

Research Article

Xue Zhang and Jing Zhang*

Existence and properties of soliton solution for the quasilinear Schrödinger system

<https://doi.org/10.1515/math-2024-0022>

received November 7, 2023; accepted May 16, 2024

Abstract: In this article, we consider the following quasilinear Schrödinger system:

$$\begin{cases} -\varepsilon\Delta u + u + \frac{k}{2}\varepsilon[\Delta|u|^2]u = \frac{2\alpha}{\alpha + \beta} |u|^{\alpha-2}u |v|^\beta, & x \in \mathbb{R}^N, \\ -\varepsilon\Delta v + v + \frac{k}{2}\varepsilon[\Delta|v|^2]v = \frac{2\beta}{\alpha + \beta} |u|^\alpha |v|^{\beta-2}v, & x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $k < 0$ are real constants, $N \geq 3$, α, β are integers multiple of constant 2. By using the Mountain Pass Theorem in a suitable Orlicz space proposed by Abbas Moameni [*Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in \mathbb{R}^N* , J. Differential Equations **229** (2006), 570–587], we proved the existence of soliton solution $(u_\varepsilon, v_\varepsilon)$ for the above system, and $(u_\varepsilon(x), v_\varepsilon(x)) \rightarrow (0, 0)$ as $|\varepsilon| \rightarrow 0$.

Keywords: Orlicz space, quasilinear Schrödinger system, Mountain Pass Theorem

MSC 2020: 35J10, 35J50, 35C08

1 Introduction

We consider the following quasilinear Schrödinger system:

$$\begin{cases} -\varepsilon\Delta u + u + \frac{k}{2}\varepsilon[\Delta|u|^2]u = \frac{2\alpha}{\alpha + \beta} |u|^{\alpha-2}u |v|^\beta, & x \in \mathbb{R}^N, \\ -\varepsilon\Delta v + v + \frac{k}{2}\varepsilon[\Delta|v|^2]v = \frac{2\beta}{\alpha + \beta} |u|^\alpha |v|^{\beta-2}v, & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $N \geq 3$, $\varepsilon > 0$, $k < 0$ are the real constants.

In recent years, much more attention has been devoted to the quasilinear Schrödinger system

$$\begin{cases} -\Delta u + V_1(x)u + \frac{k}{2}[\Delta|u|^2]u = \lambda f(x, u, v), & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v + \frac{k}{2}[\Delta|v|^2]v = \lambda h(x, u, v), & x \in \mathbb{R}^N. \end{cases} \quad (1.2)$$

* **Corresponding author: Jing Zhang**, College of Mathematics Science, Inner Mongolia Normal University, Hohhot, P.R. China; Key Laboratory of Infinite-dimensional Hamiltonian System and Its Algorithm Application, Ministry of Education, Inner Mongolia Normal University, Hohhot, P.R. China; Center for Applied Mathematics Inner Mongolia, Inner Mongolia Normal University, Hohhot, China, e-mail: jinshizhangjing@163.com

Xue Zhang: College of Mathematics Science, Inner Mongolia Normal University, Hohhot, P.R. China, e-mail: zhangxue230210@163.com

Li proved the existence of nontrivial solution by using a change of variable and the Mountain Pass Theorem when $k > 0$ is large enough [1]. Let $V_1(x) = \lambda a(x) + 1$, $V_2(x) = \lambda b(x) + 1$, $k = -1$ and $\lambda f(x, u, v) = \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^\beta$, $\lambda h(x, u, v) = \frac{2\beta}{\alpha+\beta} |u|^\alpha |v|^{\beta-2} v$, Guo and Tang proved the existence of ground state solution by using the Nehari manifold method and concentration compactness principle in [2], which is localized near the potential well $\text{int}\{a^{-1}(0)\} = \text{int}\{b^{-1}(0)\}$ for λ large enough. In [3], minimax methods in a suitable Orlicz space were employed to establish the existence of standing wave solution for a quasilinear Schrödinger system involving subcritical nonlinearities.

For some systems similar to (1.2), Chen and Zhang have done several contributions. In [4], they obtained the existence of positive ground state solution by using Morse iteration to define a Pohožaev manifold. They obtained the existence of positive solution by using monotonicity trick and the Morse iteration in [5]. They proved the existence of ground state solution by minimization under a convenient constraint and concentration compactness lemma in [6]. They found the existence of ground state solution by establishing a suitable constraint set and studying related minimization problem in [7].

By establishing a suitable Nehari-Pohožaev-type constraint set and considering related minimization problem, the existence of ground state solution for a class of systems was proved in [8]. The symmetric Mountain Pass Theorem was employed to establish the existence of infinitely many solutions for the quasilinear Schrödinger system in \mathbb{R}^N in [9], which involves a parameter α and subcritical nonlinearities. By developing a new iterative technique and suitable estimation, the existence of the entire radial large solution was established for the modified quasilinear Schrödinger elliptic system in [10].

The study of System (1.1) was in part motivated by the nonlinear Schrödinger equation:

$$i\varepsilon \partial_t z = -\varepsilon \Delta z + W(x)z - l(|z|^2)z - k\varepsilon \Delta h(|z|^2)h'(|z|^2)z, \quad x \in \mathbb{R}^N, \quad (1.3)$$

where $W(x)$ is a given potential, k is a real constant, and l and h are real functions that are essentially pure power forms. The quasilinear Schrödinger equation (1.3) describes several physical phenomena with different h , see [11–13] and references therein. We consider the case $h(s) = s$, $l(s) = \mu s^{\frac{p-1}{2}}$, and $k < 0$. Setting $z(t, x) = \exp(-iFt)u(x)$, one obtains a corresponding equation of elliptic type which has the formal variational structure

$$-\varepsilon \Delta u + V(x)u - \varepsilon k(\Delta |u|^2)u = \mu |u|^{p-1}u, \quad u > 0, \quad x \in \mathbb{R}^N, \quad (1.4)$$

where $V(x) = W(x) - F$ is the new potential function. Problem (1.4) has caused a heated discussion. When $k > 0$ is small enough, the existence result of multiple solutions was studied via dual approach techniques and variational methods [14]. The paper [15] established the existence of soliton solution by a minimization argument. The minimax principles for lower semicontinuous functionals, developed by Szulkin [16], were used to find solutions in [17]. The Mountain Pass Theorem combined with the principle of symmetric criticality to establish multiplicity of solutions in [18]. Moameni [19] changed variables to remove nonconvex term and created a suitable Orlicz space to meet the Mountain Pass Theorem, and proved the existence of soliton solution for a quasilinear Schrödinger equation involving critical exponent in \mathbb{R}^N .

Inspired by the above results, we apply the methods in [19] to solve system (1.1) and try to find nonnegative soliton solution. During this process, it is required that α, β are integers multiple of constant 2.

The main result of this article is the following:

Theorem 1.1. *For system (1.1), $k < 0$, $N \geq 3$, $\alpha > 2$, $\beta > 2$, $\alpha + \beta < 2^*$, and α, β are integers multiple of constant 2, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, system (1.1) has a nonnegative solution $(u_\varepsilon, v_\varepsilon) \in H_r^1$ and*

$$(u_\varepsilon(x), v_\varepsilon(x)) \rightarrow (0, 0), \quad \text{as } |\varepsilon| \rightarrow 0. \quad (1.5)$$

The article is organized as follows. In Section 2, we reformulate this problem in an appropriate Orlicz space. In Section 3, we prove the existence of a solution for a special deformation of problem (1.1). Theorem 1.1 is proved in Section 4.

2 Reformulation of the problem and preliminaries

The energy functional associated with (1.1) is

$$\begin{aligned} I_\varepsilon(u, v) &= \frac{\varepsilon}{2} \int_{\mathbb{R}^N} (1 - ku^2) |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 dx \\ &\quad + \frac{\varepsilon}{2} \int_{\mathbb{R}^N} (1 - kv^2) |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 dx - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx. \end{aligned}$$

By changing variables, we treated this problem in an Orlicz space. From [20] and [19], we changed variables as follows:

$$\begin{aligned} dz &= \sqrt{-k} \sqrt{1 - ku^2} du, \quad z = h(u) = \frac{1}{2} \sqrt{-k} u \sqrt{1 - ku^2} + \frac{1}{2} \ln(\sqrt{-k} u + \sqrt{1 - ku^2}), \\ dw &= \sqrt{-k} \sqrt{1 - kv^2} dv, \quad w = h(v) = \frac{1}{2} \sqrt{-k} v \sqrt{1 - kv^2} + \frac{1}{2} \ln(\sqrt{-k} v + \sqrt{1 - kv^2}). \end{aligned}$$

