Research Article

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Solving multi-point problem for Volterra-Fredholm integro-differential equations using Dzhumabaev parameterization method#

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Abstract: In this study, a multipoint boundary value problem for Volterra-Fredholm integro-differential equations is considered. The addition of a new function converts the system of Volterra-Fredholm integro-differential equations to a system of Fredholm integro-differential equations. In contrast to the original problem, the dimension of a Fredholm integro-differential equation is determined by the number of matrices in the degenerate kernel of the Volterra integral. A numerical algorithm of Dzhumabaev parameterization method for addressing a multipoint boundary value problem for Volterra-Fredholm integro-differential equations is proposed. The main advantage of the proposed method is splitting the problem into auxiliary Cauchy problems for ordinary differential equations and a system of algebraic equations with respect to the parameters. The conditions for the unique solvability of the multipoint boundary value problem for Fredholm integro-differential equations are established. Finally, various numerical examples are provided to demonstrate the efficiency and correctness of the suggested technique.

Keywords: boundary value problem, Volterra-Fredholm integro-differential equation, parameterization method, algorithm, numerical solution

MSC 2020: 34K10, 34K28, 45J99

1 Introduction

In this study, we pay our attention to the numerical algorithm solving the system of Volterra-Fredholm integro-differential equations with degenerate kernels

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + \sum_{k=1}^{m}\varphi_k(t)\int_0^T\psi_k(s)x(s)\mathrm{d}s + \sum_{k=1}^{m}\phi_k(t)\int_0^t\chi_k(s)x(s)\mathrm{d}s + f(t), \quad t \in (0, T), \tag{1}$$

Dedicated to the 70th anniversary and bright memory of Professor Dulat S. Dzhumabaev.

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with boundary condition

$$\sum_{i=0}^{N} B_i x(t_i) = d, \quad x \in \mathbb{R}^n, \ d \in \mathbb{R}^n, \tag{2}$$

where the $(n \times n)$-matrices $A(t)$, $\varphi_k(t)$, $\phi_k(t)$, $\psi_k(s)$, $\chi_k(s)$, $k = \overline{1, m}$, and n-vector $f(t)$ are continuous on $[0, T]$, B_i, $i = \overline{0, N}$, are constant $(n \times n)$-matrices, $t_0 = 0 < t_1 < \ldots < t_{N-1} < t_N = T$, $||x|| = \max_{i=\overline{1,n}}|x_i|$.

(1) and (2) problem solution is a vector function $x(t)$, continuous on $[0, T]$, and continuously differentiable on $(0, T)$. It satisfies the integro-differential equation (1) and multi-point condition (2).

Numerous physics and technology concerns lead to the research of integro-differential equations and the creation of specific difficulties for them [1–7]. In this regard, the theory of integro-differential equations has long attracted the interest of theoretical physicists and mathematicians alike.

Volterra-Fredholm integro-differential equations have received a lot of attention due to their enormous importance in many areas of research and engineering. Numerous problems in chemistry, biology, economics, finance, astronomy, and mechanics lead to these equations [8–15]. Because it is difficult to solve the Volterra-Fredholm integro-differential equation analytically, a numerical approach must be presented. Several numerical methods [16–25], for examples, Taylor method [16], Tau method [17], Chebyshev method [18], method based on block pulse functions [19], He's homotopy perturbation method [20], Bessel collocation method [21], integral collocation approximation method [22], Bernestein polynomials method [23], Legendre collocation method [24], homotopy analysis method [26], Haar collocation method [27], weighted residual scheme reminiscent of the Galerkin method [25], have been used.

Dzhumabaev parameterization method is one of the constructive methods for solving boundary value problems for various classes of differential equations. This method was originally developed for investigating and solving boundary value problems for ordinary differential equations [28]. Dzhumabaev parameterization method is based on partitioning the interval $[0, T]$ into N parts and introducing additional parameters as the value of the desired function at the internal points of the partition. The main advantage of the Dzhumabaev parameterization method for solving boundary value problems is its splitting into auxiliary Cauchy problems for ordinary differential equations and a system of algebraic equations with respect to the introduced parameters. Solutions to the original problem are determined through solutions of Cauchy problems and systems of algebraic equations. Numerical methods for solving Cauchy problems, as well as numerical integration methods, are used for an approximate and numerical solution of the considering problem.

On the basis of the Dzhumabaev parameterization method, a new approach to the general solution of the linear Fredholm integro-differential equations is suggested in the work [6]. The Dzhumabaev parameterization approach is used to solve nonlinear boundary value problems for loaded differential equations [29] and delay differential equations [30]. The Dzhumabaev parameterization method was successfully used by the authors of this article to solve problems for the integro-differential equations [31–33], the problems for differential equations with piecewise constant argument of generalized type [34,35], the problems for class of hyperbolic equations [36,37], the boundary-value problem for impulsive systems of loaded differential equations [38,39], the problem for a partial differential equation [40], the control problem for a differential equation with a parameter [41], and the problem for essentially loaded differential equations [42]. The acquired results prompted the authors of this study to look into problem (1), (2).

We expand the approach provided in [28] to solve the multi-point boundary value problem for a system of Volterra-Fredholm integro-differential equations in this study. This article is structured as follows: Section 2 describes a way for addressing problem (1), (2). Section 3 presents a method for finding a solution to problem (1), (2). Finally, in Section 4, several numerical examples are provided. There are numerical examples to show the reliability and practicality of the proposed approach, as well as comparisons with previous findings (standard collocation method [SCM] [43], the Chebyshev-Gauss-Lobatto collocation method [CGLCM] [43], and Bessel collocation method [21]).

2 Method for solving problem (1), (2)

In this section for solving problem (1), (2), we set

$$v_k(t)=\int_0^t \chi_k(s)x(s)ds,\quad k=\overline{1,m},$$

and we obtain the following multi-point problem for the system of Fredholm integro-differential equations with degenerate kernel

$$\frac{dy}{dt}=\widetilde{A}(t)y+\sum_{k=1}^{m}\widetilde{\varphi}_k(t)\int_0^T\widetilde{\psi}_k(s)y(s)ds+\widetilde{f}(t),\quad t\in(0,T),\tag{3}$$

$$\sum_{i=0}^{N}\widetilde{B}_i y(t_i)=\widetilde{d},\quad y\in\mathbb{R}^{(m+1)n},\quad \widetilde{d}\in\mathbb{R}^{(m+1)n}.\tag{4}$$

