Research Article

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Characterizations of minimal elements of upper support with applications in minimizing DC functions

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Abstract: In this study, we discuss on the problem of minimizing the differences of two non-positive valued increasing, co-radiant and quasi-concave (ICRQC) functions defined on X (where X is a real locally convex topological vector space). For this purpose, we first gave different characterizations of the upper support set's minimal elements of non-positive co-radiant functions. Then, we presented sufficient and necessary conditions for the global minimizers of the differences of two non-positive ICRQC functions.

Keywords: ICRQC function, minimal element, DC-function, upper support set

MSC 2020: 26B25, 26A48, 90C46

1 Introduction

The theoretical establishment of optimality conditions has been recently discussed extensively for certain classes of global optimization problems [1–3]. Minimizing a DC function (difference of two convex functions) is one of the most important global optimization problems, as

$$\text{minimize } h(x) \text{ subject to } x \in X,$$

where $h(x) = g(x) - f(x)$ and f, g represent the convex functions. Generally, DC functions can be substituted by difference of two abstract convex functions, e.g., minimizing the difference of two topical functions and minimizing the difference of two increasing and co-radiant functions [4,5]. Characterization of minimizing the difference of two non-negative increasing, co-radiant and quasi-concave (ICRQC) functions has been presented in [6]. Also, optimality conditions for the difference of two inverse co-radiant and decreasing functions were obtained in [7,8]. ICRQC functions have attracted numerous applications in microeconomic analysis. Commonly, production theory assumes that the production function is quasi-concave and increasing. Similarly, these properties are assumed for the utility function in consumer theory [9,10]. In the present study, f and g were replaced by non-positive ICRQC functions. In fact, we first gave different characterizations of the minimal elements of the upper support set of non-positive co-radiant functions, using a type of duality. We obtained sufficient and necessary conditions for the global minimum of the difference of two non-positive ICRQC functions defined on X.

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The layout of the paper is organized as follows: Section 2 deals with descriptions, primary results, and notations, which will be used later. In Section 3, we characterize minimal elements of the upper support set of non-positive co-radiant functions. In Section 4, we found sufficient and necessary conditions for the global minimum of the differences of two non-positive ICRQC functions.

2 Preliminaries

Suppose that X be a real locally convex topological vector space with the dual space X^*. It is assumed that X is armed with a convex closed pointed cone S ($S \cap (-S) = \{0\}$). We say $x \leq y$ if and only if $y - x \in S$. Moreover, we considered the natural (point-wise) order relation on X^*, defined by $y^* \leq z^*$ if and only if for all $x \in X$, $\langle x, y^* \rangle \leq \langle x, z^* \rangle$, where $\langle \cdot,\cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$ is the duality pairing. Furthermore, supposing that X^* is armed with the weak-star topology, and put $S^* \coloneqq \{x^* \in X^* | \langle x, x^* \rangle \geq 0, \quad \forall x \in S\}$.

The function $f : X \longrightarrow [-\infty, +\infty]$ is called co-radiant when for all $0 < \lambda \leq 1$ and all $x \in X$ we have $\lambda f(x) \leq f(\lambda x)$. Thus, it is simply observed that f is co-radiant when for all $x \in X$ and all $\lambda \geq 1$, $\lambda f(x) \geq f(\lambda x)$. If $x \geq y \ \Rightarrow \ f(x) \geq f(y)$, the function f would be increasing.

When for all $x \in X$ and all $0 < \lambda \leq 1$, $f(\lambda x) \leq \frac{1}{\lambda} f(x)$, then the function $f : X \longrightarrow [-\infty, +\infty]$ would be inverse co-radiant. Then, it is simply seen that f is inverse co-radiant when for all $\lambda \geq 1$ and all $x \in X$, $f(\lambda x) \geq \frac{1}{\lambda} f(x)$.

Definition 2.1. [11] Let $f : X \longrightarrow (-\infty, +\infty]$ be a function, and H be a non-empty set of functions $h : X \longrightarrow (-\infty, +\infty]$.

(1) The upper support set of f with respect to H is

$$\operatorname{supp}_u(f, H) = \{h \in H : f(x) \leq h(x), \quad \forall x \in X\}.$$

(2) The function f would be abstract concave with respect to H (or H-concave) when there exists a subset U of H such that

$$f(x) = \inf_{h \in U} h(x), \quad (x \in X).$$

(3) Consider $x_0 \in X$ such that $-\infty < f(x_0) < +\infty$. The superdifferential of f at x_0 with respect to H (or H-superdifferential of f) is defined by

$$\partial_H^+ f(x_0) \coloneqq \{h \in H : h(x_0) \in \mathbb{R}, \quad f(x) - f(x_0) \leq h(x) - h(x_0), \quad \forall x \in X\}.$$

Definition 2.2. Let $f : X \longrightarrow [-\infty, 0]$ be a function. The dual function $f^\sharp : X^* \longrightarrow [-\infty, 0]$ of f is defined by

$$f^\sharp(x^*) = \sup\{f(x) : x \in X, \langle x, x^* \rangle \geq 1\}, \quad \forall x^* \in X^* \tag{1}$$

(we used the convention $\sup\emptyset = -\infty$).

The dual function $f^\sharp$ is a quasi-convex and decreasing function [12]. Furthermore, when f is co-radiant, then $f^\sharp$ is inverse co-radiant [12].

Now, we consider the function $k : X \times (-S^*) \times (-\infty, 0) \longrightarrow [-\infty, 0]$ as

$$k(x, y^*, \beta) = \inf\{v_{(y^*,\beta)}(x^*) : x^* \in (-S^*), \quad 1 \leq \langle x, x^* \rangle\},$$

where $v_{(y^*,\beta)}(x^*) = \inf\{\lambda : \lambda \leq \beta, \ y^* \leq -\lambda x^*\}$ (using the convention $\inf\emptyset = 0$).