Since h is strictly monotone and has a well-defined inverse function $u = f(z)$, $v = f(w)$. Note that

$$\begin{aligned} h(u) &\sim \begin{cases} \sqrt{-k} u, & |u| \ll \sqrt{\frac{1}{-k}}, \\ \frac{-k}{2} u|u|, & |u| \gg \sqrt{\frac{1}{-k}}, \end{cases} & h'(u) &= \sqrt{-k} \sqrt{1 - ku^2}, \\ h(v) &\sim \begin{cases} \sqrt{-k} v, & |v| \ll \sqrt{\frac{1}{-k}}, \\ \frac{-k}{2} v|v|, & |v| \gg \sqrt{\frac{1}{-k}}, \end{cases} & h'(v) &= \sqrt{-k} \sqrt{1 - kv^2}, \end{aligned}$$

and

$$\begin{aligned} f(z) &\sim \begin{cases} \frac{1}{\sqrt{-k}} z, & |z| \ll \sqrt{\frac{1}{-k}}, \\ \sqrt{\frac{2}{-k|z|}} z, & |z| \gg \sqrt{\frac{1}{-k}}, \end{cases} \\ f'(z) &= \frac{1}{h'(u)} = \frac{1}{\sqrt{-k} \sqrt{1 - kv^2}} = \frac{1}{\sqrt{-k} \sqrt{1 - kf(z)^2}}, \\ f(w) &\sim \begin{cases} \frac{1}{\sqrt{-k}} w, & |w| \ll \sqrt{\frac{1}{-k}}, \\ \sqrt{\frac{2}{-k|w|}} w, & |w| \gg \sqrt{\frac{1}{-k}}, \end{cases} \\ f'(w) &= \frac{1}{h'(v)} = \frac{1}{\sqrt{-k} \sqrt{1 - kv^2}} = \frac{1}{\sqrt{-k} \sqrt{1 - kf(w)^2}}. \end{aligned}$$

Also, for some $C_0 > 0$ it holds

$$G(t) = f(t)^2 \sim \begin{cases} \frac{1}{-k} t^2, & |t| \ll \sqrt{\frac{1}{-k}}, \\ \frac{2}{-k} |t|, & |t| \gg \sqrt{\frac{1}{-k}}, \end{cases} \quad G(2t) \leq C_0 G(t),$$

$G(t)$ is convex, $G''(t) = \frac{2}{(1+f(t)^2)^2} > 0$, $i = 1, 2$. Now we introduce the Orlicz space (see [21]):

$$E_G(\mathbb{R}^N) = \left\{ \mu : \int_{\mathbb{R}^N} G(\mu) dx < \infty \right\}$$

equipped with the norm:

$$|\mu|_G = \inf_{\zeta > 0} \zeta \left(1 + \int_{\mathbb{R}^N} G(\zeta^{-1}\mu(x)) dx \right).$$

Using this change of variable, we can rewrite the functional $I_\varepsilon(u, v)$ as

$$\begin{aligned} \bar{I}_\varepsilon(z, w) &= \frac{\varepsilon}{2} \int_{\mathbb{R}^N} \frac{1}{\sqrt{-k}} |\nabla z|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |z|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^N} \frac{1}{\sqrt{-k}} |\nabla w|^2 dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^N} |w|^2 dx - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^N} |f(z)|^\alpha |f(w)|^\beta dx \end{aligned}$$

defined in the space

$$H_G^1 = \left\{ (z, w) : \int_{\mathbb{R}^N} |\nabla z|^2 dx < \infty, \int_{\mathbb{R}^N} G(z) dx < \infty, \int_{\mathbb{R}^N} |\nabla w|^2 dx < \infty, \int_{\mathbb{R}^N} G(w) dx < \infty \right\}.$$

From the definition of G , only radially symmetric functions are in this space, and equipped with the norm:

$$\|(z, w)\| = \|\nabla z\|_{L^2} + \|\nabla w\|_{L^2} + |z|_G + |w|_G,$$

where L^r is the Lebesgue function space with the norm

$$\|u\|_{L^r} = \left(\int_{\mathbb{R}^N} |u|^r dx \right)^{\frac{1}{r}}, \quad 1 \leq r < \infty.$$

Here are some related facts:

Proposition 2.1.

- (i) $E_G(\mathbb{R}^N)$ is a Banach space.
- (ii) If $\mu_n \rightarrow \mu$ in $E_G(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} |G(\mu_n) - G(\mu)| dx \rightarrow 0$ and $\int_{\mathbb{R}^N} |f(\mu_n) - f(\mu)|^2 dx \rightarrow 0$.
- (iii) If $\mu_n(x) \rightarrow \mu(x)$ a.e. in \mathbb{R}^N and $\int_{\mathbb{R}^N} G(\mu_n) dx \rightarrow \int_{\mathbb{R}^N} G(\mu) dx$, then $\mu_n \rightarrow \mu$ in $E_G(\mathbb{R}^N)$.
- (iv) The dual space $E_G^*(\mathbb{R}^N) = L^\infty \cap L^2 = \{v : v \in L^\infty, \int_{\mathbb{R}^N} v^2 dx < \infty\}$.
- (v) If $\mu \in E_G(\mathbb{R}^N)$, then $v = G'(\mu) = 2f(\mu)f'(\mu)$, and $|v|_{E_G^*} = \sup_{|\phi|_{G^1} \leq 1} (v, \phi) \leq C_1(1 + \int_{\mathbb{R}^N} G(\mu) dx)$, where C_1 is a constant independent of μ .
- (vi) For $N > 2$, the map: $(\mu_1, \mu_2) \rightarrow (f(\mu_1), f(\mu_2))$ from H_G^1 into $L^q(\mathbb{R}^N)$ is continuous for $2 \leq q \leq 22^*$ and is compact for $2 < q < 22^*$.
- (vii) Suppose B_k is the ball with center at the coordinate origin and radius $k > 0$. Let $r < s$ and $Q = B_s \setminus B_r$. The map: $(\mu_1, \mu_2) \rightarrow (f(\mu_1), f(\mu_2))$ from H_G^1 into $L^q(Q)$ is compact for $q \leq 2$.

Proof. See [20] for the proof of parts (i)–(vi). The proof of part (vii) is similar to the proof of [19]. \square

Denote by H_r^1 the space of radially symmetric functions in

$$H^{1,2} = \{u : u \in L^2(\mathbb{R}^N), \nabla u \in L^2(\mathbb{R}^N)\}.$$

Throughout this article, we use the standard notations. \int, H^1, E_G, L^t , and $\|\cdot\|$ stand for $\int_{\mathbb{R}^N}, H^{1,2}, E_G(\mathbb{R}^N), L^t(\mathbb{R}^N)$, and $\|\cdot\|_{H_G^1(\mathbb{R}^N)}$, respectively. We use C to denote any constant that is independent of the sequences considered, and the operation: $(a, b) * (a, b) = a * a, b * b$, where $*$ represents any operation.

3 Auxiliary problem

In this section, we show some results needed to prove Theorem 1.1. Indeed, we consider a special deformation $\bar{H}_\varepsilon(z, w)$ (see (3.1) in the following) of $I_\varepsilon(z, w)$ first. The functional $\bar{H}_\varepsilon(z, w)$ satisfies the PS condition and to which we can apply the Mountain Pass Theorem. Consequently, $\bar{H}_\varepsilon(z, w)$ has a critical point for each $\varepsilon > 0$. We can use it to prove Theorem 1.1 in the next section. In fact, we will see that the functionals $I_\varepsilon(z, w)$ and $\bar{H}_\varepsilon(z, w)$ will coincide for the small values of ε . This idea was explored in [19,22].

To do this, we shall consider constants θ and l satisfying

$$4 < \theta < 22^*, \quad l > \frac{\theta}{\theta - 2}.$$

Let $a_i > 0$ be the value at which $\frac{\xi_i(a_i)}{a_i} = \frac{1}{l}$, $i = 1, 2$, where $\xi_1 = \alpha|u|^{\alpha-2}u|v|^\beta$, $\xi_2 = \beta|u|^\alpha|v|^{\beta-2}v$. Set

$$\bar{\xi}_i(s) = \begin{cases} \xi_i(s), & \text{if } s \leq a_i, \\ \frac{1}{l}s, & \text{if } s > a_i, \end{cases}$$

and define

$$y_i(x, s_i) = \chi_\Lambda \xi_i(s_i) + (1 - \chi_\Lambda) \bar{\xi}_i(s_i),$$

where χ_Λ denotes the characteristic function of the set Λ , which is a bounded domain. For convenience of later calculation, let Λ be a ring domain. Set $Y_i(x, t_i) = \int_0^{t_i} y_i(x, \zeta) d\zeta$, where $i = 1, 2$. Set $y(x, s_1, s_2) = y_1(x, s_1) + y_2(x, s_2)$, $Y(x, s_1, s_2) = Y_1(x, s_1) + Y_2(x, s_2)$. According to [19], the functions y, Y satisfy the following conditions:

$$0 \leq \theta Y(x, s_1, s_2) \leq y(x, s_1, s_2)(s_1, s_2), \quad \forall x \in \Lambda, s_i \leq 0, i = 1, 2, \quad (y_1)$$

$$0 \leq 2Y(x, s_1, s_2) \leq y(x, s_1, s_2)(s_1, s_2) \leq \frac{1}{l}(s_1^2, s_2^2), \quad \forall x \in \Lambda^c, s_i \in \mathbb{R}, i = 1, 2. \quad (y_2)$$

Using the modification, the energy functional $I_\varepsilon(u, v)$ becomes

$$\begin{aligned} H_\varepsilon(u, v) &= \frac{\varepsilon}{2} \int (1 - ku^2) |\nabla u|^2 dx + \frac{1}{2} \int |u|^2 dx \\ &\quad + \frac{\varepsilon}{2} \int (1 - kv^2) |\nabla v|^2 dx + \frac{1}{2} \int |v|^2 dx - \frac{2}{\alpha + \beta} \int Y(x, u, v) dx. \end{aligned}$$

As in Section 2, we can rewrite the functional $H_\varepsilon(u, v)$ as a new functional $\bar{H}_\varepsilon(z, w)$ as follows:

$$\begin{aligned} \bar{H}_\varepsilon(z, w) &= \frac{\varepsilon}{2} \int \frac{1}{\sqrt{-k}} |\nabla z|^2 dx + \frac{1}{2} \int |f(z)|^2 dx \\ &\quad + \frac{\varepsilon}{2} \int \frac{1}{\sqrt{-k}} |\nabla w|^2 dx + \frac{1}{2} \int |f(w)|^2 dx - \frac{2}{\alpha + \beta} \int Y(x, f(z), f(w)) dx. \end{aligned} \quad (3.1)$$

$\bar{H}_\varepsilon(z, w)$ is defined on the Orlicz space H_G^1 . In this section, we shall assume $\varepsilon = 1$, $H_1 = H$, $\bar{H}_1 = \bar{H}$.