Here,

$$y(t)=\begin{pmatrix}x(t)\\ v_1(t)\\ v_2(t)\\ \cdots\\ v_m(t)\end{pmatrix},\quad \widetilde{A}(t)=\begin{pmatrix}A(t) & \phi_1(t) & \phi_2(t) & \dots & \phi_m(t)\\ \chi_1(t) & O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\\ \chi_2(t) & O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\\ \dots & \dots & \dots & \ddots & \dots\\ \chi_m(t) & O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\end{pmatrix},$$

$$\widetilde{\varphi}_k(t)=\begin{pmatrix}\varphi_k(t) & O_{n\times n} & \dots & O_{n\times n}\\ O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\\ \dots & \dots & \ddots & \dots\\ O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\end{pmatrix},\quad \widetilde{\psi}_k(t)=\begin{pmatrix}\psi_k(t) & O_{n\times n} & \dots & O_{n\times n}\\ O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\\ \dots & \dots & \ddots & \dots\\ O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\end{pmatrix},\quad k=\overline{1,m},$$

$$\widetilde{B}_0=\begin{pmatrix}B_0 & O_{n\times n} & \dots & O_{n\times n}\\ O_{n\times n} & I_{n\times n} & \dots & O_{n\times n}\\ \dots & \dots & \ddots & \dots\\ O_{n\times n} & O_{n\times n} & \dots & I_{n\times n}\end{pmatrix},\quad \widetilde{B}_p=\begin{pmatrix}B_p & O_{n\times n} & \dots & O_{n\times n}\\ O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\\ \dots & \dots & \ddots & \dots\\ O_{n\times n} & O_{n\times n} & \dots & O_{n\times n}\end{pmatrix},\quad p=\overline{1,N},$$

$$\widetilde{f}(t)=(f(t),\underbrace{O_n,O_n,\dots,O_n}_{m})\in\mathbb{R}^{(m+1)n},\quad \widetilde{d}=(d,\underbrace{O_n,O_n,\dots,O_n}_{m})\in\mathbb{R}^{(m+1)n},$$

where I and O are the identity and zero matrices or vectors, respectively.

In contrast to the original problem (1), (2), the system has dimension $(m+1)n$ in problem (3), (4). We received a multi-point boundary value problem for the system of Fredholm integro-differential equations (3), (4) instead of a multi-point boundary value problem for the system of Volterra-Fredholm integro-differential equations (1), (2). This problem was studied in [5–7,31–33]. Following that, we solve problem (3), (4) using the Dzhumabaev parameterization method.

Given the points: $t_0=0<t_1<\dots<t_{N-1}<t_N=T$, and let Δ_N denote the partition of interval $[0,T)$ into N subintervals $[0,T)=\cup_{r=1}^{N}[t_{r-1},t_r)$. Δ_1 is the case, when the interval $[0,T]$ is not divided into parts.

Let $C([0,T],\mathbb{R}^{(m+1)n})$ be the space of continuous functions $y:[0,T]\to\mathbb{R}^{(m+1)n}$ with the norm $\|y\|_1=\max(\max_{t\in[0,T]}\|x(t)\|$ and $\max_{t\in[0,T]}\max_{k=\overline{1,m}}\|v_k(t)\|)$.

Let $C([0,T],\Delta_N,\mathbb{R}^{(m+1)nN})$ be the space of functions systems $y[t]=(y_1(t),y_2(t),\dots,y_N(t))'$, where $y_r:[t_{r-1},t_r)\to\mathbb{R}^{(m+1)n}$ are continuous and have finite left-hand side limits $\lim_{t\to t_r-0}y_r(t)$ for all $r=\overline{1,N}$ with the norm $\|y[\cdot]\|_2=\max_{r=\overline{1,N}}\sup_{t\in[t_{r-1},t_r)}|y_r(t)|$.

Denote by $y_r(t)$ a restriction of function $y(t)$ on rth interval $[t_{r-1},t_r)$, i.e.,

$y_r(t)=y(t)$, for $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$.

Introducing the parameters $\xi_r = y_r(t_{r-1})$ and performing a replacement of the function $u_r(t) = y_r(t) - \xi_r$, on every rth interval $[t_{r-1}, t_r)$, we obtain the boundary value problem with parameter for the system of Fredholm integro-differential equations:

$$\frac{du_r}{dt} = \widetilde{A}(t)(u_r + \xi_r) + \sum_{j=1}^{N}\sum_{k=1}^{m}\widetilde{\varphi}_k(t)\int_{t_{j-1}}^{t_j}\widetilde{\psi}_k(s)[u_j(s) + \xi_j]ds + \widetilde{f}(t), \quad t \in [t_{r-1}, t_r), r = \overline{1,N}, \tag{5}$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}, \tag{6}$$

$$\sum_{i=0}^{N-1}\widetilde{B}_i\xi_{i+1} + \widetilde{B}_N\xi_N + \widetilde{B}_N\lim_{t\to T-0}u_N(t) = \widetilde{d}, \tag{7}$$

$$\xi_p + \lim_{t\to t_p-0}u_p(t) = \xi_{p+1}, \quad p = \overline{1, N-1}. \tag{8}$$

A pair $(\xi, u(t))$ is called a solution to problem (5)–(8), where parameter $\xi \in \mathbb{R}^{(m+1)n}$, vector function $u(t)$ continuous on $[0, T]$ and continuously differentiable on $(0, T)$, if it satisfies the integro-differential equation (5), initial condition (6), and conditions (7) and (8).

If $y^*(t)$ is a solution to problem (3), (4), then the pair $(\xi^*, u^*[t])$ with elements $\xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_N^*) \in \mathbb{R}^{(m+1)nN}$, $u^*[t] = (u_1^*(t), u_2^*(t), \ldots, u_N^*(t)) \in C([0, T], \Delta_N, \mathbb{R}^{(m+1)nN})$, where $\xi_r^* = y^*(t_{r-1}), u_r^*(t) = y^*(t) - y^*(t_{r-1}), t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, is a solution to problem (5)–(8). Conversely, if the pair $(\widetilde{\xi}, \widetilde{u}[t])$ with elements $\widetilde{\xi} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \ldots, \widetilde{\xi}_N) \in \mathbb{R}^{(m+1)nN}$, $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_N(t)) \in C([0, T], \Delta_N, \mathbb{R}^{(m+1)nN})$ is a solution to problem (5)–(8), then the function $\widetilde{y}(t)$ defined by the equalities $\widetilde{y}(t) = \widetilde{u}_r(t) + \widetilde{\xi}_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, and $\widetilde{y}(T) = \widetilde{\xi}_N + \lim_{t\to T-0}\widetilde{u}_N(t)$, is a solution to the origin boundary value problem (3), (4).

We have Cauchy problem for the system of Fredholm integro-differential equations (5), (6) for fixed ξ.