Suppose that $y^* \in (-S^*)$ and $\beta < 0$ is an arbitrary number. We define the functions $k_{(y^*,\beta)} : X \to [-\infty, 0]$ by $k_{(y^*,\beta)}(x) \coloneqq k(x, y^*, \beta)$. These functions were presented and assessed and the results were obtained as follows [12].

Proposition 2.1. *The function $k_{(y^*,\beta)}$ possesses a simpler shape. We consider $x \in X$, $y^* \in -S^*$ and $\beta < 0$. Then, we have*

$$k_{(y^*,\beta)}(x) = \begin{cases} 0, & \langle x, -y^* \rangle > \beta, \\ \langle x, -y^* \rangle, & \langle x, -y^* \rangle \leq \beta. \end{cases}$$

Now, let $K \coloneqq \{k_{(y^*,\beta)} : y^* \in -S^*, \beta < 0\}$ be the set of elementary functions.

Theorem 2.1. *We suppose that $f : X \longrightarrow [-\infty, 0]$ is an upper semi-continuous function. Then, f is ICRQC if and only if a non-empty set $B \subseteq (-S^*) \times (-\infty, 0)$ exist so that*

$$f(x) = \inf_{(y^*,\beta)\in B} k_{(y^*,\beta)}(x), \quad (x \in X).$$

Here, taking $B \coloneqq \left\{(y^*, \beta) \in (-S^*) \times (-\infty, 0) : f^\sharp\left(\frac{-y^*}{\beta}\right) \leq \beta\right\}$. Thus, f is ICRQC if and only if f is K-concave.

Proposition 2.2. *Suppose that $f : X \longrightarrow [-\infty, 0]$ is a co-radiant function. Then we obtain*

$$\operatorname{supp}_u(f, K) = \{k_{(y^*,\beta)} \in K \ : \ f^\sharp\left(\frac{-y^*}{\beta}\right) \leq \beta\}.$$

3 On minimal elements of the upper support set of non-positive co-radiant functions

Here we characterized minimal elements of the upper support set of non-positive co-radiant functions. Considering a set V of functions defined on a set W, it is assumed that V is armed with the natural order relation (point-wise) of functions. Also, a function $\overline{f} \in V$ is a minimal element of the set V, when $f(w) \leq \overline{f}(w)$ for all $w \in W$ and $f \in V$, then $\overline{f} = f$ on W.

Remark 3.1. Regarding Proposition 2.1, for each $k_{(y^*,\beta)} \in K$ and each $\lambda > 0$, we have $k_{(\lambda y^*,\lambda\beta)}(x) = \lambda k_{(y^*,\beta)}(x)$ for all $x \in X$. Moreover, if $0 \neq y^* \in X^*$, then $x_0 \in X$ exists such that $\langle x_0, y^* \rangle = -1$.

Lemma 3.1. *Suppose that $f : X \to [-\infty, 0]$ is a co-radiant function and $y^* \in (-S^*)\backslash\{0\}$. Let $k_{(y^*,\beta)} \in K$ be a minimal element of $\operatorname{supp}_u(f, K)$. Thus, we have $f^\sharp\left(\frac{-y^*}{\beta}\right) > -\infty$.*

Proof. On the contrary, assume that $f^\sharp\left(\frac{-y^*}{\beta}\right) = -\infty$. Then, $f^\sharp\left(\frac{-2y^*}{2\beta}\right) = f^\sharp\left(\frac{-y^*}{\beta}\right) = -\infty \leq 2\beta$. So, by Proposition 2.2, $k_{(2y^*,2\beta)} \in \operatorname{supp}_u(f, K)$. Therefore, by Remark 3.1, for all $x \in X$, we obtain $k_{(2y^*,2\beta)}(x) = 2k_{(y^*,\beta)}(x)$. Because for all $x \in X$, $k_{(y^*,\beta)}(x) \leq 0$ it follows that for all $x \in X$, $k_{(2y^*,2\beta)}(x) = 2k_{(y^*,\beta)}(x) \leq k_{(y^*,\beta)}(x)$. Using that $k_{(y^*,\beta)}$ is a minimal element of $\operatorname{supp}_u(f, K)$, we obtain

$$k_{(y^*,\beta)}(x) = k_{(2y^*,2\beta)}(x) = 2k_{(y^*,\beta)}(x), \qquad \forall x \in X.$$

So,

$$k_{(y^*,\beta)}(x) = 2k_{(y^*,\beta)}(x), \quad \forall x \in X. \tag{2}$$

Because $y^* \in (-S^*)\backslash\{0\}$, based on Remark 3.1, $x_0 \in X$ exists so that $\langle x_0, y^* \rangle = -1$. Put $t \coloneqq \beta x_0 \in X$. Then, we find that $\langle t, -y^* \rangle = \beta$. Thus, by Proposition 2.1, $k_{(y^*,\beta)}(t) = \langle t, -y^* \rangle = \beta$, and therefore, by putting $x \coloneqq t$ in (2), we have $\beta = 2\beta$. So, $\beta = 0$, which is a contradiction. Thus, $f^\sharp\left(\frac{-y^*}{\beta}\right) > -\infty$. □

Proposition 3.1. *Assume that* $f : X \to [-\infty, 0]$ *is a co-radiant function and* $0 \neq y^* \in (-S^*)$. *Suppose that* $k_{(y^*,\beta)} \in K$ *is a minimal element of* $\operatorname{supp}_u(f, K)$. *Thus,* $f^\sharp\left(\frac{-y^*}{\beta}\right) = \beta$.

