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Research Article

Mohamed Jleli and Bessem Samet*

On discrete inequalities for some classes of sequences

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Abstract: For a given sequence $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, our aim is to obtain an estimate of $E_n \coloneqq \left|\frac{a_1 + a_n}{2} - \frac{1}{n}\sum_{i=1}^{n} a_i\right|$. Several classes of sequences are studied. For each class, an estimate of E_n is obtained. We also introduce the class of convex matrices, which is a discrete version of the class of convex functions on the coordinates. For this set of matrices, new discrete Hermite-Hadamard-type inequalities are proved. Our obtained results are extensions of known results from the continuous case to the discrete case.

Keywords: discrete inequalities, convex sequences, convex matrices, Fejér inequality, Hermite-Hadamard inequality

MSC 2020: 26D15, 39A12, 40B05

1 Introduction

A great attention has been paid on the extensions of known inequalities from the continuous case to the discrete case, e.g., [1–11] and the references therein. In particular, the study of inequalities involving convex sequences has been considered by many authors, e.g., [12–16] and the references therein. The notion of convex sequences is a discrete version of the concept of convex functions. Namely, a sequence $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, where $n \geq 3$, is said to be convex, if (e.g., [17])

$$a_i \leq \frac{a_{i-1} + a_{i+1}}{2}, \quad i = 2, \ldots, n-1.$$

Latreuch and Belaïdi [12] established a discrete version of the Fejér double inequality for the class of convex sequences. We recall that the Fejér double inequality [18] states that, if $f : [a, b] \to \mathbb{R}$ is convex and $p : [a, b] \to \mathbb{R}$ is integrable, nonnegative, and symmetric with respect to the midpoint $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\mathrm{d}x \leq \int_a^b f(x)p(x)\mathrm{d}x \leq \frac{f(a) + f(b)}{2}\int_a^b p(x)\mathrm{d}x. \tag{1.1}$$

In particular, if $p \equiv 1$, (1.1) reduces the Hermite-Hadamard double inequality [19,20]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a) + f(b)}{2}. \tag{1.2}$$

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* **Corresponding author: Bessem Samet**, Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia, e-mail: bsamet@ksu.edu.sa

Mohamed Jleli: Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia, e-mail: jleli@ksu.edu.sa

For some results related to inequalities (1.1) and (1.2), see e.g., [21–37] and the references therein. Latreuch and Belaïdi [12] proved that, if $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is a convex sequence and $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ is a positive sequence, which is symmetric with respect to $\frac{n+1}{2}$, then

$$\frac{a_N + a_{n+1-N}}{2}\sum_{i=1}^{n} p_i \le \sum_{i=1}^{n} p_i a_i \le \frac{a_1 + a_n}{2}\sum_{i=1}^{n} p_i, \tag{1.3}$$

where $N = \left[\frac{n+1}{2}\right]$ is the integer part of $\frac{n+1}{2}$. The double inequality (1.3) is a discrete version of the Fejér double inequality (1.1). If $p_i = 1$ for all i, then (1.3) reduces to the double inequality

$$\frac{a_N + a_{n+1-N}}{2} \le \frac{1}{n}\sum_{i=1}^{n} a_i \le \frac{a_1 + a_n}{2}, \tag{1.4}$$

which is a discrete version of the Hermite-Hadamard double inequality (1.2). We also refer to [14], where some generalizations of the above results have been obtained.