Using the fundamental matrix $\Phi_r(t)$ of differential equation $\frac{dy}{dt} = \widetilde{A}(t)y(t)$ on $[t_{r-1}, t_r]$, $r = \overline{1,N}$, we reduce Cauchy problem for the system of Fredholm integro-differential equations with parameters (5), (6) to the equivalent system of integral equations

$$\begin{aligned} u_r(t) = {} & \Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\widetilde{A}(\tau)d\tau\xi_r + \Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\sum_{j=1}^{N}\sum_{k=1}^{m}\widetilde{\varphi}_k(\tau)\int_{t_{j-1}}^{t_j}\widetilde{\psi}_k(s)[u_j(s) + \xi_j]dsd\tau \\ & + \Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\widetilde{f}(\tau)d\tau, \quad t \in [t_{r-1}, t_r), r = \overline{1,N}. \end{aligned} \tag{9}$$

Set $\theta_k = \sum_{j=1}^{N}\int_{tj-1}^{t_j}\widetilde{\psi}_k(s)u_j(s)ds$ and re-write the system of integral equations (9) in the following form:

$$\begin{aligned} u_r(t) = {} & \sum_{k=1}^{m}\Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\widetilde{\varphi}_k(\tau)d\tau\theta_k + \Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)[\widetilde{A}(\tau)\xi_r + \widetilde{f}(\tau)]d\tau \\ & + \Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\sum_{k=1}^{m}\widetilde{\varphi}_k(\tau)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\widetilde{\psi}_k(s)dsd\tau\xi_j, \quad t \in [t_{r-1}, t_r), r = \overline{1,N}. \end{aligned} \tag{10}$$

Multiplying both sides of (10) by $\widetilde{\psi}_p(t)$, integrating on the interval $[t_{r-1}, t_r]$, and summing up over r, we have the system of linear algebraic equations with respect to $\theta = (\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^{(m+1)nm}$:

$$\theta_p = \sum_{k=1}^{m}\Theta_{p,k}(\Delta_N)\theta_k + \sum_{r=1}^{N}V_{p,r}(\Delta_N)\xi_r + g_p(f, \Delta_N), \quad p = \overline{1,m}, \tag{11}$$

with the $((m + 1)n \times (m + 1)n)$ matrices

$$\Theta_{p,k}(\Delta_N) = \sum_{r=1}^{N}\int_{t_{r-1}}^{t_r}\widetilde{\psi}_p(\tau)\Phi_r(\tau)\int_{t_{r-1}}^{\tau}\Phi_r^{-1}(s)\widetilde{\varphi}_k(s)dsd\tau, \quad p, k = \overline{1,m}, \tag{12}$$

$$V_{p,r}(\Delta_N) = \int_{t_{r-1}}^{t_r} \widetilde{\psi}_p(\tau)\Phi_r(\tau)\int_{t_{r-1}}^{\tau} \Phi_r^{-1}(s)\widetilde{A}(s)\mathrm{d}s\mathrm{d}\tau + \sum_{j=1}^{N}\sum_{k=1}^{m}\int_{t_{j-1}}^{t_j} \widetilde{\psi}_p(\tau)\Phi_j(\tau)\int_{t_{j-1}}^{\tau} \Phi_j^{-1}(\tau_1)\widetilde{\varphi}_k(\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau \int_{t_{r-1}}^{t_r}\widetilde{\psi}_k(s)\mathrm{d}s, \quad p=\overline{1,m}, \tag{13}$$

and vectors of dimension $((m+1)n)$

$$g_p(f,\Delta_N) = \sum_{r=1}^{N}\int_{t_{r-1}}^{t_r} \widetilde{\psi}_p(\tau)\Phi_r(\tau)\int_{t_{r-1}}^{\tau}\Phi_r^{-1}(s)\widetilde{f}(s)\mathrm{d}s\mathrm{d}\tau, \quad p=\overline{1,m}. \tag{14}$$

Using the matrices $\Theta_{p,k}(\Delta_N)$, $V_{p,r}(\Delta_N)$, compose the matrices $\Theta(\Delta_N) = (\Theta_{p,k}(\Delta_N))$, $p,k=\overline{1,m}$, and $V(\Delta_N) = (V_{p,r}(\Delta_N))$, $p=\overline{1,m}$, $r=\overline{1,N}$. Then, re-write system (11) in the form

$$[I_{(m+1)nm\times(m+1)nm} - \Theta(\Delta_N)]\theta = V(\Delta_N)\mu + g(f,\Delta_N). \tag{15}$$

where $g(f,\Delta_N) = (g_1(f,\Delta_N), g_2(f,\Delta_N), \ldots, g_m(f,\Delta_N)) \in \mathbb{R}^{(m+1)nm}$.

Take $\Delta_N \in \sigma(m,[0,T])$ [6] and present $[I_{(m+1)nm\times(m+1)nm} - \Theta(\Delta_N)]^{-1}$ in the form $[I_{(m+1)nm\times(m+1)nm} - \Theta(\Delta_N)]^{-1} = (M_{k,p}(\Delta_N))$, $k,p=\overline{1,m}$, where $M_{k,p}(\Delta_N)$ are the $((m+1)nm\times(m+1)nm)$ square matrices.

Then, according to (15), the elements of vector $\theta\in\mathbb{R}^{(m+1)nm}$ can be determined by the equalities

$$\theta_k = \sum_{j=1}^{N}\left(\sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,j}(\Delta_N)\right)\xi_j + \sum_{p=1}^{m} M_{k,p}(\Delta_N)g_p(f,\Delta_N), \quad k=\overline{1,m}. \tag{16}$$

In (10), replacing the right-hand side of the previous expression (16) instead of θ_k, we obtain the representation of functions $u_r(t)$ via ξ_j, $j=\overline{1,N}$:

$$\begin{aligned} u_r(t) ={}& \sum_{j=1}^{N}\sum_{k=1}^{m}\Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\widetilde{\varphi}_k(\tau)\mathrm{d}\tau\sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,j}(\Delta_N)\xi_j \\ &+ \sum_{j=1}^{N}\sum_{k=1}^{m}\Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\widetilde{\varphi}_k(\tau)\mathrm{d}\tau\int_{t_{j-1}}^{t_j}\widetilde{\psi}_k(s)\mathrm{d}s\xi_j \\ &+ \Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\left[\sum_{k=1}^{m}\widetilde{\varphi}_k(\tau)\sum_{p=1}^{m} M_{k,p}(\Delta_N)g_p(f,\Delta_N) + \widetilde{f}(\tau)\right]\mathrm{d}\tau \\ &+ \Phi_r(t)\int_{t_{r-1}}^{t}\Phi_r^{-1}(\tau)\widetilde{A}(\tau)\mathrm{d}\tau\xi_r, \quad t\in[t_{r-1},t_r),\ r=\overline{1,N}. \end{aligned} \tag{17}$$