Some properties of the functional \bar{H} are stated as follows:

Proposition 3.1.

- (i) \bar{H} is well defined in H_G^1 .
- (ii) \bar{H} is continuous in H_G^1 .
- (iii) \bar{H} is Gateaux-differentiable in H_G^1 .

Proof. The proof is similar to the proof of [20]. □

The following theorem is the main result in this section.

Theorem 3.1. \bar{H} has a critical point in H_G^1 , that is, there exists $(0, 0) \neq (z, w) \in H_G^1$ such that

$$\begin{aligned} & \int \frac{1}{\sqrt{-k}} \nabla z \nabla \phi \, dx + \int f(z) f'(z) \phi \, dx + \int \frac{1}{\sqrt{-k}} \nabla w \nabla \psi \, dx + \int f(w) f'(w) \psi \, dx \\ & - \frac{2}{\alpha + \beta} \int Y(x, f(z), f(w)) (f'(z), f'(w)) (\phi, \psi) \, dx = 0, \end{aligned} \quad (3.2)$$

for every $(\phi, \psi) \in H_G^1$.

We will use the Mountain Pass Theorem like [23,24]. First, let us define the Mountain Pass value

$$C_0 = \inf_{\gamma \in \tau} \sup_{(t_1, t_2) \in [0,1] \times [0,1]} \bar{H}(\gamma(t_1, t_2)),$$

where

$$\tau = \{\gamma \in C([0, 1] \times [0, 1], H_G^1) : \gamma(0, 0) = 0, \bar{H}(\gamma(1, 1)) \leq 0, \gamma(1, 1) \neq 0\}.$$

Next, we will prove Theorem 3.1 by the following lemmas.

Lemma 3.1. *The functional \bar{H} satisfies the Mountain Pass geometry.*

Proof. From the definition of Y , we have $\bar{H}(0, 0) = 0$. Clearly, there exists (z_0, w_0) that satisfies $\bar{H}(z_0, w_0) > 0$. Let $e_1, e_2 \in C_{0,r}^\infty(\mathbb{R}^N)$ with $e \neq 0$ and $\text{supp}(e_i) \subset \Omega$, $i = 1, 2$. It is easy to see that $H(te_1, te_2) \leq 0$ for the large values of t . Consequently, there exists $(0, 0) \neq (z, w) \in H_G^1$ such that $\bar{H}(z, w) \leq 0$, where $z = h(te_1)$, $w = h(te_2)$. Therefore, the functional \bar{H} satisfies the Mountain Pass Geometry. \square

This lemma guaranties the existence of a $(PS)_{C_0}$ sequence (z_n, w_n) , that is, $\bar{H}(z_n, w_n) \rightarrow C_0$ and $\bar{H}'(z_n, w_n) \rightarrow 0$.

Lemma 3.2. C_0 is positive.

Proof. Set

$$S_\rho = \left\{ (z, w) \in H_G^1 : \varepsilon \int \frac{1}{\sqrt{-k}} |\nabla z|^2 \, dx + \int f(z)^2 \, dx = \rho^2, \varepsilon \int \frac{1}{\sqrt{-k}} |\nabla w|^2 \, dx + \int f(w)^2 \, dx = \rho^2 \right\},$$

where $0 < \rho \ll 1$.

For $(z, w) \in S_\rho$, we have $\int f(z)^2 \, dx \leq \rho^2$, $\int f(w)^2 \, dx \leq \rho^2$. By Hölder inequality and continuity of f , we obtain

$$\int f(z)^\alpha \, dx \leq \left(\int f(z)^2 \, dx \right)^{\frac{\alpha}{2}} \leq \rho^\alpha, \quad (3.3)$$

$$\int f(w)^\beta \, dx \leq \left(\int f(w)^2 \, dx \right)^{\frac{\beta}{2}} \leq \rho^\beta. \quad (3.4)$$

Also, it follows from (y_1) and (y_2) that

$$\begin{aligned} \int Y(x, f(z), f(w)) \, dx &= \int_{\Lambda} Y(x, f(z), f(w)) \, dx + \int_{\Lambda^c} Y(x, f(z), f(w)) \, dx \\ &\leq \frac{1}{\theta} \left(\int f(z)^\alpha \, dx + \int f(w)^\beta \, dx \right) + \frac{1}{2l} \left(\int f(z)^2 \, dx + \int f(w)^2 \, dx \right). \end{aligned} \quad (3.5)$$

From (3.3), (3.4), and (3.5), we obtain

$$\begin{aligned}
\bar{H}_\varepsilon(z, w) &= \frac{\varepsilon}{2} \int \frac{1}{\sqrt{-k}} |\nabla z|^2 dx + \frac{1}{2} \int |f(z)|^2 dx \\
&\quad + \frac{\varepsilon}{2} \int \frac{1}{\sqrt{-k}} |\nabla w|^2 dx + \frac{1}{2} \int |f(w)|^2 dx - \frac{2}{\alpha + \beta} \int Y(x, f(z), f(w)) dx \\
&\geq \rho^2 - \frac{2}{(\alpha + \beta)\theta} (\rho^\alpha + \rho^\beta) - \frac{2}{(\alpha + \beta)l} \rho^2 \\
&\geq \left(1 - \frac{2}{(\alpha + \beta)l}\right) \rho^2 - \frac{2}{(\alpha + \beta)\theta} (\rho^\alpha + \rho^\beta) \\
&\geq \bar{C} \rho^2,
\end{aligned}$$

if $0 < \rho \leq \rho_0 \ll 1$ for some ρ_0 , then $\bar{C} = (1 - \frac{2}{(\alpha + \beta)l} - \frac{2}{(\alpha + \beta)\theta}) > 0$. Hence, for $(z, w) \in S_\rho$ with $0 < \rho \leq \rho_0 \ll 1$ we have

$$\bar{H}_\varepsilon(z, w) \geq \bar{C} \rho^2. \quad (3.6)$$

If $\gamma(1) = (z, w)$ and $\bar{H}(\gamma(1)) < 0$, then it follows from (3.6) that

$$\int \frac{1}{\sqrt{-k}} |\nabla z|^2 dx + \int f(z)^2 dx > \rho_0^2, \quad \int \frac{1}{\sqrt{-k}} |\nabla w|^2 dx + \int f(w)^2 dx > \rho_0^2,$$

thereby giving

$$\sup_{(t_1, t_2) \in [0, 1] \times [0, 1]} \bar{H}(\gamma(t_1, t_2)) \geq \sup_{\gamma(t_1, t_2) \in S_\rho} \bar{H}(\gamma(t_1, t_2)) \geq \bar{C} \rho_0^2,$$

which combines the definition of C_0 , and we have

$$C_0 \geq \bar{C} \rho_0^2 > 0. \quad \square$$

Lemma 3.3. Suppose (z_n, w_n) is a $(PS)_{C_0}$ sequence. The following statements hold:

- (i) (z_n, w_n) is bounded in H_G^1 .
- (ii) For each $\delta > 0$, there exists $R > 0$, such that

$$\limsup_{n \rightarrow +\infty} \int_{B_R^c} \left(\frac{\varepsilon}{\sqrt{-k}} |\nabla z_n|^2 + f(z_n)^2 \right) dx < \delta,$$

$$\limsup_{n \rightarrow +\infty} \int_{B_R^c} \left(\frac{\varepsilon}{\sqrt{-k}} |\nabla w_n|^2 + f(w_n)^2 \right) dx < \delta.$$

- (iii) If (z_n, w_n) converges weakly to (z, w) in H_G^1 , then

$$\lim_{n \rightarrow +\infty} \int y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx = \int y(x, f(z), f(w))(f(z), f(w)) dx.$$

- (iv) If $(z_n, w_n) \geq 0$ converges weakly to (z, w) in H_G^1 , then for every nonnegative test function $(\phi, \psi) \in H_G^1$ we have

$$\lim_{n \rightarrow +\infty} \langle \bar{H}'(z_n, w_n), (\phi, \psi) \rangle = \langle \bar{H}'(z, w), (\phi, \psi) \rangle.$$