Proof. It follows from Lemma 3.1 and Proposition 2.2 that $-\infty < f^\sharp\left(\frac{-y^*}{\beta}\right) \leq \beta$. Now, considering $k_{(z^*,\beta')}$ such that $z^* = \frac{f^\sharp\left(\frac{-y^*}{\beta}\right)}{\beta} y^*$ and $\beta' = f^\sharp\left(\frac{-y^*}{\beta}\right)$, thus, we have $f^\sharp\left(\frac{-z^*}{\beta'}\right) = f^\sharp\left(\frac{-y^*}{\beta}\right) = \beta'$, which implies that by Proposition 2.2, $k_{(z^*,\beta')} \in \operatorname{supp}_u(f, K)$. Also, in view of Remark 3.1 $\left(\text{with } \lambda := \frac{f^\sharp\left(\frac{-y^*}{\beta}\right)}{\beta}\right)$, one has

$$k_{(z^*,\beta')}(x) = \frac{f^\sharp\left(\frac{-y^*}{\beta}\right)}{\beta} k_{(y^*,\beta)}(x), \quad \forall x \in X. \tag{3}$$

Since $k_{(y^*,\beta)}(x) \leq 0$ and $\frac{f^\sharp\left(\frac{-y^*}{\beta}\right)}{\beta} \geq 1$, it concludes from (3) that $k_{(z^*,\beta')}(x) \leq k_{(y^*,\beta)}(x)$ for all $x \in X$. Taking into account that $k_{(y^*,\beta)} \in K$ is a minimal element of $\operatorname{supp}_u(f, K)$, it is concluded that

$$k_{(z^*,\beta')}(x) = k_{(y^*,\beta)}(x), \quad \forall x \in X. \tag{4}$$

Because $0 \neq y^* \in (-S^*)$, there is $t \in X$ so that $\langle t, -y^* \rangle = \beta$ by a similar argument as proving Lemma 3.1. Hence, by Proposition 2.1, we have $k_{(y^*,\beta)}(t) = \langle t, -y^* \rangle = \beta$. Substituting $x := t$ in (3) and (4), it is concluded that $f^\sharp\left(\frac{-y^*}{\beta}\right) = \beta$, which completes the proof.

By further conditions, we will reveal that the converse of Proposition 3.1 holds.

Proposition 3.2. *Let* $f : X \to [-\infty, 0]$ *be a co-radiant function. Suppose that* $0 \neq y^* \in (-S^*)$ *is such that* $\varepsilon_{y^*} := \max\left\{\beta < 0 \;:\; f^\sharp\left(\frac{-y^*}{\beta}\right) \leq \beta\right\} > -\infty$. *Assume that* $f^\sharp$ *is one-to-one. Thus,* $k_{(y^*,\varepsilon_{y^*})} \in K$ *is a minimal element of* $\operatorname{supp}_u(f, K)$ *if and only if* $f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) = \varepsilon_{y^*}$.

Proof. Based on Proposition 3.1, it is indicated that if $f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) = \varepsilon_{y^*}$, then, $k_{(y^*,\varepsilon_{y^*})}$ is a minimal element of $\operatorname{supp}_u(f, K)$. Hence, assume that $k_{(z^*,\varepsilon')} \in \operatorname{supp}_u(f, K)$ is such that

$$k_{(z^*,\varepsilon')}(x) \leq k_{(y^*,\varepsilon_{y^*})}(x), \quad \forall x \in X. \tag{5}$$

So, by Proposition 2.1 and (5) we have

$$\{t \in X : \; \langle t, -y^* \rangle \leq \varepsilon_{y^*}\} \subseteq \{t \in X : \; \langle t, -z^* \rangle \leq \varepsilon'\}. \tag{6}$$

Now, for all $x \in X$, it is revealed that $k_{(z^*,\varepsilon')}(x) = k_{(y^*,\varepsilon_{y^*})}(x)$. Let $t \in X$ so that $\langle t, -y^* \rangle \leq \varepsilon_{y^*}$. By (6), $\langle t, -z^* \rangle \leq \varepsilon'$, and so, substituting $x := t$ in (5) and Proposition 2.1, we obtain

$$\langle t, -z^* \rangle = k_{(z^*,\varepsilon')}(t) \leq k_{(y^*,\varepsilon_{y^*})}(t) = \langle t, -y^* \rangle \leq \varepsilon_{y^*},$$

thus,

$$\{t \in X : \; \langle t, -y^* \rangle \leq \varepsilon_{y^*}\} \subseteq \{t \in X : \; \langle t, -z^* \rangle \leq \varepsilon_{y^*}\}. \tag{7}$$

Also, by (6), we have

$$f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) \leq f^\sharp\left(\frac{-z^*}{\varepsilon'}\right). \tag{8}$$

Thus, by hypothesis, (8) and since $k_{(z^*,\varepsilon')} \in \mathrm{supp}_u(f,K)$, we conclude that $\varepsilon_{y^*} = f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) \le f^\sharp\left(\frac{-z^*}{\varepsilon'}\right) \le \varepsilon'$. So,

$$\varepsilon_{y^*} \le \varepsilon'. \tag{9}$$

Now, it is claimed that when $x \in X$ so that $\langle x, -z^*\rangle \le \varepsilon_{y^*}$, then $f(x) \le \varepsilon_{y^*}$. First, it is indicated that if $\langle x, -z^*\rangle \le \varepsilon'$, then $f(x) \le \varepsilon'$. Suppose that $\langle x, -z^*\rangle \le \varepsilon'$, since $f^\sharp\left(\frac{-z^*}{\varepsilon'}\right) \le \varepsilon'$ based on Definition 2.2, we obtain

$$f(x) \le f^\sharp\left(\frac{-z^*}{\varepsilon'}\right) \le \varepsilon'. \tag{10}$$