Introduce the notations:

$$\begin{aligned} P_{i,i}(\Delta_N) ={}& \sum_{k=1}^{m}\Phi_i(t_i)\int_{t_{i-1}}^{t_i}\Phi_i^{-1}(\tau)\widetilde{\varphi}_k(\tau)\mathrm{d}\tau \\ &\times\left[\sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,i}(\Delta_N) + \int_{t_{i-1}}^{t_i}\widetilde{\psi}_k(s)\mathrm{d}s\right] + \Phi_i(t_i)\int_{t_{i-1}}^{t_i}\Phi_i^{-1}(\tau)\widetilde{A}(\tau)\mathrm{d}\tau, \quad i=\overline{1,N}, \end{aligned}$$

$$\begin{aligned} P_{i,j}(\Delta_N) ={}& \sum_{k=1}^{m}\Phi_i(t_i)\int_{t_{i-1}}^{t_i}\Phi_i^{-1}(\tau)\widetilde{\varphi}_k(\tau)\mathrm{d}\tau\sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,j}(\Delta_N) \\ &+ \sum_{k=1}^{m}\Phi_i(t_i)\int_{t_{i-1}}^{t_i}\Phi_i^{-1}(\tau)\widetilde{\varphi}_k(\tau)\mathrm{d}\tau\int_{t_{j-1}}^{t_j}\widetilde{\psi}_k(s)\mathrm{d}s, \quad i\neq j,\ i,j=\overline{1,N}, \end{aligned}$$

$$\begin{aligned} F_r(\Delta_N) ={}& \sum_{k=1}^{m}\Phi_r(t_r)\int_{t_{r-1}}^{t_r}\Phi_r^{-1}(\tau)\widetilde{\varphi}_k(\tau)\mathrm{d}\tau\sum_{p=1}^{m} M_{k,p}(\Delta_N)g_p(f,\Delta_N) \\ &+ \Phi_r(t_r)\int_{t_{r-1}}^{t_r}\Phi_r^{-1}(\tau)\widetilde{f}(\tau)\mathrm{d}\tau, \quad r=\overline{1,N}. \end{aligned}$$

Then, from (17), we have

$$\lim_{t\to t_r-0} u_r(t) = \sum_{j=1}^{N} P_{r,j}(\Delta_N)\xi_j + F_r(\Delta_N). \tag{18}$$

Substituting the right-hand side of (18) into the boundary condition (7) and conditions of matching solution (8), we obtain the following system of linear algebraic equations with respect to parameters ξ_r, $r = \overline{1,N}$:

$$\sum_{i=0}^{N-1} \widetilde{B}_i\xi_{i+1} + \widetilde{B}_N\xi_N + \widetilde{B}_N \sum_{j=1}^{N} P_{N,j}(\Delta_N)\xi_j = \widetilde{d} - \widetilde{B}_N F_N(\Delta_N), \tag{19}$$

$$[I_{(m+1)n\times(m+1)n} + P_{i,i}(\Delta_N)]\xi_i - [I_{(m+1)n\times(m+1)n} - P_{i,i+1}(\Delta_N)]\xi_{i+1} + \sum_{j=1, j\neq i, j\neq i+1}^{N} P_{i,j}(\Delta_N)\xi_j = -F_i(\Delta_N), \quad i = \overline{1, N-1}. \tag{20}$$

By denoting the matrix corresponding to the left-hand side of the system of equations (19) and (20) by $Q_*(\Delta_N)$, the system can be written as

$$Q_*(\Delta_N)\xi = -F_*(\Delta_N), \quad \xi \in \mathbb{R}^{(m+1)nN}, \tag{21}$$

where $F_*(\Delta_N) = (-\widetilde{d} + \widetilde{B}_N F_N(\Delta_N), F_1(\Delta_N), \ldots, F_{N-1}(\Delta_N)) \in \mathbb{R}^{(m+1)nN}$.

Lemma 1. *For $\Delta_N \in \sigma(m, [0,T])$, the following assertions hold:*

(*a*) *The vector $\xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_N^*) \in R^{(m+1)nN}$, composed by the values of solution $y^*(t)$ to problem* (3), (4) *at the partition points $\xi^* = y^*(t_{r-1})$, $r = \overline{1,N}$, satisfies system* (21);

(*b*) *if $\widetilde{\xi} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \ldots, \widetilde{\xi}_N) \in R^{(m+1)nN}$ is a solution to system* (21) *and the function system $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_N(t))$ is a solution to the special Cauchy problem* (5), (6) *with $\xi_r = \widetilde{\xi}_r$, $r = \overline{1,N}$, then the function $\widetilde{y}(t)$, defined by the equalities: $\widetilde{y}(t) = \widetilde{\xi}_r + \widetilde{u}_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, $\widetilde{y}(T) = \widetilde{\xi}_N + \lim_{t\to T-0}\widetilde{u}_N(t)$ is a solution to problem* (3), (4).

The proof with minor changes is similar to the proof of Lemma 1 [44].

Let us introduce the notations $\alpha = \max_{t\in[0,T]}\|\widetilde{A}(t)\| = \max(\max_{t\in[0,T]}\|A(t)\| + \sum_{j=1}^{m}\max_{t\in[0,T]}\|\phi_j(t)\|, \max_{t\in[0,T]}\max_{j=\overline{1,m}}\|\chi_j(t)\|)$, $\overline{h} = \max_{r=\overline{1,N}}(t_r - t_{r-1})$, $\overline{\varphi}(m) = \max_{r=\overline{1,N}}\int_{t_{r-1}}^{t_r}\sum_{k=1}^{m}\|\varphi_k(t)\|dt$, and $\overline{\psi}(T) = \max_{p=\overline{1,m}}\int_0^T\|\psi_p(t)\|dt$.

Theorem 1. *Let $\Delta_N \in \sigma(m, [0,T])$ and the matrix $Q_*(\Delta_N) : R^{(m+1)nN} \to R^{(m+1)nN}$ be invertible. Then, problem* (3), (4) *have a unique solution $y^*(t)$ for any $\widetilde{f}(t) \in C([0,T], R^{(m+1)n})$, $\widetilde{d} \in R^{(m+1)n}$ and the estimate holds:*

$$\|y^*(t)\| \leq \mathcal{K}(m, \Delta_N)\max(\|\widetilde{d}\|, \|\widetilde{f}\|_1)$$

where

$$\begin{aligned}\mathcal{K}(m,\Delta_N) = e^{\alpha\overline{h}}\{\overline{\varphi}(m)[\|[I - \Theta(\Delta_N)]^{-1}\|\overline{\psi}(T)(e^{\alpha\overline{h}} - 1 + e^{\alpha\overline{h}}\overline{\varphi}(m)\overline{\psi}(T)) + \overline{\psi}(T)] + 1\}\\ \times \gamma_*(\Delta_N)(1 + \|C\|)\max\{1, \overline{h}e^{\alpha\overline{h}}[1 + e^{\alpha\overline{h}}\overline{\varphi}(m)\|[I - \Theta(\Delta_N)]^{-1}\|\overline{\psi}(T)]\}\\ + e^{\alpha\overline{h}}\overline{h}[\overline{\varphi}(m)\|[I - \Theta(\Delta_N)]^{-1}\|\overline{\psi}(T)e^{\alpha\overline{h}} + 1].\end{aligned}$$

The proof with minor changes is similar to the proof of Theorem 2.1 [6].