Proof. Since (z_n, w_n) is a $(PS)_{C_0}$ sequence, we have

$$\begin{aligned}
\bar{H}_\varepsilon(z_n, w_n) &= \frac{\varepsilon}{2} \int \frac{1}{\sqrt{-k}} |\nabla z_n|^2 dx + \frac{1}{2} \int |f(z_n)|^2 dx + \frac{\varepsilon}{2} \int \frac{1}{\sqrt{-k}} |\nabla w_n|^2 dx \\
&\quad + \frac{1}{2} \int |f(w_n)|^2 dx - \frac{2}{\alpha + \beta} \int Y(x, f(z_n), f(w_n)) dx = C_0 + o(1)
\end{aligned} \quad (3.7)$$

and

$$\begin{aligned} \langle \bar{H}_\varepsilon'(z_n, w_n), (\phi, \psi) \rangle &= \varepsilon \int \frac{1}{\sqrt{-k}} \nabla z \nabla \phi \, dx + \int f(z_n) f'(z_n) \phi \, dx + \varepsilon \int \frac{1}{\sqrt{-k}} \nabla w_n \nabla \psi \, dx \\ &\quad + \int f(w_n) f'(w_n) \psi \, dx - \frac{2}{\alpha + \beta} \int y(x, f(z_n), f(w_n)) (f'(z_n), f'(w_n)) (\phi, \psi) \, dx \\ &= o(\|(\phi, \psi)\|). \end{aligned} \quad (3.8)$$

For part (i), pick

$$(\phi, \psi) = \left(\frac{f(z_n)}{f'(z_n)}, \frac{f(w_n)}{f'(w_n)} \right) = (\sqrt{-k} \sqrt{1 - f^2(z_n)} f(z_n), \sqrt{-k} \sqrt{1 - f^2(w_n)} f(w_n)).$$

It is easy to see that $\|(\phi, \psi)\|_G \leq C \|(z_n, w_n)\|_G$ and

$$(\nabla \phi, \nabla \psi) = \left(\left(1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right) |\nabla z_n|, \left(1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right) |\nabla w_n| \right) \leq (2|\nabla z_n|, 2|\nabla w_n|),$$

hence, $\|(\phi, \psi)\| \leq C \|(z_n, w_n)\|$. Substituting (ϕ, ψ) in (3.8), then

$$\begin{aligned} \left\langle \bar{H}_\varepsilon'(z_n, w_n), \left(\frac{f(z_n)}{f'(z_n)}, \frac{f(w_n)}{f'(w_n)} \right) \right\rangle &= \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right) |\nabla z_n|^2 \, dx + \int f(z_n)^2 \, dx \\ &\quad + \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right) |\nabla w_n|^2 \, dx + \int f(w_n)^2 \, dx \\ &\quad - \frac{2}{\alpha + \beta} \int y(x, f(z_n), f(w_n)) (f'(z_n), f'(w_n)) \, dx \\ &= o(\|(f(z_n), f(w_n))\|). \end{aligned} \quad (3.9)$$

It follows from (y_1) and (y_2) that

$$\begin{aligned} &-\int G(x, f(z_n), f(w_n)) \, dx + \frac{1}{\theta} \int y(x, f(z_n), f(w_n)) (f'(z_n), f'(w_n)) (\phi, \psi) \, dx \\ &\geq \frac{1}{l} \left(\frac{1}{\theta} - \frac{1}{2} \right) \int (|f(z_n)|^2 + |f(w_n)|^2) \, dx. \end{aligned} \quad (3.10)$$

Taking into account (3.7), (3.9), and (3.10), we obtain

$$\begin{aligned} C_0 + o(1) + o(\|(\phi, \psi)\|) &= \bar{H}_\varepsilon(z_n, w_n) - \frac{1}{\theta} \left\langle \bar{H}_\varepsilon'(z_n, w_n), \left(\frac{f(z_n)}{f'(z_n)}, \frac{f(w_n)}{f'(w_n)} \right) \right\rangle \\ &\geq \left(\frac{\varepsilon}{2} - \frac{2\varepsilon}{\theta} \right) \int \frac{1}{\sqrt{-k}} |\nabla z_n|^2 \, dx + \left(\frac{1}{2} - \frac{1}{\theta} \right) \int |f(z_n)|^2 \, dx \\ &\quad + \left(\frac{\varepsilon}{2} - \frac{2\varepsilon}{\theta} \right) \int \frac{1}{\sqrt{-k}} |\nabla w_n|^2 \, dx + \left(\frac{1}{2} - \frac{1}{\theta} \right) \int |f(w_n)|^2 \, dx \\ &\quad + \frac{2}{\alpha + \beta} \int \left(\frac{1}{\theta} y(x, f(z_n), f(w_n)) (f'(z_n), f'(w_n)) - Y(x, f(z_n), f(w_n)) \right) \, dx \\ &\geq \left(\frac{\varepsilon}{2} - \frac{2\varepsilon}{\theta} \right) \int \frac{1}{\sqrt{-k}} |\nabla z_n|^2 \, dx + \left(\frac{\varepsilon}{2} - \frac{2\varepsilon}{\theta} \right) \int \frac{1}{\sqrt{-k}} |\nabla w_n|^2 \, dx \\ &\quad + \left(1 - \frac{1}{l} \right) \left(\frac{1}{\theta} - \frac{1}{2} \right) \int (|f(z_n)|^2 + |f(w_n)|^2) \, dx. \end{aligned}$$

Since $\frac{\varepsilon}{2} - \frac{2\varepsilon}{\theta} > 0$ and $(1 - \frac{1}{l})(\frac{1}{\theta} - \frac{1}{2}) > 0$, it follows from the above that $\int |\nabla z_n|^2 \, dx + \int |f(z_n)|^2 \, dx + \int |\nabla w_n|^2 \, dx + \int |f(w_n)|^2 \, dx$ is bounded. It proves part (i).

For part (ii), let $\eta_R \in C^\infty(\mathbb{R}^N)$ be a function satisfying $\eta_R = 0$ on $B_{R/2}$, $\eta_R = 1$ on B_R^c , and $|\nabla \eta_R(x)| \leq \frac{C}{R}$. It follows from part (i) that (z_n, w_n) is bounded in H_G^1 . Hence, from (3.8) we have

$$\left\langle \bar{H}_\varepsilon'(z_n, w_n), \left(\frac{f(z_n)}{f'(z_n)} \eta_R, \frac{f(w_n)}{f'(w_n)} \eta_R \right) \right\rangle = o(1),$$

which stands for

$$\begin{aligned} & \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right) |\nabla z_n^2| \eta_R dx + \int f(z_n)^2 \eta_R dx + \frac{\varepsilon}{\sqrt{-k}} \int \frac{f(z_n)}{f'(z_n)} \nabla z_n \nabla \eta_R dx \\ & + \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right) |\nabla w_n^2| \eta_R dx + \int f(w_n)^2 \eta_R dx + \frac{\varepsilon}{\sqrt{-k}} \int \frac{f(w_n)}{f'(w_n)} \nabla w_n \nabla \eta_R dx \\ & = \frac{2}{\alpha + \beta} \int y(x, f(z_n), f(w_n))(f(z_n) \eta_R, f(w_n) \eta_R) dx. \end{aligned}$$

By (y_2) , we obtain

$$y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) \leq \frac{1}{l}(f^2(z_n), f^2(w_n)), \quad \forall x \in B_R^c.$$

Therefore,

$$\begin{aligned} & \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right) |\nabla z_n^2| \eta_R dx + \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right) |\nabla w_n^2| \eta_R dx \\ & + \left(1 - \frac{1}{l} \right) \left(\frac{2}{\alpha + \beta} \right) \int (f(z_n)^2 \eta_R, f(w_n)^2 \eta_R) dx \\ & \leq \frac{\varepsilon}{\sqrt{-k}} \frac{C}{R} \int \left(\frac{|f(z_n)|}{f'(z_n)} |\nabla z_n|, \frac{|f(w_n)|}{f'(w_n)} |\nabla w_n| \right) dx + o(1) \\ & \leq \frac{\varepsilon}{\sqrt{-k}} \frac{C}{R} \int (\nabla z_n, \nabla w_n) dx + \frac{\varepsilon}{\sqrt{-k}} \frac{C}{R} \int (|f(z_n)|^2 + |f(z_n)|^4, |f(w_n)|^2 + |f(w_n)|^4) dx + o(1). \end{aligned} \tag{3.11}$$

Also, like [19], it follows from Proposition 2.1(vi) that $\{f(v_n)\}_n$ is a bounded sequence in $L^2 \cap L^{22^*}$. Hence, $\int (|f(z_n)|^2 + |f(z_n)|^4, |f(w_n)|^2 + |f(w_n)|^4) dx$ is bounded. Therefore, it follows from (3.11) that

$$\limsup_{n \rightarrow +\infty} \int_{B_R^c} \left(\frac{\varepsilon}{\sqrt{-k}} |\nabla z_n|^2 + f(z_n)^2 \right) dx < \delta,$$

$$\limsup_{n \rightarrow +\infty} \int_{B_R^c} \left(\frac{\varepsilon}{\sqrt{-k}} |\nabla w_n|^2 + f(w_n)^2 \right) dx < \delta.$$

It proves part (ii).