Now, let $x \in X$ so that $\langle x, -z^*\rangle \le \varepsilon_{y^*}$. Then, $\langle \frac{\varepsilon' x}{\varepsilon_{y^*}}, -z^*\rangle \le \varepsilon'$. So, (10) implies that $f\left(\frac{\varepsilon' x}{\varepsilon_{y^*}}\right) \le \varepsilon'$. Since, by (9) and $\varepsilon_{y^*} < 0$, we have $0 < \frac{\varepsilon'}{\varepsilon_{y^*}} < 1$ and because f is co-radiant, we obtain $\frac{\varepsilon'}{\varepsilon_{y^*}} f(x) \le f\left(\frac{\varepsilon' x}{\varepsilon_{y^*}}\right) \le \varepsilon'$. Thus, for all $x \in X$, we obtain $f(x) \le \varepsilon_{y^*}$ so that $\langle x, -z^*\rangle \le \varepsilon_{y^*}$. Therefore,

$$f^\sharp\left(\frac{-z^*}{\varepsilon_{y^*}}\right) = \sup\{f(x) : x \in X, \quad \langle x, -z^*\rangle \le \varepsilon_{y^*}\} \le \varepsilon_{y^*}. \tag{11}$$

So, by hypotheses, (7) and (11), we conclude that $\varepsilon_{y^*} = f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) \le f^\sharp\left(\frac{-z^*}{\varepsilon_{y^*}}\right) \le \varepsilon_{y^*}$, hence, it is indicated that $f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) = f^\sharp\left(\frac{-z^*}{\varepsilon_{y^*}}\right)$. Because $f^\sharp$ is one-to-one, thus, $y^* = z^*$. Moreover, since $\varepsilon_{y^*} \coloneqq \max\left\{\beta < 0 \ : \ f^\sharp\left(\frac{-y^*}{\beta}\right) \le \beta\right\}$ and $f^\sharp(\frac{-y^*}{\varepsilon'}) = f^\sharp\left(\frac{-z^*}{\varepsilon'}\right) \le \varepsilon'$, then $\varepsilon_{y^*} \ge \varepsilon'$. So, by (9), we obtain $\varepsilon_{y^*} = \varepsilon'$. Therefore, for all $x \in X$, we have $k_{(z^*,\varepsilon')}(x) = k_{(y^*,\varepsilon_{y^*})}(x)$ which completes the proof. □

By Example 3.1, it is revealed that the one-to-one of the function $f^\sharp$ in Proposition 3.2 cannot be omitted.

Example 3.1. Let $X \coloneqq \mathbb{R}^2$ and $S \coloneqq \mathbb{R}^2_+$. Consider $f : X \to (-\infty, 0]$ defined as follows:

$$f(x) = \begin{cases} 0, & x_2 > -1, \\ x_2, & x_2 \le -1, \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. Here, $-S^* = \mathbb{R}^2_-$. Obviously, f is a non-positive co-radiant function. Put $x^* \coloneqq (-1,-1)$ and $z^* \coloneqq (-2,-1)$, then $f^\sharp(x^*) = 0 = f^\sharp(z^*)$. So, $f^\sharp$ is not one-to-one. Let $y^* \coloneqq (0,-1)$, then $\varepsilon_{y^*} \coloneqq \max\left\{\beta < 0 \ : \ f^\sharp\left(\frac{-y^*}{\beta}\right) \le \beta\right\} = -1$ and $f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) = -1$. Thus, $f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) = \varepsilon_{y^*}$. Now, it is presented that $k_{(y^*,\varepsilon_{y^*})}$ is not a minimal element of $\mathrm{supp}_u(f,K)$. Let $\beta_0 \coloneqq \frac{1}{2}\varepsilon_{y^*} = -\frac{1}{2}$. It is clear that $f^\sharp\left(\frac{-y^*}{\beta_0}\right) = \beta_0 = -\frac{1}{2}$. So, $k_{(y^*,\beta_0)} \in \mathrm{supp}_u(f,K)$. And also, by Proposition 2.1 we have

$$k_{(y^*,\varepsilon_{y^*})}(x) \coloneqq \begin{cases} x_2, & x_2 \le -1, \\ 0, & x_2 > -1, \end{cases}$$

and

$$k_{(y^*,\beta_0)}(x) \coloneqq \begin{cases} x_2, & x_2 \le -\dfrac{1}{2}, \\ 0, & x_2 > -\dfrac{1}{2}. \end{cases}$$

Therefore, we obtain $k_{(y^*,\beta_0)} \lneqq k_{(y^*,\varepsilon_{y^*})}$.

Lemma 3.2. *Suppose that $f : X \to [-\infty, 0]$ is a function such that $f^\sharp$ is one-to-one. Then, $f^\sharp(y^*) > -\infty$ for each $y^* \in (-S^*)\backslash\{0\}$.*

Proof. Assuming that there is $x^* \in (-S^*)\backslash\{0\}$ so that $f^\sharp(x^*) = -\infty$. Then, by (1) we obtain that

$$\begin{aligned} -\infty \le f^\sharp\left(\frac{1}{2}x^*\right) &= \sup\left\{f(x) : x \in X, \quad \left\langle x, \frac{1}{2}x^*\right\rangle \ge 1\right\} \\ &= \sup\{f(x) : x \in X, \quad \langle x, x^*\rangle \ge 2\} \\ &\le \sup\{f(x) : x \in X, \quad \langle x, x^*\rangle \ge 1\} \\ &= f^\sharp(x^*) = -\infty. \end{aligned}$$

So, it will be obtained $f^\sharp(x^*) = f^\sharp\left(\frac{1}{2}x^*\right) = -\infty$. As $f^\sharp$ is one-to-one, it is concluded that $x^* = \frac{1}{2}x^*$, then $x^* = 0$, which is a contradiction. □