3 An algorithm for solving problem (3), (4)

The following Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt} = \widetilde{A}(t)z + S(t), \quad z(t_{r-1}) = 0, \ t \in [t_{r-1}, t_r], \ r = \overline{1,N} \tag{22}$$

are a significant part of the proposed algorithm. Here, $S(t)$ is either $((m+1)n \times (m+1)n)$ matrix, or $(m+1)n$ vector, both continuous on $[t_{r-1}, t_r]$, $r = \overline{1, N}$. Consequently, solution to Problem (22) is a square matrix or a vector of dimension $(m+1)n$. Denote by $a(S, t)$ the solution to the Cauchy Problem (22). Obviously,

$$a_r(L(s), t) = \Phi_r(t) \int_{t_{r-1}}^{t} \Phi_r^{-1}(s) L(s) ds, \quad t \in [t_{r-1}, t_r], \tag{23}$$

where $\Phi_r(t)$ is a fundamental matrix of differential equation (22) on the rth interval.

We offer the following numerical algorithm for solving problem (3), (4).

(1) Suppose we have a partition Δ_N: $t_0 = 0 < t_1 < \ldots < t_N = T$. Divide each rth interval $[t_{r-1}, t_r]$, $r = \overline{1, N}$ into N_r parts with step $h_r = (t_r - t_{r-1})/N_r$. Assume on each interval $[t_{r-1}, t_r]$ the variable $\hat{t}$ takes its discrete values: $\hat{t} = t_{r-1}$, $\hat{t} = t_{r-1} + h_r, \ldots, \hat{t} = t_{r-1} + (N_r - 1)h_r$, $\hat{t} = t_r$, and denote by $\{t_{r-1}, t_r\}$ the set of such points.

(2) Using the Runge-Kutta method of fourth order, we find the numerical solutions to Cauchy problem (22) and define the values of $((m+1)n \times (m+1)n)$ matrices $a_r^{h_r}(\widetilde{\varphi}_k(s), \hat{t})$ on the set $\{t_{r-1}, t_r\}$, $r = \overline{1, N}$, $k = \overline{1, m}$.

(3) Using the values of $((m+1)n \times (m+1)n)$ matrices $\widetilde{\psi}_k(s)$, $a_r^{h_r}(\widetilde{\varphi}_k(s), \hat{t})$ on $\{t_{r-1}, t_r\}$ and Simpson's method, we find the $((m+1)n \times (m+1)n)$ matrices

$$\hat{\psi}_{p,r}^{h_r}(\widetilde{\varphi}_k) = \int_{t_{r-1}}^{t_r} \widetilde{\psi}_p(\tau) a_r^{h_r}(\widetilde{\varphi}_k(s), \tau) d\tau, \quad p, k = \overline{1, m}, \ r = \overline{1, N}.$$

Summing up the matrices $\hat{\psi}_{p,r}^{h_r}(\widetilde{\varphi}_k)$ over r, we find the $((m+1)n \times (m+1)n)$ matrices $\Theta_{p,k}^{\tilde{h}}(\Delta_N) = \sum_{r=1}^{N} \hat{\psi}_{p,r}^{h_r}(\widetilde{\varphi}_k)$, where $\tilde{h} = (h_1, h_2, \ldots, h_N) \in \mathbb{R}^n$. Using them, we compose the $((m+1)mn \times (m+1)mn)$ matrix $\Theta^{\tilde{h}}(\Delta_N) = \big(\Theta_{p,k}^{\tilde{h}}(\Delta_N)\big)$, $p, k = \overline{1, m}$. Check the invertibility of matrix $[I_{(m+1)nm \times (m+1)nm} - \Theta^{\tilde{h}}(\Delta_N)]$: $\mathbb{R}^{(m+1)mn} \to \mathbb{R}^{(m+1)mn}$. If this matrix is invertible, we find $[I_{(m+1)nm \times (m+1)nm} - \Theta^{\tilde{h}}(\Delta_N)]^{-1} = \big(M_{p,k}^{\tilde{h}}(\Delta_N)\big)$, $p, k = \overline{1, m}$. If it has no the inverse, then we take a new partition. In particular, each subinterval can be divided into two.

(4) Solving the Cauchy Problem (22) using the Runge-Kutta method of fourth order again, we find the values of $((m+1)n \times (m+1)n)$ matrices $a_r^{h_r}(\widetilde{A}(s), \hat{t})$ and n vector $a_r^{h_r}(\widetilde{f}(s), \hat{t})$ on the set $\{t_{r-1}, t_r\}$, $r = \overline{1, N}$.

(5) By applying Simpson's method on the set $\{t_{r-1}, t_r\}$, $r = \overline{1, N}$, we evaluate the definite integrals

$$\hat{\psi}_{p,r}^{h_r}(\widetilde{A}) = \int_{t_{r-1}}^{t_r} \widetilde{\psi}_p(\tau) a_r^{h_r}(\widetilde{A}(s), \tau) d\tau, \quad \hat{\psi}_{p,r}^{h_r} = \int_{t_{r-1}}^{t_r} \widetilde{\psi}_p(s) ds,$$

$$\hat{\psi}_{p,r}^{h_r}(\widetilde{f}) = \int_{t_{r-1}}^{t_r} \widetilde{\psi}_p(\tau) a_r^{h_r}(\widetilde{f}(s), \tau) d\tau, \quad p = \overline{1, m}, r = \overline{1, N}.$$

By the equalities

$$V_{p,r}^{\tilde{h}}(\Delta_N) = \hat{\psi}_{p,r}^{h_r}(\widetilde{A}) + \sum_{j=1}^{N} \sum_{k=1}^{m} \hat{\psi}_{p,j}^{h_j}(\widetilde{\varphi}_k) \cdot \hat{\psi}_{k,r}^{h_r}, \quad g_p^{\tilde{h}}(\widetilde{f}, \Delta_N) = \sum_{r=1}^{N} \hat{\psi}_{p,r}^{h_r}(\widetilde{f}),$$

we define the $(m+1)n \times (m+1)n$ matrices $V_{p,r}^{\tilde{h}}(\Delta_N)$ and $(m+1)n$ vectors $g_p^{\tilde{h}}(\widetilde{f}, \Delta_N)$, $r = \overline{1, N}$, $p = \overline{1, m}$.