For part (iii), from part (ii) we know that for each $\delta > 0$, there exists $R > 0$, such that

$$\limsup_{n \rightarrow +\infty} \int_{B_R^c} \left(\frac{\varepsilon}{\sqrt{-k}} |\nabla z_n|^2 + f(z_n)^2 \right) dx < \frac{l\delta}{4}, \tag{3.12}$$

$$\limsup_{n \rightarrow +\infty} \int_{B_R^c} \left(\frac{\varepsilon}{\sqrt{-k}} |\nabla w_n|^2 + f(w_n)^2 \right) dx < \frac{l\delta}{4}. \tag{3.13}$$

We might as well make $B_R^c \subseteq \Lambda^c$, $R_1 < R$, $B_{R_1} \subset \Lambda^c$, and it follows from (y_2) that

$$y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) \leq \frac{1}{l}(f(z_n)^2, f(w_n)^2), \quad \forall x \in B_R^c,$$

and from (3.12), (3.13), we obtain

$$\limsup_{n \rightarrow +\infty} \int_{B_R^c} y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx \leq \left(\frac{\delta}{4}, \frac{\delta}{4} \right) \quad (3.14)$$

and

$$\int_{B_R^c} y(x, f(z), f(w))(f(z), f(w)) dx \leq \left(\frac{\delta}{4}, \frac{\delta}{4} \right). \quad (3.15)$$

By (3.14) and (3.15), we have

$$\begin{aligned} & \left| \int y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx - \int y(x, f(z), f(w))(f(z), f(w)) dx \right| \\ & \leq \frac{\delta}{2} + \left| \int_{B_{R_1}} y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) - y(x, f(z), f(w))(f(z), f(w)) dx \right| \\ & + \left| \int_{B_R \setminus B_{R_1}} y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) - y(x, f(z), f(w))(f(z), f(w)) dx \right|. \end{aligned} \quad (3.16)$$

Because of $B_{R_1} \subset \Lambda^c$,

$$y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) \leq \frac{1}{l}(f(z_n)^2, f(w_n)^2), \quad \forall x \in B_{R_1}.$$

Then, by the compact embedding theorem and Lebesgue theorem, we obtain a subsequence still denoted by $(f(z_n), f(w_n))$, such that

$$\int_{B_{R_1}} y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx \rightarrow \int_{B_{R_1}} y(x, f(z), f(w))(f(z), f(w)) dx. \quad (3.17)$$

It follows from part (vii) of Proposition 2.1 that the map: $(\mu_1, \mu_2) \rightarrow (f(\mu_1), f(\mu_2))$ from H_G^1 into $L^q(B_R \setminus B_{R_1})$ is compact for $q \leq 2$, combined with (y_2) , hence

$$\int_{B_R \setminus B_{R_1}} y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx \rightarrow \int_{B_R \setminus B_{R_1}} y(x, f(z), f(w))(f(z), f(w)) dx. \quad (3.18)$$

Considering (3.17) and (3.18), it follows from (3.16) that

$$\limsup_{n \rightarrow +\infty} \left| \int y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx - \int y(x, f(z), f(w))(f(z), f(w)) dx \right| \leq \frac{\delta}{2}$$

for every $\delta > 0$. Consequently,

$$\int y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx \rightarrow \int y(x, f(z), f(w))(f(z), f(w)) dx$$

as $n \rightarrow \infty$. It proves part (iii).

For part (iv), we know f is increasing and $f = 0$, hence $f(z_n) \geq 0, f(w_n) \geq 0$ and $f(z) \geq 0, f(w) \geq 0$. For the second term and the fourth term on the right-hand side of (3.8), we have

$$f(z_n)f'(z_n)\phi \leq f(z_n)\phi,$$

$$f(w_n)f'(w_n)\psi \leq f(w_n)\psi,$$

and since $(z_n, w_n) \rightharpoonup (z, w)$ weakly in H_G^1 , for the right-hand side of the above two inequalities we have

$$\lim_{n \rightarrow +\infty} \int f(z_n)\phi dx = \int f(z)\phi dx,$$

$$\lim_{n \rightarrow +\infty} \int f(w_n)\psi dx = \int f(w)\psi dx.$$

By the dominated convergence theorem and the fact that $(z_n, w_n) \rightarrow (z, w)$ a.e. in \mathbb{R}^N , we obtain

$$\lim_{n \rightarrow +\infty} \int f(z_n) f'(z_n) \phi dx = \int f(z) f'(z) \phi dx \quad (3.19)$$

and

$$\lim_{n \rightarrow +\infty} \int f(w_n) f'(w_n) \psi dx = \int f(w) f'(w) \psi dx. \quad (3.20)$$

For the fifth term on the right-hand side of (3.8), we have

$$y(x, f(z_n), f(w_n))(f'(z_n), f'(w_n))(\phi, \psi) \leq \frac{1}{l}(f(z_n), f(w_n))(\phi, \psi), \quad \forall x \in \Lambda^c,$$

and similarly by the dominated convergence theorem, we obtain

$$\lim_{n \rightarrow +\infty} \int_{\Lambda^c} y(x, f(z_n), f(w_n))(f'(z_n), f'(w_n))(\phi, \psi) dx = \int_{\Lambda^c} y(x, f(z), f(w))(f'(z), f'(w))(\phi, \psi) dx. \quad (3.21)$$

Also, we know

$$y(x, f(z_n), f(w_n))(f'(z_n), f'(w_n))(\phi, \psi) \leq (|u|^{\alpha-2}|v|^\beta, |u|^\alpha|v|^{\beta-2})(\phi, \psi), \quad \forall x \in \Lambda$$

and follows from part (vii) of Proposition 2.1 that the map: $(\mu_1, \mu_2) \rightarrow (f(\mu_1), f(\mu_2))$ from H_G^1 into $L^q(\Lambda)$ is compact for $q \leq 2$, hence it follows from the dominated convergence theorem that

$$\lim_{n \rightarrow +\infty} \int_{\Lambda} y(x, f(z_n), f(w_n))(f'(z_n), f'(w_n))(\phi, \psi) dx = \int_{\Lambda} y(x, f(z), f(w))(f'(z), f'(w))(\phi, \psi) dx. \quad (3.22)$$

It follows from (3.8) and (3.24)–(3.27) that

$$\lim_{n \rightarrow +\infty} \langle \bar{H}_\varepsilon'(z_n, w_n), (\phi, \psi) \rangle = \langle \bar{H}_\varepsilon'(z, w), (\phi, \psi) \rangle.$$

It proves part (iv). □

Lemma 3.4. *If (z_n, w_n) is a $(PS)_{C_0}$ sequence, then $(z_n, w_n) \geq 0$ converges to (z, w) in H_G^1 . Consequently, $\bar{H}(z, w) = \lim_{n \rightarrow +\infty} \bar{H}(z_n, w_n)$ and $\bar{H}'(z, w) = 0$.*

Proof. It follows from part (i) of Lemma 3.3 that (z_n, w_n) is bounded in H_G^1 . Hence, there exists $(z, w) \in H_G^1$ such that, up to a subsequence, $(z_n, w_n) \rightharpoonup (z, w)$ weakly in H_G^1 and $(z_n, w_n) \rightarrow (z, w)$ a.e. in \mathbb{R}^N . We may replace (z_n, w_n) by $|z_n, w_n|$, hence $(z_n, w_n) \geq 0$ and $(z, w) \geq 0$. Since (z_n, w_n) is a $(PS)_{C_0}$ sequence, we have

$$\begin{aligned} o(\|(f_1(z_n), f_2(w_n))\|) &= \left\langle \bar{H}_\varepsilon'(z_n, w_n), \left(\frac{f(z_n)}{f'(z_n)}, \frac{f(w_n)}{f'(w_n)} \right) \right\rangle \\ &= \varepsilon \int \frac{1}{\sqrt{-k}} \left[1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right] |\nabla z_n|^2 dx + \int f(z_n)^2 dx \\ &\quad + \varepsilon \int \frac{1}{\sqrt{-k}} \left[1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right] |\nabla w_n|^2 dx + \int f(w_n)^2 dx \\ &\quad - \frac{2}{\alpha + \beta} \int y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx \end{aligned} \quad (3.23)$$

and

$$o(\|(f(z), f(w))\|) = \left\langle \bar{H}_\varepsilon'(z, w), \left(\frac{f(z)}{f'(z)}, \frac{f(w)}{f'(w)} \right) \right\rangle. \quad (3.24)$$

It follows from part (iv) of Lemma 3.3 and (3.24) that

$$\begin{aligned}
\left\langle \bar{H}_\varepsilon'(z_n, w_n), \left(\frac{f(z)}{f'(z)}, \frac{f(w)}{f'(w)} \right) \right\rangle &= \left\langle \bar{H}_\varepsilon'(z, w), \left(\frac{f(z)}{f'(z)}, \frac{f(w)}{f'(w)} \right) \right\rangle + o(\|(f(z), f(w))\|) \\
&= \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(z)}{1 - kf^2(z)} \right) |\nabla z|^2 dx + \int f(z)^2 dx \\
&\quad + \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(w)}{1 - kf^2(w)} \right) |\nabla w|^2 dx + \int f(w)^2 dx \\
&\quad - \frac{2}{\alpha + \beta} \int y(x, f(z), f(w))(f(z), f(w)) dx \\
&\quad + o(\|(f(z), f(w))\|).
\end{aligned} \tag{3.25}$$