Dragomir and Agarwal [26] established two interesting inequalities for the class of differentiable mappings. They first proved that, if $f\colon [a, b] \to \mathbb{R}$ is differentiable and $|f'|$ is convex on $[a, b]$, then

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{1.5}$$

Next they proved that, if $f\colon [a, b] \to \mathbb{R}$ is differentiable and $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ for some $p > 1$, then

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}. \tag{1.6}$$

Dragomir et al. [27] considered the class of L-Lipschitzian functions $f\colon [a, b] \to \mathbb{R}$, namely, the class of functions f satisfying

$$|f(x) - f(y)| \le L|x - y|, \quad x, y \in [a, b],$$

where $L > 0$ is a constant. They proved (among many other results) that

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{L}{3}(b-a). \tag{1.7}$$

Dragomir [25] considered the class of convex functions on the coordinates. Namely, the class of functions $f\colon [a, b] \times [c, d] \to \mathbb{R}$ satisfying

(i) for all $x \in [a, b]$, the function $f(x, \cdot) : [c, d] \ni y \mapsto f(x, y) \in \mathbb{R}$ is convex;
(ii) for all $y \in [c, d]$, the function $f(\cdot, y) : [a, b] \ni x \mapsto f(x, y) \in \mathbb{R}$ is convex.

Note that, if f is convex on $[a, b] \times [c, d]$, then f is convex on the coordinates. However, the converse is not true in general [25, Lemma 1]. Dragomir [25] proved that, if $f\colon [a, b] \times [c, d] \to \mathbb{R}$ is convex on the coordinates, then

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)\mathrm{d}x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)\mathrm{d}y\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x \\ &\le \frac{1}{4(b-a)}\int_a^b (f(x, c) + f(x, d))\,\mathrm{d}x + \frac{1}{4(d-c)}\int_c^d (f(a, y) + f(b, y))\,\mathrm{d}y \\ &\le \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \tag{1.8}$$

The aim of this study is to establish discrete versions of inequalities (1.5), (1.6), (1.7), and (1.8).

2 Main results and proofs

We first fix some notations. For $n \geq 4$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, we denote by $a' = (a'_1, \dots, a'_{n-1}) \in \mathbb{R}^{n-1}$ the sequence defined by

$$a'_i = |a_{i+1} - a_i|, \quad i = 1, \dots, n-1.$$

For a real number $q > 1$, we denote by $a'^q \in \mathbb{R}^{n-1}$ the sequence defined by

$$a'^q = (a'^q_1, \dots, a'^q_{n-1}).$$

2.1 Discrete version of inequality (1.5)

Let us consider the set of sequences

$$A_n = \{a \in \mathbb{R}^n : a' \text{ is convex}\},$$

i.e., $a \in A_n$, if and only if

$$a'_i \leq \frac{a'_{i+1} + a'_{i-1}}{2}, \quad i = 2, \dots, n-2.$$

Our first main result is a discrete version of inequality (1.5), which was established in [26].

Theorem 2.1. *Let $n \geq 4$. If $a = (a_1, \dots, a_n) \in A_n$, then*

$$\left| \frac{a_1 + a_n}{2} - \frac{1}{n} \sum_{i=1}^{n} a_i \right| \leq \frac{1}{n} \left[\frac{n}{2}\right] \left(n - 1 - \left[\frac{n}{2}\right]\right) \frac{a'_1 + a'_{n-1}}{2}, \tag{2.1}$$

where $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$.

Proof. We first claim that

$$\frac{a_1 + a_n}{2} - \frac{1}{n} \sum_{i=1}^{n} a_i = \frac{1}{n} \sum_{i=1}^{n-1} \left(i - \frac{n}{2}\right)(a_{i+1} - a_i). \tag{2.2}$$