(6) Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\tilde{h}}(\Delta_N)\xi = -F_*^{\tilde{h}}(\Delta_N), \quad \xi \in \mathbb{R}^{(m+1)nN}. \tag{24}$$

The elements of matrix $Q_*^{\tilde{h}}(\Delta_N)$ and vector $F_*^{\tilde{h}}(\Delta_N) = \big(-\widetilde{d} + \widetilde{B}_N F_N^{\tilde{h}}(\Delta_N), F_1^{\tilde{h}}(\Delta_N), \ldots, F_{N-1}^{\tilde{h}}(\Delta_N)\big)$ are defined by the equalities

$$P_{i,i}^{\tilde{h}}(\Delta_N) = \sum_{k=1}^{m} a_i^{h_i}(\widetilde{\varphi}_k(s), t_i) \left[\sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) V_{p,i}^{\tilde{h}}(\Delta_N) + \hat{\psi}_{k,i}^{h_i} \right] + a_i^{h_i}(\widetilde{A}(s), t_i), \quad i = \overline{1, N},$$

$$P_{i,j}^{\tilde{h}}(\Delta_N) = \sum_{k=1}^{m} a_i^{h_i}(\widetilde{\varphi}_k(s), t_i) \times \left[\sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) V_{p,j}^{\tilde{h}}(\Delta_N) + \hat{\psi}_{k,j}^{h_j}\right], \quad i \neq j, i, j = \overline{1, N},$$

$$F_r^{\tilde{h}}(\Delta_N) = \sum_{k=1}^{m} a_r^{h_r}(\widetilde{\varphi}_k(s), t_r) \sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) g_p^{\tilde{h}}(f, \Delta_N) + a_r^{h_r}(\widetilde{f}(s), t_r), \quad r = \overline{1, N}.$$

Solving system (24), we find ξ.

(7) We first find

$$\theta_k^{\tilde{h}} = \sum_{j=1}^{N} \left(\sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) V_{p,j}^{\tilde{h}}(\Delta_N)\right) \xi_j + \sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) g_p^{\tilde{h}}(f, \Delta_N), \quad k = \overline{1, m},$$

and then solve the Cauchy problems

$$\frac{dy}{dt} = \widetilde{A}(t) y + \sum_{k=1}^{m} \widetilde{\varphi}_k(t) \left(\theta_k^{\tilde{h}} + \sum_{j=1}^{N} \hat{\psi}_{k,j}^{h_j} \xi_j\right) + \widetilde{f}(t), \quad t \in [t_{r-1}, t_r), \tag{25}$$

$$y(t_{r-1}) = \xi_r^{\tilde{h}}, \quad r = \overline{1, N}. \tag{26}$$

The Runge-Kutta method of fourth order is used to solve Cauchy problem (25), (26). As a result, the algorithm enables us to determine the numerical solution to problem (3), (4).

Since $y(t) = (x(t), v_1(t), v_2(t), \ldots, v_m(t))'$, the proposed algorithm makes it possible to find a numerical solution to the original problem (1), (2).

The implementation of this algorithm is shown in Section 4.

4 Illustrative examples

In this section, we look at several numerical examples to show how accurate and efficient the suggested algorithm is. The method described in Section 2 is used to solve all instances. To demonstrate the efficacy of the suggested technique, all numerical results are compared to the precise answers and given in the accompanying tables. The MathCad computer system is used to complete the necessary computations.

Example 1. Consider the following Volterra-type integro-differential equation with degenerate kernel

$$\frac{dx}{dt} = -x + (t^2 + 2t + 1)e^{-t} + 5t^2 + 8 - \int_0^t s x(s) ds, \quad x(0) = 10, \tag{27}$$

with the exact solution $x(t) = 10 - te^{-t}$.

In Table 1, the computational results derived from the Dzhumabaev parameterization method are compared to the precise answer. Table 2 shows the absolute errors achieved by the SCM, the CGLCM [43], and the idsolver program (a general-purpose MATLAB solver) [45] and the current method. The results achieved by the suggested method are better than those obtained by the other methods, as shown in Table 2.

Example 2. Let us now consider the second-order Volterra-Fredholm integro-differential equation with degenerate kernels given by

$$x''(t) = -tx'(t) + tx(t) + e^t - \sin(t) + \frac{1}{2} t \cos(t) + \int_0^1 \sin(t) e^{-s} x(s) ds - \frac{1}{2} \int_0^x \cos(t) e^{-s} x(s) ds, \tag{28}$$

Table 1: Comparison of exact solution with numerical solutions for problem (27)

t_i	Exact solution	Presented method	t_i	Exact solution	Presented method
0	10	10	0.5	9.696734670144	9.696734956525
0.05	9.952438528775	9.952438566076	0.55	9.682677604291	9.682677912255
0.1	9.909516258196	9.909516330182	0.6	9.670713018344	9.670713347122
0.15	9.870893803536	9.870893907913	0.65	9.660670245105	9.660670594003
0.2	9.836253849384	9.836253984137	0.7	9.652390287346	9.652390655727
0.25	9.805299804232	9.805299967592	0.75	9.645725085444	9.645725472723
0.3	9.777754533795	9.777754724205	0.8	9.640536828706	9.640537234338
0.35	9.753359168598	9.753359384686	0.85	9.636697307844	9.636697731316
0.4	9.731871981586	9.731872222139	0.9	9.634087306233	9.634087747058
0.45	9.713067331770	9.713067595715	0.95	9.632596027718	9.632596485425
0.5	9.696734670144	9.696734956525	1	9.632120558829	9.632121032961

with the initial conditions

$$x(0) = 1 \quad \text{and} \quad x'(0) = 1, \tag{29}$$

which is the exact solution $x(t) = e^t$.

Here, $A(t) = \begin{pmatrix} 0 & 1 \\ t & -t \end{pmatrix}$, $\varphi_1(t) = \begin{pmatrix} 0 & 0 \\ \sin(t) & 0 \end{pmatrix}$, $\psi_1(s) = \begin{pmatrix} e^{-s} & 0 \\ 0 & 0 \end{pmatrix}$, $\phi_1(t) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2}\cos(t) & 0 \end{pmatrix}$, $\chi_1(s) = \begin{pmatrix} e^{-s} & 0 \\ 0 & 0 \end{pmatrix}$, $B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $f(t) = \begin{pmatrix} 0 \\ e^t - \sin(t) + \frac{1}{2}t\cos(t) \end{pmatrix}$.

Table 3 compares the numerical results of the Bessel collocation method and the Dzhumabaev parameterization method with the actual solution of problem (28), (29). The absolute errors acquired by the Bessel

Table 2: Comparison of absolute errors for problem (27)