According to [19], we also have

$$\varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(z)|\nabla z|^2}{1 - kf^2(z)} \right) dx \leq \liminf_{n \rightarrow +\infty} \varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(z_n)|\nabla z_n|^2}{1 - kf^2(z_n)} \right) dx, \tag{3.26}$$

$$\varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(w)|\nabla w|^2}{1 - kf^2(w)} \right) dx \leq \liminf_{n \rightarrow +\infty} \varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(w_n)|\nabla w_n|^2}{1 - kf^2(w_n)} \right) dx. \tag{3.27}$$

Also, lower semi continuity and Fatou lemma imply

$$\varepsilon \int |\nabla z|^2 dx \leq \liminf_{n \rightarrow +\infty} \varepsilon \int |\nabla z_n|^2 dx, \tag{3.28}$$

$$\varepsilon \int |\nabla w|^2 dx \leq \liminf_{n \rightarrow +\infty} \varepsilon \int |\nabla w_n|^2 dx, \tag{3.29}$$

$$\int G(z)^2 dx = \liminf_{n \rightarrow +\infty} \int G(z_n)^2 dx, \tag{3.30}$$

$$\int G(w)^2 dx = \liminf_{n \rightarrow +\infty} \int G(w_n)^2 dx. \tag{3.31}$$

Up to a subsequence one can assume

$$\liminf_{n \rightarrow +\infty} \varepsilon \int |\nabla z_n|^2 dx = \lim_{n \rightarrow +\infty} \varepsilon \int |\nabla z_n|^2 dx, \tag{3.32}$$

$$\liminf_{n \rightarrow +\infty} \int G(z_n)^2 dx = \lim_{n \rightarrow +\infty} \int G(z_n)^2 dx, \tag{3.33}$$

$$\liminf_{n \rightarrow +\infty} \varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(z_n)|\nabla z_n|^2}{1 - kf^2(z_n)} \right) dx = \lim_{n \rightarrow +\infty} \varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(z_n)|\nabla z_n|^2}{1 - kf^2(z_n)} \right) dx. \tag{3.34}$$

There exist nonnegative numbers δ_1 , δ_2 , and δ_3 such that

$$\lim_{n \rightarrow +\infty} \varepsilon \int |\nabla z_n|^2 dx = \varepsilon \int |\nabla z|^2 dx + \delta_1, \tag{3.35}$$

$$\lim_{n \rightarrow +\infty} \int G(z_n)^2 dx = \int G(z)^2 dx + \delta_2, \tag{3.36}$$

$$\lim_{n \rightarrow +\infty} \varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(z_n)|\nabla z_n|^2}{1 - kf^2(z_n)} \right) dx = \varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(z)|\nabla z|^2}{1 - kf^2(z)} \right) dx + \delta_3. \tag{3.37}$$

Continuing in this way, we can obtain

$$\lim_{n \rightarrow +\infty} \varepsilon \int |\nabla w_n|^2 dx = \varepsilon \int |\nabla w|^2 dx + \delta'_1, \tag{3.38}$$

$$\lim_{n \rightarrow +\infty} \int G(w_n)^2 dx = \int G(w)^2 dx + \delta'_2, \tag{3.39}$$

$$\lim_{n \rightarrow +\infty} \varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(w_n)|\nabla w_n|^2}{1 - kf^2(w_n)} \right) dx = \varepsilon \int \frac{1}{\sqrt{-k}} \left(\frac{-kf^2(w)|\nabla w|^2}{1 - kf^2(w)} \right) dx + \delta'_3, \quad (3.40)$$

where δ'_1 , δ'_2 , and δ'_3 are the nonnegative numbers.

It follows from part (iii) of Lemma 3.3 that

$$\lim_{n \rightarrow +\infty} \int y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx = \int y(x, f(z), f(w))(f(z), f(w)) dx,$$

which together with (3.23) and (3.25) imply

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \left\{ \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right) |\nabla z_n|^2 dx + \int f(z_n)^2 dx + \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right) |\nabla w_n|^2 dx \right. \\ & \quad \left. + \int f(w_n)^2 dx \right\} \\ & = \lim_{n \rightarrow +\infty} \frac{2}{\alpha + \beta} \int y(x, f(z_n), f(w_n))(f(z_n), f(w_n)) dx. \end{aligned} \quad (3.41)$$

Taking into account (3.35)–(3.40), the above limit implies $\delta_1 = \delta_2 = \delta_3 = \delta'_1 = \delta'_2 = \delta'_3 = 0$. Therefore, it follows from (3.35), (3.36), (3.38), and (3.39) that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \varepsilon \int |\nabla z_n|^2 dx &= \varepsilon \int |\nabla z|^2 dx, \\ \lim_{n \rightarrow +\infty} \int G(z_n)^2 dx &= \int G(z)^2 dx, \\ \lim_{n \rightarrow +\infty} \varepsilon \int |\nabla w_n|^2 dx &= \varepsilon \int |\nabla w|^2 dx, \\ \lim_{n \rightarrow +\infty} \int G(w_n)^2 dx &= \int G(w)^2 dx. \end{aligned}$$

By Proposition 2.1, $(z_n, w_n) \rightarrow (z, w)$ in E_G and we have $(\nabla z_n, \nabla w_n) \rightarrow (\nabla z, \nabla w)$ in $L^2 \times L^2$. Hence, $(z_n, w_n) \rightarrow (z, w)$ in H_G^1 . \square

By Lemmas 3.1, 3.2, and 3.4, we obtain Theorem 3.1.

4 Proof of Theorem 1.1

To prove Theorem 1.1, we need to find a critical point for the functional \bar{I}_ε . It follows from [19] that the functionals \bar{I}_ε and \bar{H}_ε will coincide for the small values of ε , every critical point of \bar{H}_ε will be a critical point of \bar{I}_ε .

Without loss of generality, assume ε^2 instead of ε in the functionals \bar{H}_ε and \bar{I}_ε , i.e.,

$$\begin{aligned} \bar{H}_\varepsilon(z, w) &= \frac{\varepsilon^2}{2} \int \frac{1}{\sqrt{-k}} |\nabla z|^2 dx + \frac{1}{2} \int |f(z)|^2 dx + \frac{\varepsilon^2}{2} \int \frac{1}{\sqrt{-k}} |\nabla w|^2 dx \\ & \quad + \frac{1}{2} \int |f(w)|^2 dx - \frac{2}{\alpha + \beta} \int Y(x, f(z), f(w)) dx, \end{aligned}$$

and

$$\begin{aligned} \bar{I}_\varepsilon(z, w) &= \frac{\varepsilon^2}{2} \int \frac{1}{\sqrt{-k}} |\nabla z|^2 dx + \frac{1}{2} \int |z|^2 dx + \frac{\varepsilon^2}{2} \int \frac{1}{\sqrt{-k}} |\nabla w|^2 dx \\ & \quad + \frac{1}{2} \int |w|^2 dx - \frac{2}{\alpha + \beta} \int |f(z)|^\alpha |f(w)|^\beta dx. \end{aligned}$$

It follows from Theorem 3.1 that there exists a critical point $(z_\varepsilon, w_\varepsilon) \in H_G^1$ of \bar{H}_ε for each $\varepsilon > 0$. Set $(u_\varepsilon, v_\varepsilon) = (f(z_\varepsilon), f(w_\varepsilon))$.

The following lemmas are crucial for the proof of Theorem 1.1.

Lemma 4.1. *The sequence $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0}$ is strongly convergent to $(0, 0)$ when $\varepsilon \rightarrow 0$, in $H^1 \times H^1$, i.e.,*

$$\|(u_\varepsilon, v_\varepsilon)\|_{H^1 \times H^1} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. Let $(0, 0) \neq (\phi, \psi) \in C_{0,r}^\infty$ be a nonnegative function with $(\text{supp}(\phi), \text{supp}(\psi)) \subset \Omega$, and $H_1((\phi, \psi)) \leq 0$. Set $\gamma_1(t_1, t_2) = h(t_1\phi, t_2\psi)$. Hence, we have

$$\bar{H}_\varepsilon(\gamma_1(1, 1)) = \bar{H}_\varepsilon(h(\phi, \psi)) = H_\varepsilon(\phi, \psi) \leq H_1(\phi, \psi) \leq 0.$$

It follows from the definition of the Mountain Pass value that

$$\begin{aligned} \bar{H}_\varepsilon(z_\varepsilon, w_\varepsilon) &= \inf_{\gamma \in \tau} \sup_{(t_1, t_2) \in [0,1] \times [0,1]} \bar{H}_\varepsilon(\gamma(t_1, t_2)) \\ &\leq \sup_{(t_1, t_2) \in [0,1] \times [0,1]} \bar{H}_\varepsilon(\gamma_1(t_1, t_2)) \\ &= \sup_{(t_1, t_2) \in [0,1] \times [0,1]} \bar{H}_\varepsilon(h(t_1\phi, t_2\psi)) \\ &= \sup_{(t_1, t_2) \in [0,1] \times [0,1]} H_\varepsilon(t_1\phi, t_2\psi). \end{aligned}$$