Indeed, we have

$$\begin{aligned}
&\sum_{i=1}^{n-1} \left(i - \frac{n}{2}\right)(a_{i+1} - a_i) + \sum_{i=1}^{n} a_i \\
&= \sum_{i=1}^{n-1} \left(i a_{i+1} - i a_i - \frac{n}{2} a_{i+1} + \frac{n}{2} a_i\right) + \sum_{i=1}^{n} a_i \\
&= \left(\sum_{i=1}^{n-1} i a_{i+1} - \sum_{i=1}^{n-1} i a_i\right) + \left(-\frac{n}{2} \sum_{i=1}^{n-1} a_{i+1} + \frac{n}{2} \sum_{i=1}^{n-1} a_i + \sum_{i=1}^{n} a_i\right) \\
&= \left(\sum_{i=1}^{n-1} (i+1) a_{i+1} - \sum_{i=1}^{n-1} i a_i\right) + \left(-\frac{n}{2} \sum_{i=1}^{n-1} a_{i+1} + \frac{n}{2} \sum_{i=1}^{n-1} a_i + \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_{i+1}\right) \\
&= \left(\sum_{i=2}^{n} i a_i - \sum_{i=1}^{n-1} i a_i\right) + \left(-\frac{n}{2} \sum_{i=2}^{n} a_i + \frac{n}{2} \sum_{i=1}^{n-1} a_i + \sum_{i=1}^{n} a_i - \sum_{i=2}^{n} a_i\right) \\
&= n a_n - a_1 + \left(-\frac{n}{2} a_n + \frac{n}{2} a_1 + a_1\right) \\
&= n \frac{a_1 + a_n}{2}.
\end{aligned}$$

Then, multiplying the above identity by $\frac{1}{n}$, we obtain (2.2).

We next use (2.2) to obtain

$$\left|\frac{a_1+a_n}{2}-\frac{1}{n}\sum_{i=1}^{n}a_i\right| \le \frac{1}{n}\sum_{i=1}^{n-1}\left|i-\frac{n}{2}\right||a_{i+1}-a_i|,$$

i.e.,

$$\left|\frac{a_1+a_n}{2}-\frac{1}{n}\sum_{i=1}^{n}a_i\right| \le \frac{1}{n}\sum_{i=1}^{n-1}p_i a'_i, \tag{2.3}$$

where $p=(p_1,\dots,p_{n-1})\in\mathbb{R}^{n-1}$ is the sequence defined by

$$p_i=\left|i-\frac{n}{2}\right|,\quad i=1,\dots,n-1.$$

Observe that p is symmetric with respect to $\frac{n-1}{2}$. Moreover, the sequence a' is convex. Hence, by the right discrete Fejér inequality (1.3), we have

$$\sum_{i=1}^{n-1}p_i a'_i \le \left(\sum_{i=1}^{n-1}p_i\right)\frac{a'_1+a'_{n-1}}{2},$$

i.e.,

$$\sum_{i=1}^{n-1}p_i a'_i \le \left(\sum_{i=1}^{n-1}\left|\,i-\frac{n}{2}\,\right|\right)\frac{a'_1+a'_{n-1}}{2}. \tag{2.4}$$

On the other hand, we have

$$\begin{aligned}
\sum_{i=1}^{n-1}\left|i-\frac{n}{2}\right| &= \sum_{i=1}^{\left[\frac{n}{2}\right]}\left(\frac{n}{2}-i\right)+\sum_{i=\left[\frac{n}{2}\right]+1}^{n-1}\left(i-\frac{n}{2}\right)\\
&=\frac{n}{2}\left[\frac{n}{2}\right]-\sum_{i=1}^{\left[\frac{n}{2}\right]}i+\sum_{i=\left[\frac{n}{2}\right]+1}^{n-1}i-\frac{n}{2}\left(n-1-\left[\frac{n}{2}\right]\right)\\
&=\frac{n}{2}\left[\frac{n}{2}\right]-\frac{1}{2}\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)+\frac{1}{2}\left(n-1-\left[\frac{n}{2}\right]\right)\left(n+\left[\frac{n}{2}\right]\right)-\frac{n}{2}\left(n-1-\left[\frac{n}{2}\right]\right)\\
&=\frac{n}{2}\left[\frac{n}{2}\right]-\frac{1}{2}\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)+\frac{1}{2}\left(n-1-\left[\frac{n}{2}\right]\right)\left[\frac{n}{2}\right]\\
&=\frac{1}{2}\left[\frac{n}{2}\right]\left(n-1-\left[\frac{n}{2}\right]\right)+\frac{1}{2}\left(n-1-\left[\frac{n}{2}\right]\right)\left[\frac{n}{2}\right]\\
&=\frac{1}{2}\left(n-1-\left[\frac{n}{2}\right]\right)\left(\left[\frac{n}{2}\right]+\left[\frac{n}{2}\right]\right)\\
&=\left[\frac{n}{2}\right]\left(n-1-\left[\frac{n}{2}\right]\right).
\end{aligned}$$