Corollary 3.1. *If $f : X \to [-\infty, 0]$ is a co-radiant function. Assume that $y^* \in (-S^*)\backslash\{0\}$ is such that $\varepsilon_{y^*} := \max\left\{\delta < 0 \ : \ f^\sharp\left(\frac{-y^*}{\delta}\right) \le \delta\right\} > -\infty$. Suppose that $f^\sharp$ is one-to-one. Thus, for each $k_{(y^*,\beta)} \in \operatorname{supp}_u(f, K)$, a minimal element $k_{(\tilde{y}^*,\tilde{\beta})}$ of $\operatorname{supp}_u(f, K)$ exists so that $k_{(\tilde{y}^*,\tilde{\beta})}(x) \le k_{(y^*,\beta)}(x)$ for all $x \in X$. Here, one has $\tilde{y}^* = \frac{f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right)}{\varepsilon_{y^*}}y^*$ and $\tilde{\beta} = f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right)$.*

Proof. By Lemma 3.2 and hypothesis, we have $-\infty < f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) < 0$. So, $-\infty < \tilde{\beta} < 0$ and $\tilde{y}^* \in (-S^*)\backslash\{0\}$. Consider $k_{(y^*,\beta)} \in \operatorname{supp}_u(f, K)$, clearly $f^\sharp\left(\frac{-\tilde{y}^*}{\tilde{\beta}}\right) = \tilde{\beta}$. Therefore, by Proposition 2.2, $k_{(\tilde{y}^*,\tilde{\beta})} \in \operatorname{supp}_u(f, K)$. Since $k_{(y^*,\beta)} \in \operatorname{supp}_u(f, K)$, regarding Proposition 2.2, $f^\sharp\left(\frac{-y^*}{\beta}\right) \le \beta$, therefore, $\varepsilon_{y^*} \ge \beta$. This indicates that $k_{(y^*,\beta)}(x) \ge k_{(y^*,\varepsilon_{y^*})}(x)$ for each $x \in X$, which together with $\frac{f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right)}{\varepsilon_{y^*}} \ge 1$, $k_{(y^*,\varepsilon_{y^*})}(x) \le 0$, and Remark 3.1 implies that

$$k_{(\tilde{y}^*,\tilde{\beta})}(x) = \frac{f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right)}{\varepsilon_{y^*}} k_{(y^*,\varepsilon_{y^*})}(x) \le k_{(y^*,\varepsilon_{y^*})}(x) \le k_{(y^*,\beta)}(x), \quad \forall x \in X.$$

Therefore, $k_{(y^*,\beta)}(x) \ge k_{(\tilde{y}^*,\tilde{\beta})}(x)$ for each $x \in X$. Now, it is observed that $k_{(\tilde{y}^*,\tilde{\beta})}$ is a minimal element of $\operatorname{supp}_u(f, K)$. For this aim, consider $N := \left\{\zeta < 0 \ : \ f^\sharp\left(\frac{-\tilde{y}^*}{\zeta}\right) \le \zeta\right\}$. Because $f^\sharp\left(\frac{-\tilde{y}^*}{\tilde{\beta}}\right) = \tilde{\beta}$, then $\tilde{\beta} \in N$. Now, it is indicated that $\tilde{\beta} \ge \zeta$ for all $\zeta \in N$. Let $\zeta \in N$ be an arbitrary number. Then, $f^\sharp\left(\frac{-\tilde{y}^*}{\zeta}\right) \le \zeta$. Let $\gamma := \frac{\varepsilon_{y^*}}{f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right)}$. Hypothesis implies that $f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) \le \varepsilon_{y^*}$, so $0 < \gamma \le 1$. Since $f^\sharp\left(\frac{-\tilde{y}^*}{\zeta}\right) \le \zeta$, thus $f^\sharp\left(\frac{-\tilde{y}^*}{\zeta}\right) \le \gamma\zeta$, that is $f^\sharp(\frac{-y^*}{\gamma\zeta}) \le \gamma\zeta$. Then, we obtain $\gamma\zeta \in \{\delta < 0 \ : \ f^\sharp\left(\frac{-y^*}{\delta}\right) \le \delta\}$. Therefore, by hypothesis, we have $\varepsilon_{y^*} \ge \gamma\zeta$. It is indicated that $\zeta \le \tilde{\beta}$. Thus, $\tilde{\beta} = \max N$. Since $f^\sharp$ is one-to-one and $f^\sharp\left(\frac{-\tilde{y}^*}{\tilde{\beta}}\right) = \tilde{\beta}$. So, the result is based on Proposition 3.2. □

Theorem 3.1. *Let $g, f : X \to [-\infty, 0]$ be co-radiant functions. Assume that* $\varepsilon_{y^*} \coloneqq \max\left\{\delta < 0 \ : \ f^\sharp\left(\frac{-y^*}{\delta}\right) \le \delta\right\} > -\infty$ *and* $\eta_{y^*} \coloneqq \max\left\{\theta < 0 \ : \ g^\sharp\left(\frac{-y^*}{\theta}\right) \le \theta\right\} > -\infty$ $(y^* \in (-S^*)\backslash\{0\})$. *Moreover, let $f^\sharp$ and $g^\sharp$ be one-to-one. Then, the following assertions are equivalent*:

(*i*) $\operatorname{supp}_u(f, K) \subseteq \operatorname{supp}_u(g, K)$.

(*ii*) *For each minimal element $k_{(y_1^*, \beta_1)}$ of $\operatorname{supp}_u(f, K)$, a minimal element $k_{(y_2^*, \beta_2)}$ of $\operatorname{supp}_u(g, K)$ exists so that $k_{(y_1^*, \beta_1)}(x) \ge k_{(y_2^*, \beta_2)}(x)$ for all $x \in X$.*

(*iii*) $g^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) \le \varepsilon_{y^*}$ *for each* $y^* \in (-S^*)\backslash\{0\}$.