And following the maximum property of function, we obtain

$$\begin{aligned} \bar{H}_\varepsilon(z_\varepsilon, w_\varepsilon) &\leq \sup_{(t_1, t_2) \in [0,1] \times [0,1]} H_\varepsilon(t_1\phi, t_2\psi) \\ &= \sup_{(t_1, t_2) \in [0,1] \times [0,1]} \left\{ \frac{\varepsilon^2 t_1^4}{2} \int (-k\phi^2) |\nabla\phi|^2 dx + \frac{\varepsilon^2 t_1^2}{2} \int |\nabla\phi|^2 dx + \frac{t_1^2}{2} \int \phi^2 dx \right. \\ &\quad \left. + \frac{\varepsilon^2 t_2^4}{2} \int (-k\psi^2) |\nabla\psi|^2 dx + \frac{\varepsilon^2 t_2^2}{2} \int |\nabla\psi|^2 dx + \frac{t_2^2}{2} \int \psi^2 dx - \frac{2(t_1^\alpha t_2^\beta)}{\alpha + \beta} \int \phi^\alpha \psi^\beta dx \right\} \\ &\leq \sup_{(t_1, t_2) \in [0,1] \times [0,1]} \left\{ \frac{\varepsilon^2 t_1^2}{2} \int (1 - k\phi^2) |\nabla\phi|^2 dx + \frac{t_1^2}{2} \int \phi^2 dx \right. \\ &\quad \left. + \frac{\varepsilon^2 t_2^2}{2} \int (1 - k\psi^2) |\nabla\psi|^2 dx + \frac{t_2^2}{2} \int \psi^2 dx - \frac{2t_1^\alpha t_2^\beta}{\alpha + \beta} \int \phi^\alpha \psi^\beta dx \right\} \\ &\leq \varepsilon^C A(\phi, \psi), \end{aligned} \tag{4.1}$$

where $A(\phi, \psi)$ is a function about (ϕ, ψ) , $C > 4$. Now, as in the proof of part (i) of Lemma 3.3 we obtain

$$\begin{aligned} \bar{H}_\varepsilon(z_n, w_n) &= \bar{H}_\varepsilon(z_n, w_n) - \frac{1}{\theta} \left\langle \bar{H}_\varepsilon'(z_n, w_n), \begin{pmatrix} f(z_n) & f(w_n) \\ f'(z_n) & f'(w_n) \end{pmatrix} \right\rangle \\ &\geq \varepsilon^2 \left(\frac{1}{2} - \frac{2}{\theta} \right) \int \frac{1}{\sqrt{-k}} |\nabla z_n|^2 dx + \varepsilon^2 \left(\frac{1}{2} - \frac{2}{\theta} \right) \int \frac{1}{\sqrt{-k}} |\nabla w_n|^2 dx \\ &\quad + \left(1 - \frac{1}{l} \right) \left(\frac{1}{\theta} - \frac{1}{2} \right) \int (|f(z_n)|^2 + |f(w_n)|^2) dx. \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.3), we obtain

$$\begin{aligned} &\varepsilon^2 \left(\frac{1}{2} - \frac{2}{\theta} \right) \int \frac{1}{\sqrt{-k}} |\nabla z_n|^2 dx + \varepsilon^2 \left(\frac{1}{2} - \frac{2}{\theta} \right) \int \frac{1}{\sqrt{-k}} |\nabla w_n|^2 dx \\ &\quad + \left(1 - \frac{1}{l} \right) \left(\frac{1}{\theta} - \frac{1}{2} \right) \int (|f(z_n)|^2 + |f(w_n)|^2) dx \leq \varepsilon^C A(\phi, \psi). \end{aligned}$$

Therefore,

$$\begin{aligned} & \left(\frac{1}{2} - \frac{2}{\theta}\right) \int \frac{1}{\sqrt{-k}} |\nabla z_n|^2 dx + \varepsilon^2 \left(\frac{1}{2} - \frac{2}{\theta}\right) \int \frac{1}{\sqrt{-k}} |\nabla w_n|^2 dx \\ & + \left(1 - \frac{1}{l}\right) \left[\frac{1}{\theta} - \frac{1}{2}\right] \int (|f(z_n)|^2 + |f(w_n)|^2) dx \leq \varepsilon^{C-2} A(\phi, \psi). \end{aligned} \quad (4.3)$$

Hence, substituting $(u_\varepsilon, v_\varepsilon) = (f(z_\varepsilon), f(w_\varepsilon))$ in (4.3), then

$$\int (1 + |u_\varepsilon|^2) |\nabla u_\varepsilon|^2 dx + \int (1 + |v_\varepsilon|^2) |\nabla v_\varepsilon|^2 dx + \int (|u_\varepsilon|^2 + |v_\varepsilon|^2) dx \leq C \varepsilon^{C-2} A(\phi, \psi). \quad (4.4)$$

Therefore,

$$\|(u_\varepsilon, v_\varepsilon)\|_{H^1 \times H^1} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0. \quad \square$$

Lemma 4.2. *Let $N > 2$. There is a constant $C = C_N$, such that*

$$\begin{aligned} 0 \leq u_\varepsilon(x) &\leq \frac{C}{|x|^{\frac{N-2}{N}}} \|u_\varepsilon(x)\|_{H^1}, \quad \forall x \neq 0, \\ 0 \leq v_\varepsilon(x) &\leq \frac{C}{|x|^{\frac{N-2}{N}}} \|v_\varepsilon(x)\|_{H^1}, \quad \forall x \neq 0, \end{aligned}$$

for any $(u, v) \in H_r^1 \times H_r^1$.

This lemma is liking [25].

Lemma 4.3. *For every compact set $Q \subset \mathbb{R}^N$ such that Q is nonempty, $\|(u_\varepsilon, v_\varepsilon)\|_{L^\infty(Q) \times L^\infty(Q)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.*

Proof. For each $\varepsilon > 0$, it follows from Lemma 4.2 that

$$\begin{aligned} 0 \leq u_\varepsilon(x) &\leq \frac{C}{|x|^{\frac{N-2}{N}}} \|u_\varepsilon(x)\|_{H^1}, \quad \forall x \neq 0, \\ 0 \leq v_\varepsilon(x) &\leq \frac{C}{|x|^{\frac{N-2}{N}}} \|v_\varepsilon(x)\|_{H^1}, \quad \forall x \neq 0, \end{aligned}$$

which together with the result of Lemma 4.1 obviously means

$$\|(u_\varepsilon, v_\varepsilon)\|_{L^\infty(Q) \times L^\infty(Q)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0. \quad \square$$

Next, we prove that Theorem 1.1.

Proof. By Lemma 4.3, we have

$$\max_{x \in \Lambda} (f(z_\varepsilon), f(w_\varepsilon)) \rightarrow (0, 0), \quad \text{as } \varepsilon \rightarrow 0. \quad (4.5)$$

From (4.5), $\forall N_1, N_2 > 0$, there exists $\varepsilon_0 > 0$, such that $\max_{x \in \Lambda} (f(z_\varepsilon), f(w_\varepsilon)) < (N_1, N_2)$ for every $0 < \varepsilon < \varepsilon_0$.

Using the test function $(\phi, \psi) = \left(\frac{(f(z_\varepsilon) - N_1)_+}{f'(z_\varepsilon)}, \frac{(f(w_\varepsilon) - N_2)_+}{f'(w_\varepsilon)} \right)$, we obtain

$$\begin{aligned} 0 &= \left\langle \bar{H}_\varepsilon'(z_n, w_n), \left(\frac{(f(z_\varepsilon) - N)_+}{f'(z_\varepsilon)}, \frac{(f(w_\varepsilon) - N)_+}{f'(w_\varepsilon)} \right) \right\rangle \\ &= \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right) |\nabla z_n|^2 dx + \int f(z_n) (f(z_\varepsilon) - N)_+ dx \\ &\quad + \varepsilon \int \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right) |\nabla w_n|^2 dx + \int f(w_n) (f(w_\varepsilon) - N)_+ dx \\ &\quad - \frac{2}{\alpha + \beta} \int y(x, f(z_n), f(w_n)) ((f(z_\varepsilon) - N)_+, (f(w_\varepsilon) - N)_+) dx \end{aligned}$$

$$\begin{aligned}
 &= \varepsilon \int_F \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right) |\nabla z_n|^2 dx + \int_{\mathbb{R}^N \setminus \bar{\Lambda}} f(z_n)(f(z_\varepsilon) - N)_+ dx \\
 &\quad + \varepsilon \int_F \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right) |\nabla w_n|^2 dx + \int_{\mathbb{R}^N \setminus \bar{\Lambda}} f(w_n)(f(w_\varepsilon) - N)_+ dx \\
 &\quad - \frac{2}{\alpha + \beta} \int_{\mathbb{R}^N \setminus \bar{\Lambda}} y(x, f(z_n), f(w_n))(f(z_\varepsilon) - N)_+, (f(w_\varepsilon) - N)_+ dx,
 \end{aligned}$$

where $F = (\mathbb{R}^N \setminus \bar{\Lambda}) \cup \{x : (f(z_\varepsilon), f(w_\varepsilon)) \geq (N_1, N_2)\}$. From (y_2) , we have

$$(f(z_n), f(w_n))(f(z_\varepsilon) - N)_+, (f(w_\varepsilon) - N)_+ - y(x, f(z_n), f(w_n))(f(z_\varepsilon) - N)_+, (f(w_\varepsilon) - N)_+ \geq 0, \quad \forall x \in \Lambda^c.$$

