}}{3}\cos\left(\frac{\sqrt{3}}{3}s\right),-\frac{\sqrt{3}}{3}\right).$$

Let $\sin\theta = u, \cos\theta = \sqrt{1-u^2}$. Thus, the spherical surface of γ is given by

$$S_\gamma(s,u)=\left(-\sqrt{1-u^2},-\frac{\sqrt{6}}{3}\sin\left(\frac{\sqrt{3}}{3}s\right)u,-\frac{\sqrt{6}}{3}\cos\left(\frac{\sqrt{3}}{3}s\right)u,-\frac{\sqrt{3}}{3}u\right).$$

We draw the projection of the image of the spherical surface to 3-space (Figure 3).

Example 5.2. We suppose $\Sigma = S^3$. For $\gamma : I \to S^3$, we have $n_\gamma = \gamma(s), t(s) = \gamma'(s), n_1(s)$ and $n_2(s)$. Here, $\{t, \gamma, n_1, n_2\}$ is the orthonormal frame. In this case, $k_n(s) = -1$, $\tau_1(s) = \tau_2(s) = 0$, $k_g(s) = k_b(s)$ and $\tau_g(s) = \tau_b(s)$.

$$\begin{cases}\gamma'(s) = t(s),\\ t'(s) = -\gamma(s) + k_b(s)n_1(s),\\ n_1'(s) = -k_b(s)t(s) + \tau_b(s)n_2(s),\\ n_2'(s) = -\tau_b(s)n_1(s).\end{cases}$$

Therefore, the spherical surface of γ is given by

$$S_\gamma(s,\theta)=\frac{\cos\theta}{\sqrt{1+k_b^2(s)}}(-k_b(s)\gamma(s)-n_1(s))+\sin\theta n_2(s).$$

Let $\gamma : I \to S^3$ be a curve defined by

$$\gamma(s)=n_\gamma(s)=\left(\frac{1}{\sqrt{3}}\cos\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),\frac{1}{\sqrt{3}}\sin\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),\frac{\sqrt{2}}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{6}}s\right),\frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{6}}s\right)\right),$$

we have

$$t(s)=\gamma'(s)=\left(-\frac{2\sqrt{2}}{3}\sin\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),\frac{2\sqrt{2}}{3}\cos\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),-\frac{1}{3}\sin\left(\frac{1}{\sqrt{6}}s\right),\frac{1}{3}\cos\left(\frac{1}{\sqrt{6}}s\right)\right),$$

$$t'(s)=\gamma''(s)=\left(-\frac{8}{3\sqrt{3}}\cos\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),-\frac{8}{3\sqrt{3}}\sin\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),-\frac{1}{3\sqrt{6}}\cos\left(\frac{1}{\sqrt{6}}s\right),-\frac{1}{3\sqrt{6}}\sin\left(\frac{1}{\sqrt{6}}s\right)\right).$$

By the direct computation, we obtain $||\gamma(s)|| = 1$, $||t(s)|| = 1$, and $k_n(s) = \langle t'(s), n_\gamma(s)\rangle = -1$. Thus, we obtain

$$t'(s)+n_\gamma(s)=\left(-\frac{5}{3\sqrt{3}}\cos\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),-\frac{5}{3\sqrt{3}}\sin\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),\frac{5}{3\sqrt{6}}\cos\left(\frac{1}{\sqrt{6}}s\right),\frac{5}{3\sqrt{6}}\sin\left(\frac{1}{\sqrt{6}}s\right)\right),$$

and $k_g(s) = ||t'(s) + n_\gamma(s)|| = \frac{5}{3\sqrt{2}}$. We obtain the normal vector $n_1(s)$, which is given by

$$n_1(s)=\left(-\frac{\sqrt{2}}{\sqrt{3}}\cos\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),-\frac{\sqrt{2}}{\sqrt{3}}\sin\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right),\frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{6}}s\right),\frac{1}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{6}}s\right)\right).$$

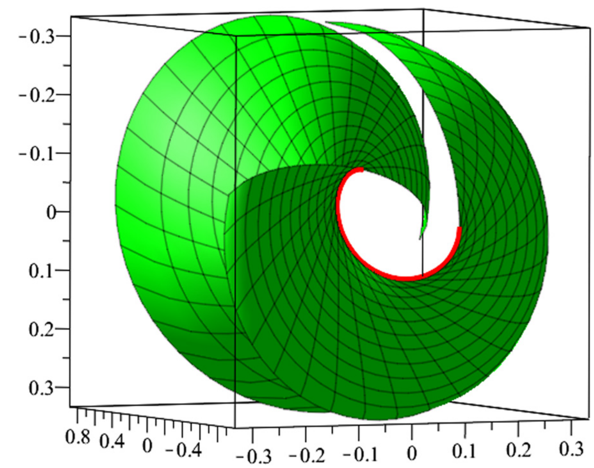

Figure 4: Projection of the image of the spherical surface on $x_1x_2x_3$-space (in green) and its critical value set (in red).