t_i	SCM [43]	CGLCM [43]	Idsolver of [45]	Presented method
0	0	0	0	0
0.05	1.4526×10^{-6}	2.8995×10^{-6}	3.0727×10^{-5}	3.7301×10^{-8}
0.1	1.8968×10^{-6}	1.1526×10^{-5}	8.4166×10^{-6}	7.1986×10^{-8}
0.15	1.9169×10^{-6}	2.5374×10^{-5}	5.6410×10^{-6}	1.0438×10^{-7}
0.2	1.8115×10^{-6}	4.3394×10^{-5}	9.0705×10^{-6}	1.3475×10^{-7}
0.25	1.7020×10^{-6}	6.4011×10^{-5}	8.4009×10^{-6}	1.6336×10^{-7}
0.3	1.6129×10^{-6}	8.5183×10^{-5}	7.1799×10^{-6}	1.9041×10^{-7}
0.35	1.5265×10^{-6}	1.0451×10^{-4}	6.0076×10^{-6}	2.1609×10^{-7}
0.4	1.4180×10^{-6}	1.1938×10^{-4}	5.1799×10^{-6}	2.4055×10^{-7}
0.45	1.2742×10^{-6}	1.2719×10^{-4}	4.9377×10^{-6}	2.6395×10^{-7}
0.5	1.0994×10^{-6}	1.2562×10^{-4}	3.2948×10^{-6}	2.8638×10^{-7}
0.55	9.1328×10^{-7}	1.1293×10^{-4}	9.4362×10^{-7}	3.0796×10^{-7}
0.6	7.4216×10^{-7}	8.8431×10^{-5}	3.6631×10^{-7}	3.2878×10^{-7}
0.65	6.0849×10^{-7}	5.2842×10^{-5}	5.8380×10^{-8}	3.4890×10^{-7}
0.7	5.2077×10^{-7}	8.9188×10^{-6}	2.1033×10^{-6}	3.6838×10^{-7}
0.75	4.6696×10^{-7}	3.7978×10^{-5}	4.3888×10^{-6}	3.8728×10^{-7}
0.8	4.1397×10^{-7}	7.9189×10^{-5}	5.3578×10^{-6}	4.0563×10^{-7}
0.85	3.1575×10^{-7}	1.0199×10^{-4}	6.1403×10^{-6}	4.2347×10^{-7}
0.9	1.3241×10^{-7}	8.8737×10^{-5}	8.0110×10^{-6}	4.4082×10^{-7}
0.95	1.3740×10^{-7}	1.5945×10^{-5}	1.0440×10^{-5}	4.5771×10^{-7}
1	4.0883×10^{-7}	1.4674×10^{-4}	1.2924×10^{-5}	4.7413×10^{-7}

Table 3: Numerical results for Example 2 for the t values

t_i	Exact solution	Bessel collocation method	Proposed method	t_i	Exact solution	Bessel collocation method	Proposed method
0	1	1	1	0.5	1.648721271	1.650869848	1.648721281
0.05	1.051271096	1.051271099	1.051271097	0.55	1.733253018	1.737059354	1.733253029
0.1	1.105170918	1.105171056	1.105170920	0.6	1.822118800	1.828534418	1.822118812
0.15	1.161834243	1.161835809	1.161834246	0.65	1.915540829	1.925911601	1.915540841
0.2	1.221402758	1.221411559	1.221402763	0.7	2.013752707	2.029930497	2.013752719
0.25	1.284025417	1.284058988	1.284025422	0.75	2.117000017	2.141473654	2.117000029
0.3	1.349858808	1.349959052	1.349858814	0.8	2.225540928	2.261588073	2.225540940
0.35	1.419067549	1.419320327	1.419067556	0.85	2.339646852	2.391508283	2.339646863
0.4	1.491824698	1.492387934	1.491824706	0.9	2.459603111	2.532681001	2.459603122
0.45	1.568312185	1.569454028	1.568312195	0.95	2.585709659	2.686791368	2.585709669
0.5	1.648721271	1.650869848	1.648721281	1	2.718281828	2.855790771	2.718281837

collocation method [21] and Dzhumabaev parameterization method are shown in Figure 1. According to the results, the proposed method produced satisfactory results for problem (28), (29).

Example 3. Consider the system of Volterra-Fredholm integro-differential equations given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{pmatrix} t & -2 \\ 1 & 2t^2 \end{pmatrix} x + \begin{pmatrix} 3 & 4t^2 \\ 1 & 0 \end{pmatrix} \int_0^1 \begin{pmatrix} s & 4 \\ 0 & -5 \end{pmatrix} x(s)\mathrm{d}s + \begin{pmatrix} 2t & 1 \\ 0 & t-2 \end{pmatrix} \int_0^t \begin{pmatrix} 4 & 0 \\ 15s & 5 \end{pmatrix} x(s)\mathrm{d}s + \begin{pmatrix} t^4 - 6t^5 + 330t^2 + 4t - 172 \\ 8t^5 - 6t^6 + 12t^3 - 37t^2 + 10t - 52 \end{pmatrix}, \quad t \in (0, 2), \tag{30}$$

with boundary condition

$$\begin{pmatrix} -1 & 0 \\ 4 & 8 \end{pmatrix} x(0) + \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix} x(1) + \begin{pmatrix} -2 & 5 \\ 0 & 1 \end{pmatrix} x(2) = \begin{pmatrix} 112 \\ 59 \end{pmatrix}, \tag{31}$$

with the exact solution $x_1(t) = t^3 - 3$, $x_2(t) = t^4 + 3t + 1$.

Table 4 compares the maximum error of our method with the different partitioning of the interval $(0, 2)$.

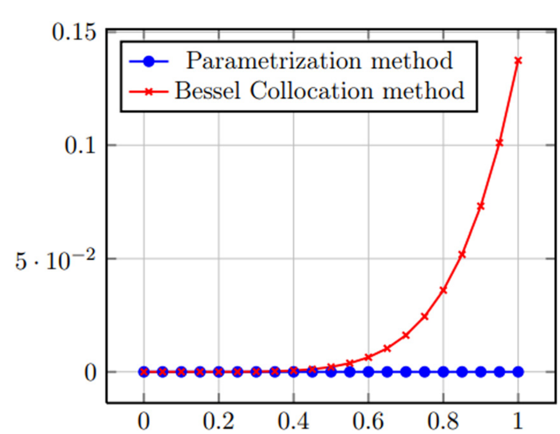

Figure 1: Comparison of the absolute errors for Example 2.

Table 4: Maximum error with various h for Example 3

h	Proposed method
0.125	0.005013140560
0.05	0.000140299615
0.025	0.000009013011
0.0125	0.000000570844
0.00625	0.000000035912
0.003125	0.000000002252
0.0015625	0.000000000140

Example 4. Consider the system of Volterra-Fredholm integro-differential equations with degenerate kernels

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{2}\varphi_k(t)\int_0^1 \psi_k(s)x(s)ds + \sum_{k=1}^{2}\phi_k(t)\int_0^t \chi_k(s)x(s)ds + f(t), \quad t \in (0,1), \tag{32}$$

with boundary condition

$$B_0x(0) + B_1x(0.5) + B_2x(1) = d, \quad x \in R^2, \ d \in R^2. \tag{33}$$

Here,

$$A(t) = \begin{pmatrix} 3t & 1 \\ 0 & e^t \end{pmatrix}, \quad \varphi_1(t) = \begin{pmatrix} 2 & 0 \\ 3 & 4t^2 \end{pmatrix}, \quad \psi_1(t) = \begin{pmatrix} -1 & 4 \\ 2t & 0 \end{pmatrix}, \quad \varphi_2(t) = \begin{pmatrix} -1 & 5t \\ 2 & 6 \end{pmatrix}, \quad \psi_2(t) = \begin{pmatrix} 6 & 0 \\ t & 2 \end{pmatrix},$$

$$\phi_1(t) = \begin{pmatrix} 1 & 2t \\ 0 & 4 \end{pmatrix}, \quad \chi_1(t) = \begin{pmatrix} 2t & 4 \\ 0 & -3 \end{pmatrix}, \quad \phi_2(t) = \begin{pmatrix} -t & 1 \\ 0 & 2t^2 \end{pmatrix}, \quad \chi_2(t) = \begin{pmatrix} 0 & 2 \\ -1 & 18t \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 & 2 \\ 7 & 11 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 12 & 0 \\ 0 & 8 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 3 & -4 \\ 9 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} -52 \\ 19 \end{pmatrix}, \quad f(t) = \begin{pmatrix} t^2 - 18t^3 - 3t^4 - 7t - 46 \\ 24t - 2e^t + 14t^2 - 10t^3 - 36t^4 - 8t^5 - te^t - 26 \end{pmatrix}.$$