Thus,

$$\varepsilon \int_F \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(z_n)}{1 - kf^2(z_n)} \right) |\nabla z_n|^2 dx + \varepsilon \int_F \frac{1}{\sqrt{-k}} \left(1 + \frac{-kf^2(w_n)}{1 - kf^2(w_n)} \right) |\nabla w_n|^2 dx = 0,$$

from which we obtain

$$(f(z_\varepsilon), f(w_\varepsilon)) \leq (N_1, N_2), \quad \forall x \in \mathbb{R}^N \setminus \bar{\Lambda}.$$

Let $(N_1, N_2) = (a_1, a_2)$, therefore

$$y(x, f(z_n), f(w_n)) = (\alpha f^{\alpha-1}(z_n) f^\beta(w_n), \beta f^\alpha(z_n) f^{\beta-1}(w_n)), \quad \forall x \in \mathbb{R}^N \setminus \bar{\Lambda},$$

and it follows that $\langle \bar{H}_\varepsilon'(z_n, w_n), (\phi, \psi) \rangle = 0$, we have

$$\begin{aligned}
 &\varepsilon^2 \int \frac{1}{\sqrt{-k}} \nabla z_\varepsilon \nabla \phi dx + \int f(z_\varepsilon) f'(z_\varepsilon) \phi dx + \varepsilon^2 \int \frac{1}{\sqrt{-k}} \nabla w_\varepsilon \nabla \psi dx + \int f(w_\varepsilon) f'(w_\varepsilon) \psi dx \\
 &= \frac{2}{\alpha + \beta} \int (\alpha |f(z_n)|^{\alpha-2} f(z_n) |f(w_n)|^\beta \phi, \beta |f(z_n)|^\alpha |f(w_n)|^{\beta-2} f(w_n) \psi)(f'(z_n), f'(w_n)) dx,
 \end{aligned}$$

for every $(\phi, \psi) \in H_G^1$ and $0 < \varepsilon < \varepsilon_0$. Therefore, $\bar{I}_\varepsilon(z, w)$ has a critical point $(z_\varepsilon, w_\varepsilon)$ in H_G^1 for every $\varepsilon \in (0, \varepsilon_0)$. □

Funding information: Jing Zhang was supported by the Natural Science Foundation of Inner Mongolia Autonomous Region (No. 2022MS01001, 2023LHMS01005), Key Laboratory of Infinite-dimensional Hamiltonian System and Its Algorithm Application (Inner Mongolia Normal University), Ministry of Education (No. 2023KFZD01), Research Program of science and technology at Universities of Inner Mongolia Autonomous Region (No. NJYT23100), Mathematics First-class Disciplines Cultivation Fund of Inner Mongolia Normal University (No. 2024YLKY14), and the Fundamental Research Funds for the Inner Mongolia Normal University (No. 2022JBQN072). Xue Zhang was supported by the Fundamental Research Funds for the Inner Mongolia Normal University (2022JBXC03) and Graduate students' research innovation fund of Inner Mongolia Normal University (CXJJS22100).

Author contributions: The authors declare that they have equal contributions.

Conflict of interest: The authors state no conflicts of interest.

References

- [1] G. Li, *On the existence of nontrivial solutions for quasilinear Schrödinger systems*, Bound. Value Probl. **2022** (2022), no. 1, 40, DOI: <https://doi.org/10.1186/s13661-022-01623-z>.
- [2] Y. Guo and Z. Tang, *Ground state solutions for quasilinear Schrödinger systems*, J. Math. Anal. Appl. **389** (2012), no. 1, 322–339, DOI: <http://doi.org/10.1016/j.jmaa.2011.11.064>.

- [3] U. Severo and E. Silva, *On the existence of standing wave solutions for a class of quasilinear Schrödinger systems*, J. Math. Anal. Appl. **412** (2014), no. 2, 763–775, DOI: <http://doi.org/10.1016/j.jmaa.2013.11.012>.
- [4] J. Chen and Q. Zhang, *Existence of positive ground state solutions for quasilinear Schrödinger system with positive parameter*, Appl. Anal. **102** (2023), no. 10, 2676–2691, DOI: <https://doi.org/10.1080/00036811.2022.2033232>.
- [5] J. Chen and Q. Zhang, *Positive solutions for quasilinear Schrödinger system with positive parameter*, Z. Angew. Math. Phys. **73** (2022), no. 4, 144, DOI: <https://doi.org/10.1007/s00033-022-01781-1>.
- [6] J. Chen and Q. Zhang, *Ground state solution of Nehari-Pohožaev type for periodic quasilinear Schrödinger system*, J. Math. Phys. **61** (2020), no. 10, 101510, DOI: <https://doi.org/10.1063/5.0014321>.
- [7] J. Chen and Q. Zhang, *Existence of ground state solution of Nehari-Pohožaev type for a quasilinear Schrödinger system*, Differential Integral Equations **34(1/2)** (2023), 1–20, DOI: <https://doi.org/10.57262/die/1610420451>.
- [8] Y. Wang and X. Huang, *Ground states of Nehari-Pohožaev type for a quasilinear Schrödinger system with superlinear reaction*, Electron. Res. Arch. **31** (2023), no. 4, 2071–2094, DOI: <https://doi.org/10.3934/era.2023106>.
- [9] C. Chen and H. Yang, *Multiple solutions for a class of quasilinear Schrödinger systems in \mathbb{R}^N* , Bull. Malays. Math. Sci. Soc. **42** (2019), no. 2, 611–636, DOI: <https://doi.org/10.1007/s40840-017-0502-z>.
- [10] X. Zhang, L. Liu, Y. Wu, and Y. Cui, *The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach*, J. Math. Anal. Appl. **464** (2018), no. 2, 1089–1106, DOI: <https://doi.org/10.1016/j.jmaa.2018.04.040>.
- [11] H. Lange, B. Toomire, and P. F. Zweifel, *Time-dependent dissipation in nonlinear Schrödinger systems*, J. Math. Phys. **36** (1995), 1274–1283, DOI: <https://doi.org/10.1063/1.531120>.
- [12] E. W. Laedke, K. H. Spatschek, and L. Stenflo, *Evolution theorem for a class of perturbed envelope soliton solutions*, J. Math. Phys. **24** (1983), no. 12, 2764–2769, DOI: <https://doi.org/10.1063/1.525675>.
- [13] B. Ritchie, *Relativistic self-focusing and channel formation in laser-plasma interactions*, Phys. Rev. E **50** (1994), no. 2, 687–689, DOI: <https://doi.org/10.1103/PhysRevE.50.R687>.
- [14] J. Chen, X. Huang, B. Cheng, and C. Zhu, *Some results on standing wave solutions for a class of quasilinear Schrödinger equations*, J. Math. Phys. **60** (2019), no. 9, 091506, DOI: <https://doi.org/10.1063/1.5093720>.
- [15] J. Q. Liu, Y. Wang, and Z. Q. Wang, *Soliton solutions for quasilinear Schrödinger equations, I*, Proc. Amer. Math. Soc. **131** (2003), no. 2, 441–448.
- [16] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **3** (1986), no. 2, 77–109, DOI: [https://doi.org/10.1016/S0294-1449\(16\)30389-4](https://doi.org/10.1016/S0294-1449(16)30389-4).
- [17] C. O. Alves and D. C. de Morais Filho, *Existence and concentration of positive solutions for a Schrödinger logarithmic equation*, Z. Angew. Math. Phys. **69** (2018), no. 6, 144, DOI: <https://doi.org/10.1007/s00033-018-1038-2>.
- [18] U. Severo, *Symmetric and nonsymmetric solutions for a class of quasilinear Schrödinger equations*, Adv. Nonlinear Stud. **8** (2008), no. 2, 375–389, DOI: <https://doi.org/10.1515/ans-2008-0208>.
- [19] A. Moameni, *Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in \mathbb{R}^N* , J. Differential Equations **229** (2006), no. 2, 570–587, DOI: <https://doi.org/10.1016/j.jde.2006.07.001>.
- [20] J. Q. Liu, Y. Wang, and Z. Q. Wang, *Soliton solutions for quasilinear Schrödinger equations, II*, J. Differential Equations **187** (2003), no. 2, 473–493, DOI: [https://doi.org/10.1016/S0022-0396\(02\)00064-5](https://doi.org/10.1016/S0022-0396(02)00064-5).
- [21] M. M. Rao and Z. D. Ren, *Theory of Orlicz Space*, Marcel Dekker, New York, 1991.
- [22] M. Del Pino and P. L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations **4** (1996), no. 2, 121–137, DOI: <https://doi.org/10.1007/s005260050031>.
- [23] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), no. 4, 349–381, DOI: [https://doi.org/10.1016/0022-1236\(73\)90051-7](https://doi.org/10.1016/0022-1236(73)90051-7).
- [24] P. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, RI, 1984.
- [25] W. A. Strauss, *Existence of solitary waves in higher dimensions*, Commun. Math. Phys. **55** (1977), no. 2, 149–162, DOI: <https://doi.org/10.1007/BF01626517>.