Proof. (i) ⇒ (ii). When $k_{(y_1^*, \beta_1)}$ is a minimal element of $\operatorname{supp}_u(f, K)$, then, by hypothesis, $k_{(y_1^*, \beta_1)} \in \operatorname{supp}_u(g, K)$. Thus, by Corollary 3.1, a minimal element $k_{(y_2^*, \beta_2)}$ of $\operatorname{supp}_u(g, K)$ exists such that for all $x \in X$, $k_{(y_1^*, \beta_1)}(x) \ge k_{(y_2^*, \beta_2)}(x)$.

(ii) ⇒ (i). Suppose that $k_{(y^*, \beta)} \in \operatorname{supp}_u(f, K)$ be arbitrary. Thus, by Corollary 3.1, a minimal element $k_{(y_1^*, \beta_1)}$ of $\operatorname{supp}_u(f, K)$ exists such that for all $x \in X$, $k_{(y^*, \beta)}(x) \ge k_{(y_1^*, \beta_1)}(x)$. So, hypothesis (ii) implies that a minimal element $k_{(y_2^*, \beta_2)}$ of $\operatorname{supp}_u(g, K)$ exists such that for all $x \in X$,$k_{(y_1^*, \beta_1)}(x) \ge k_{(y_2^*, \beta_2)}(x)$. So, for all $x \in X$, $k_{(y^*, \beta)}(x) \ge k_{(y_2^*, \beta_2)}(x)$, indicating that $k_{(y^*, \beta)} \in \operatorname{supp}_u(g, K)$. Hence, we see that $\operatorname{supp}_u(f, K) \subseteq \operatorname{supp}_u(g, K)$.

(i) ⇒ (iii). Consider $y^* \in (-S^*)\backslash\{0\}$. Because $f^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) \le \varepsilon_{y^*}$, it concludes from Proposition 2.2 that $k_{(y^*, \varepsilon_{y^*})} \in \operatorname{supp}_u(f, K)$, and so, by hypothesis, $k_{(y^*, \varepsilon_{y^*})} \in \operatorname{supp}_u(g, K)$. Therefore, by Proposition 2.2, $g^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) \le \varepsilon_{y^*}$.

(iii) ⇒ (i). Assume that $k_{(y^*, \beta)} \in \operatorname{supp}_u(f, K)$ is arbitrary, then by Proposition 2.2, $f^\sharp\left(\frac{-y^*}{\beta}\right) \le \beta$. Thus, based on the hypothesis, we have $\beta \le \varepsilon_{y^*}$. Because $g^\sharp$ is inverse co-radiant, by hypothesis (iii), one has

$$g^\sharp\left(\frac{-y^*}{\beta}\right) = g^\sharp\left(\frac{\varepsilon_{y^*}}{\beta}\frac{-y^*}{\varepsilon_{y^*}}\right) \le \frac{\beta}{\varepsilon_{y^*}} g^\sharp\left(\frac{-y^*}{\varepsilon_{y^*}}\right) \le \beta.$$

Hence, Proposition 2.2 implies that $k_{(y^*, \beta)} \in \operatorname{supp}_u(g, K)$. Thus, $\operatorname{supp}_u(f, K) \subseteq \operatorname{supp}_u(g, K)$. □

4 On conditions for the global minimizers of the difference of non-positive valued ICRQC

In this section, we present sufficient and necessary conditions for the global minimum of the differences of two non-positive valued ICRQC and proper upper semi-continuous functions. Assume that $g, f : X \to [-\infty, 0]$ are ICRQC and proper upper semi-continuous functions such that $\operatorname{dom}(f) \subseteq \operatorname{dom}(g)$, where $\operatorname{dom}(f) \coloneqq \{x \in X : -\infty < f(x) < +\infty\}$. Consider the function $h \coloneqq g - f$, that is,

$$h(x) \coloneqq \begin{cases} g(x) - f(x), & x \in \operatorname{dom}(f), \\ +\infty, & x \notin \operatorname{dom}(f), \end{cases} \tag{12}$$

(with the convention $(-\infty) - (-\infty) = +\infty$). Obviously, when $\operatorname{dom}(f) \nsubseteq \operatorname{dom}(g)$, then, $\inf_{x \in X} h(x) = -\infty$. Thus, considering $\inf_{x \in X} h(x) > -\infty$, it is implied that $\operatorname{dom}(f) \subseteq$. Also, since $h(0) = f(0) - g(0) = 0$, one has $\inf_{x \in X} h(x) \le 0$. Based on Theorem 2.1, so

$$f(x) \le g(x), \quad \forall x \in X \Leftrightarrow \operatorname{supp}_u(g, K) \subseteq \operatorname{supp}_u(f, K). \tag{13}$$

First note that if for all $x \in X$, $f(x) \le g(x)$, then it is easy to see that $\operatorname{supp}_u(g, K) \subseteq \operatorname{supp}_u(f, K)$. Conversely, suppose that $\operatorname{supp}_u(g, K) \subseteq \operatorname{supp}_u(f, K)$. Suppose if possible that $x_0 \in X$ exists such that $f(x_0) > g(x_0)$. Because g is an ICRQC and proper upper semi-continuous function, in view of Theorem 2.1 and Proposition 2.2, it is concluded that

$$g(x_0) = \inf_{k_{(y^*,\beta)} \in \operatorname{supp}_u(g,K)} k_{(y^*,\beta)}(x_0). \tag{14}$$

Because $f(x_0) > g(x_0)$, it is based on (14) that $k_{(y_0^*,\beta_0)} \in \operatorname{supp}_u(g, K)$ exists such that $f(x_0) > k_{(y_0^*,\beta_0)}(x_0)$. It is indicated that $k_{(y_0^*,\beta_0)} \notin \operatorname{supp}_u(f, K)$, which contradicts the hypothesis. Hence, $f(x) \le g(x)$ for each $x \in X$.