Table 5: Numerical results for Example 4 for the t values

k	t_k	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$	k	t_k	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$
0	0	−5.0000030522	2.0000003616	20	0.5	−3.5000004352	2.4999999832
1	0.025	−4.9962529178	2.0250003521	21	0.525	−3.3462503036	2.5249999831
2	0.05	−4.9850027844	2.0500003395	22	0.55	−3.1850001718	2.5499999872
3	0.075	−4.9662526519	2.0750003239	23	0.575	−3.0162500396	2.5749999954
4	0.1	−4.9400025201	2.1000003058	24	0.6	−2.8399999070	2.6000000074
5	0.125	−4.9062523889	2.1250002853	25	0.625	−2.6562497741	2.6250000230
6	0.15	−4.8650022581	2.1500002628	26	0.65	−2.4649996408	2.6500000417
7	0.175	−4.8162521277	2.1750002386	27	0.675	−2.2662495074	2.6750000629
8	0.2	−4.7600019976	2.2000002132	28	0.7	−2.0599993738	2.7000000855
9	0.225	−4.6962518676	2.2250001870	29	0.725	−1.8462492404	2.7250001086
10	0.25	−4.6250017378	2.2500001605	30	0.75	−1.6249991073	2.7500001307
11	0.275	−4.5462516080	2.2750001342	31	0.775	−1.3962489750	2.7750001501
12	0.3	−4.4600014783	2.3000001085	32	0.8	−1.1599988440	2.8000001646
13	0.325	−4.3662513484	2.3250000840	33	0.825	−0.9162487150	2.8250001716
14	0.35	−4.2650012185	2.3500000611	34	0.85	−0.6649985891	2.8500001679
15	0.375	−4.1562510885	2.3750000404	35	0.875	−0.4062484676	2.8750001493
16	0.4	−4.0400009583	2.4000000223	36	0.9	−0.1399983522	2.9000001109
17	0.425	−3.9162508279	2.4250000071	37	0.925	0.1337517548	2.9250000463
18	0.45	−3.7850006973	2.4499999954	38	0.95	0.4150018505	2.9499999477
19	0.475	−3.6462505664	2.4749999873	39	0.975	0.7037519308	2.9749998050
20	0.5	−3.5000004352	2.4999999832	40	1	1.0000019907	2.9999996053

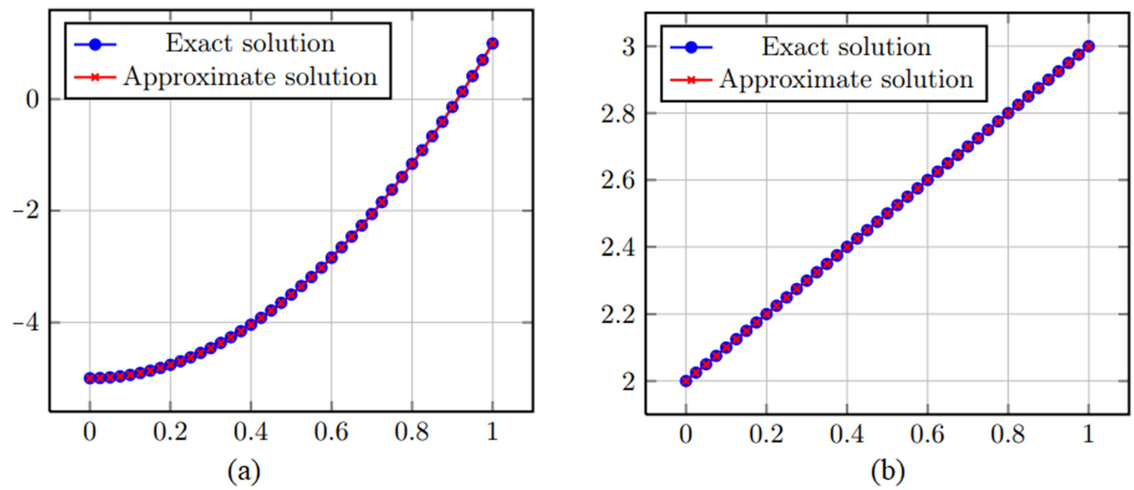

Figure 2: Comparison of the exact solutions (blue) with the numerical solutions (red) solutions of (a) $x_1(t)$ and (b) $x_2(t)$ for Example 4.

The exact solution is known to be $x^*(t) = \begin{pmatrix} 6t^2 - 5 \\ t + 2 \end{pmatrix}$.

The computational results of the approximate solution in the interval [0, 1] are reported in Table 5.

For the difference of the corresponding values of the exact and constructed solutions of the problem, the following estimate is true:

$$\max_{j=\overline{0,40}} ||x^*(t_j) - \widetilde{x}(t_j)|| < 0.000003.$$

Figure 2 depicts a comparison of numerical findings with the exact solution of problem (32), (33). We can see that the two graphs overlap, showing that our method is very accurate.

5 Conclusion

In this study, we presented a numerical algorithm for solving multi-point boundary value problem for systems of Volterra-Fredholm integro-differential equations. The proposed algorithm includes two auxiliary problems: solving the Cauchy problem for ordinary differential equation (ODE) on subintervals and calculating definite integrals. There are a number of effective ways to solving these problems. The accuracy of the numerical solution obtained by the algorithm depends on the choice of methods for solving auxiliary problems. In the given examples, the fourth-order Runge-Kutta method [46] is used to solve the Cauchy problem for ordinary differential equations, and the Simpson method [47] is used to calculate definite integrals.

To solve Cauchy Problem (22), you can use the Bulirsch-Stoer method and Runge-Kutta Fehlberg method, which will improve the convergence of numerical solution to exact solution of boundary value problem for Volterra-Fredholm integro-differential equations.

The presented method was then applied to four example problems. Furthermore, a comparison of the results produced by the suggested method, the precise solution, and the other methods demonstrates that our method is extremely effective.

Our next steps will be to apply Dzhumabaev parameterization method to boundary value problems for linear Volterra-Fredholm integro-differential equations with non-degenerate kernels, boundary value problems for Volterra-Fredholm impulsive integro-differential equations, and boundary value problems for nonlinear Volterra-Fredholm integro-differential equations.

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