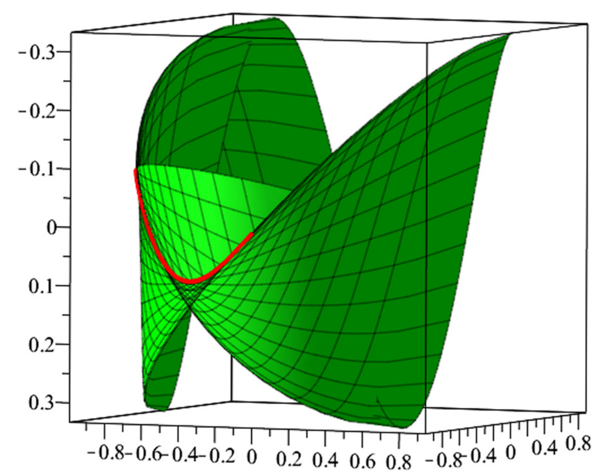

Figure 5: Projection of the image of the spherical surface on $x_2x_3x_4$-space (in green) and its critical value set (in red).

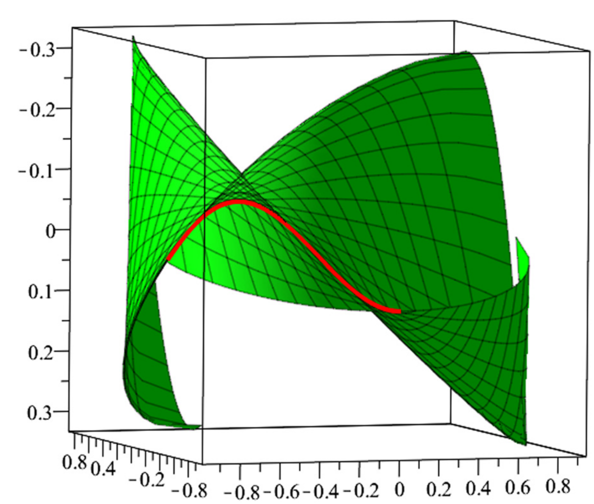

Figure 6: Projection of the image of the spherical surface on $x_1x_3x_4$-space (in green) and its critical value set (in red).

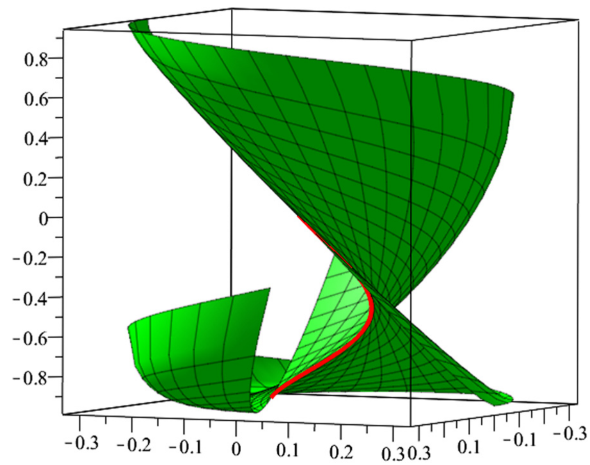

Figure 7: Projection of the image of the spherical surface on $x_1x_2x_4$-space (in green) and its critical value set (in red).

The other normal vector $n_2(s)$ is given by

$$\begin{aligned} n_2(s) &= t(s) \wedge n_\gamma(s) \wedge n_1(s) \\ &= \left(\frac{1}{3}\sin\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right), -\frac{1}{3}\cos\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right), -\frac{2\sqrt{2}}{3}\sin\left(\frac{1}{\sqrt{6}}s\right), \frac{2\sqrt{2}}{3}\cos\left(\frac{1}{\sqrt{6}}s\right)\right). \end{aligned}$$

Let $\sin\theta = u$, $\cos\theta = \sqrt{1-u^2}$. Thus, the spherical surface of γ is given by

$$S_\gamma(u, s) = (x_1(u, s), x_2(u, s), x_3(u, s), x_4(u, s)),$$

where

$$\begin{aligned} x_1(u, s) &= \frac{1}{\sqrt{129}}\sqrt{1-u^2}\cos\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right) + \frac{1}{3}u\sin\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right), \\ x_2(u, s) &= \frac{1}{\sqrt{129}}\sqrt{1-u^2}\sin\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right) - \frac{1}{3}u\cos\left(\frac{2\sqrt{2}}{\sqrt{3}}s\right), \\ x_3(u, s) &= -\frac{8\sqrt{2}}{\sqrt{129}}\sqrt{1-u^2}\cos\left(\frac{1}{\sqrt{6}}s\right) - \frac{2\sqrt{2}}{3}u\sin\left(\frac{1}{\sqrt{6}}s\right), \\ x_4(u, s) &= -\frac{8\sqrt{2}}{\sqrt{129}}\sqrt{1-u^2}\sin\left(\frac{1}{\sqrt{6}}s\right) + \frac{2\sqrt{2}}{3}u\cos\left(\frac{1}{\sqrt{6}}s\right). \end{aligned}$$

The points $(s, \theta(s)) = (s, 0)$ are the cuspidal edge-type of singularities of S_γ, where $s \in I$. We draw the projection of the image of the spherical surface S_γ (in green) and its critical value set $S_\gamma(s, 0)$ (in red) to 3-space (Figures 4–7).

Funding information: This research was funded by the Project of Science and Technology of Heilongjiang Provincial Education Department (Grant No. 1453ZD019), the Special Fund for Scientific and Technological Innovation of Graduate Students in Mudanjiang Normal University (Grant No. kjcx2022-021mdjnu), and the Reform and Development Foundation for Local Colleges and Universities of the Central Government (Grant No. ZYQN2019071).

Author contributions: All the authors contributed to this work equally and should be regarded as co-first authors.

Conflict of interest: The authors state no conflicts of interest.

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