Theorem 4.1. *Let* $f, g : X \to [-\infty, 0]$. *Assume that* $\varepsilon_{y^*} := \max\left\{\delta < 0 \ : \ f^\sharp\left(\frac{-y^*}{\delta}\right) \le \delta\right\} > -\infty$ *and* $\eta_{y^*} := \max\left\{\theta < 0 \ : \ g^\sharp\left(\frac{-y^*}{\theta}\right) \le \theta\right\} > -\infty$ $(y^* \in (-S^*)\backslash\{0\})$. *Moreover, suppose that* $f^\sharp$ *and* $g^\sharp$ *are one-to-one. Consider* $x_0 \in X$ *with* $h(x_0) \le 0$. *Thus, we have*:

(*i*) *If* x_0 *is a global minimizer of the function* $h := g - f$, *and g and f are proper co-radiant functions, then,* $f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right) \le \mu_{y^*}$ *for each* $y^* \in (-S^*)\backslash\{0\}$, *where* $\mu_{y^*} := \max\left\{\varepsilon < 0 \ : \ g^\sharp\left(\frac{-y^*}{\varepsilon}\right) - h(x_0) \le \varepsilon\right\}$.

(*ii*) *If g and f are ICRQC and proper upper semi-continuous functions, and* $f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right) \le \mu_{y^*}$ (*defined by Assertion* (*i*)) *for each* $y^* \in (-S^*)\backslash\{0\}$, *then* x_0 *is a global minimizer of the function h.*

Proof. (i) Assume that x_0 is a global minimizer for the function h. Therefore, for all $x \in X$, $h(x) = g(x) - f(x) \ge h(x_0)$, and thus, for all $x \in X$, $f(x) \le \tilde{g}(x) := g(x) - h(x_0)$. Then, by (13) we obtain $\operatorname{supp}_u(\tilde{g}, K) \subseteq \operatorname{supp}_u(f, K)$. So, based on Theorem 3.1 (the implication (i) ⇒ (iii)), $f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right) \le \mu_{y^*}$ for each $y^* \in (-S^*)\backslash\{0\}$, where $\mu_{y^*} := \max\left\{\varepsilon < 0 \ : \ (\tilde{g})^\sharp\left(\frac{-y^*}{\varepsilon}\right) \le \varepsilon\right\} = \max\left\{\varepsilon < 0 \ : \ g^\sharp\left(\frac{-y^*}{\varepsilon}\right) - h(x_0) \le \varepsilon\right\}$. It is worth noting that since $-h(x_0) \ge 0$, thus $\tilde{g}$ is a co-radiant function, and $(\tilde{g})^\sharp$ is one-to-one.

(ii) Assume that g and f are ICRQC and proper upper semi-continuous functions, and $f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right) \le \mu_{y^*}$ for each $y^* \in (-S^*)\backslash\{0\}$. So, by Theorem 3.1 (the implication (iii) ⇒ (i)), we conclude that $\operatorname{supp}_u(\tilde{g}, K) \subseteq \operatorname{supp}_u(f, K)$. Thus, (13) includes that for all $x \in X$, $f(x) \le \tilde{g}(x) = g(x) - h(x_0)$, and hence, for all $x \in X$, $h(x_0) \le h(x)$, i.e., x_0 is a global minimizer of the function h. Note that since $-h(x_0) \ge 0$, then $\tilde{g}$ is an ICRQC and upper semi-continuous function, and $(\tilde{g})^\sharp$ is one-to-one. □

Theorem 4.2. *Let* $g, f : X \to [-\infty, 0]$ *be ICRQC and proper upper semi-continuous functions. Suppose that* $\varepsilon_{y^*} := \max\left\{\delta < 0 \ : \ f^\sharp\left(\frac{-y^*}{\delta}\right) \le \delta\right\} > -\infty$ *and* $\eta_{y^*} := \max\left\{\theta < 0 \ : \ g^\sharp\left(\frac{-y^*}{\theta}\right) \le \theta\right\} > -\infty (y^* \in (-S^*)\backslash\{0\})$. *Moreover, assume that* $g^\sharp$ *and* $f^\sharp$ *are one-to-one. Consider* $x_0 \in X$ *with* $h(x_0) < 0$. *Thus, the following assertions are equivalent*:

(*i*) *The point* x_0 *is a global minimizer of the function h* (*defined by* (12)).

(*ii*) *We have* $\inf_{y^* \in (-S^*)\backslash\{0\}}\left\{\mu_{y^*} - f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right)\right\} = 0$, *where*

$$\mu_{y^*} := \max\left\{\varepsilon < 0 \ : \ g^\sharp\left(\frac{-y^*}{\varepsilon}\right) - h(x_0) \le \varepsilon\right\}.$$

Proof. (*i*) ⇒ (*ii*). Let $m := \inf_{y^* \in (-S^*)\backslash\{0\}}\left\{\mu_{y^*} - f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right)\right\}$. Suppose that x_0 is a global minimizer of the function h. We show that $m = 0$. For this purpose, suppose that $m > 0$ (note that by Theorem 4.1(i), we have that $m \ge 0$). Now, $m' > 0$ is chosen so that $0 < m' \le m$. Then, $m' \le \mu_{y^*} - f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right)$ for each $y^* \in (-S^*)\backslash\{0\}$. Since $m' > 0$, in view of Theorem 3.1 (the implication (iii) ⇒ (i)), we obtain $\mathrm{supp}_u(\tilde{g}, K) \subseteq \mathrm{supp}_u(\bar{f}, K)$, where for all $x \in X$, $\bar{f}(x) := f(x) + m'$ (it is worth noting that since $m' > 0$, it follows that $\bar{f}$ is an upper semi-continuous function in U_{iq}^{-}, and $(\bar{f})^{\sharp}$ is one-to-one). Thus, (13) implies that for all $x \in X$, $\bar{f}(x) \leq \tilde{g}(x) = g(x) - h(x_0)$. In particular, this implies that $\bar{f}(x_0) = f(x_0) + m' \leq \tilde{g}(x_0) = g(x_0) - h(x_0)$. Therefore, $m' \leq 0$, which is a contradiction. Hence, $m = 0$.

$(ii) \Rightarrow (i)$. Suppose that

$$\inf_{y^* \in (-S^*)\setminus\{0\}} \left\{ \mu_{y^*} - f^{\sharp}\left(\frac{-y^*}{\mu_{y^*}} \right) \right\} = 0.$$

So, $f^{\sharp}\left(\frac{-y^*}{\mu_{y^*}}\right) \leq \mu_{y^*}$ for each $y^* \in (-S^*)\setminus\{0\}$. Thus, by Theorem 4.1(ii), it is concluded that x_0 is a global minimizer of the function h. □

Example 4.1. Assume that $X := \mathbb{R}$ and $S := \mathbb{R}_+$. Suppose that $f, g : X \to (-\infty, 0]$ are defined as follows:

$$f(x) := \begin{cases} -x^2, & x < 0, \\ 0, & x \geq 0, \end{cases} \quad \text{and} \quad g(x) := \begin{cases} 3x + 1, & x \leq -\frac{1}{3}, \\ 0, & x \geq -\frac{1}{3}. \end{cases}$$

Here, we have $-S^* = \mathbb{R}_-$. It is obvious that g and f are non-positive IRCQC and proper upper semi-continuous functions, then

$$\varepsilon_{y^*} := \max\left\{ \delta < 0 \ : \ f^{\sharp}\left(\frac{-y^*}{\delta}\right) \leq \delta \right\} = -(y^*)^2 \quad (y^* < 0),$$

$$\eta_{y^*} := \max\left\{ \theta < 0 \ : \ g^{\sharp}\left(\frac{-y^*}{\theta}\right) \leq \theta \right\} = \begin{cases} \dfrac{y^*}{y^* + 3}, & -3 < y^* < 0, \\ -\infty, & y^* \leq -3, \end{cases}$$

and

$$\mu_{y^*} := \max\left\{ \varepsilon < 0 \ : \ g^{\sharp}\left(\frac{-y^*}{\varepsilon}\right) - h(x_0) \leq \varepsilon \right\} = \begin{cases} (1 - h(x_0))\dfrac{y^*}{y^* + 3}, & -3 < y^* < 0, \\ -\infty, & y^* \leq -3. \end{cases} \tag{15}$$

Thus, by Theorem 4.2 and utilizing (15), x_0 is a global minimizer of the function $h := g - f$ if and only if

$$\inf_{y^* \in (-S^*)\setminus\{0\}} \left\{ \mu_{y^*} - f^{\sharp}\left(\frac{-y^*}{\mu_{y^*}} \right) \right\} = 0,$$

if and only if

$$\inf_{-3<y^*<0} \left\{ (1 - h(x_0))\frac{y^*}{y^* + 3} + \left(\frac{h(x_0) - 1}{y^* + 3} \right)^2 \right\} = 0,$$

if and only if $h(x_0) = -\frac{5}{4}$, and if and only if $x_0 = -\frac{3}{2}$.

Example 4.2. Suppose that $X := C([0,1])$ is the Banach space of all real-valued continuous functions defined on $[0,1]$, and put $S := \{x \in X : \ x(t) \geq 0, \ \forall t \in [0,1]\}$. Thus, S is a closed convex and pointed cone in $C([0,1])$. Assume that $f, g : C([0,1]) \to (-\infty, 0]$ are defined as follows:

$$f(x) := \begin{cases} -\|x\|^2 + 1, & x \in -S, \|x\| \geq 1, \\ 0, & o.\,w, \end{cases} \qquad \text{and} \qquad g(x) := \begin{cases} -\|x\|, & x \in -S, \\ 0, & x \notin -S. \end{cases}$$

Note that we have $\|x\| = \max_{t\in[0,1]}|x(t)|$ for all $x \in X$ and $\|y^*\| = \sup_{x\in X,\|x\|=1}|y^*(x)|$ for all $y^* \in X^*$, then g and f are non-positive ICRQC and proper upper semi-continuous functions. Let $h \coloneqq g - f$, then

$$\mu_{y^*} \coloneqq \max\left\{\varepsilon < 0 \ : \ g^\sharp\left(\frac{-y^*}{\varepsilon}\right) - h(x_0) \le \varepsilon\right\} = \begin{cases} \dfrac{h(x_0)\|y^*\|}{1-\|y^*\|}, & \|y^*\| < 1, \\ -\infty, & \|y^*\| \ge 1, \end{cases} \tag{16}$$

and

$$f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right) = \frac{-h^2(x_0)}{(1-\|y^*\|)^2} + 1, \quad (\|y^*\| < 1). \tag{17}$$

Now, let $x_0 :\equiv -1$ (constant function), then $h(x_0) = -1$. So (16) and (17) imply that

$$\inf_{y^*\in(-S^*)\setminus\{0\}}\left\{\mu_{y^*} - f^\sharp\left(\frac{-y^*}{\mu_{y^*}}\right)\right\} = \inf_{\|y^*\|<1}\left\{\frac{-\|y^*\|}{1-\|y^*\|} + \frac{1}{(1-\|y^*\|)^2} - 1\right\} = 0.$$

Therefore, following Theorem 4.2, x_0 is a global minimizer of the function h.

5 Conclusion

We present different characterizations of the minimal elements of the upper support set of non-positive co-radiant functions, using a type of duality. We obtained sufficient and necessary conditions for the global minimum of the difference of two non-positive ICRQC functions defined on X.

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