Finally, (2.1) follows from (2.3), (2.4), and the above identity. □

If n is even, then $\left[\frac{n}{2}\right]=\frac{n}{2}$. Hence, from Theorem 2.1, we deduce the following result.

Corollary 2.2. *Let $n\ge 4$ be even. If $a=(a_1,\dots,a_n)\in A_n$, then*

$$\left|\frac{a_1+a_n}{2}-\frac{1}{n}\sum_{i=1}^{n}a_i\right| \le \frac{n-2}{8}(a_1'+a_{n-1}').$$

If n is odd, then $\left[\frac{n}{2}\right] = \frac{n-1}{2}$. Hence, from Theorem 2.1, we deduce the following result.

Corollary 2.3. *Let $n \geq 4$ be odd. If $a = (a_1, \dots, a_n) \in A_n$, then*

$$\left| \frac{a_1 + a_n}{2} - \frac{1}{n} \sum_{i=1}^{n} a_i \right| \leq \frac{(n-1)^2}{8n} (a_1' + a_{n-1}').$$

2.2 Discrete version of inequality (1.6)

For a real number $q > 1$ and $n \geq 4$, let us consider the set of sequences

$$A_{n,q} = \{a \in \mathbb{R}^n : a'^q \text{ is convex}\},$$

i.e., $a \in A_{n,q}$ if and only if

$$a_i'^q \leq \frac{a_{i+1}'^q + a_{i-1}'^q}{2}, \quad i = 2, \dots, n-2.$$

Our second main result is a discrete version of inequality (1.6), which was established in [26].

Theorem 2.4. *Let $n \geq 4$, $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $a = (a_1, \dots, a_n) \in A_{n,q}$, then*

$$\left| \frac{a_1 + a_n}{2} - \frac{1}{n} \sum_{i=1}^{n} a_i \right| \leq \frac{1}{n} \left(\frac{n-1}{2} \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n-1} \left| i - \frac{n}{2} \right|^p \right)^{\frac{1}{p}} (a_1'^q + a_{n-1}'^q)^{\frac{1}{q}}. \tag{2.5}$$

Proof. Making use of (2.3) and Hölder's inequality, we obtain

$$\left| \frac{a_1 + a_n}{2} - \frac{1}{n} \sum_{i=1}^{n} a_i \right| \leq \frac{1}{n} \sum_{i=1}^{n-1} \left| i - \frac{n}{2} \right| a'_i \leq \frac{1}{n} \left(\sum_{i=1}^{n-1} \left| i - \frac{n}{2} \right|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} a'^q_i \right)^{\frac{1}{q}}. \tag{2.6}$$

On the other hand, since $a \in A_{n,q}$, making use of the right discrete Hermite-Hadamard inequality (1.4), we obtain

$$\sum_{i=1}^{n-1} a'^q_i \leq (n-1) \frac{a'^q_1 + a'^q_{n-1}}{2}. \tag{2.7}$$

Then, combining (2.6) with (2.7), we obtain (2.5). □

Remark that

$$\sum_{i=1}^{n-1} \left| i - \frac{n}{2} \right|^p = \sum_{i=1}^{\left[\frac{n}{2}\right]} \left(\frac{n}{2} - i \right)^p + \sum_{i=\left[\frac{n}{2}\right]+1}^{n-1} \left(i - \frac{n}{2} \right)^p = \sum_{i=1}^{\left[\frac{n}{2}\right]} \left(\frac{n}{2} - i \right)^p + \sum_{j=1}^{n-\left[\frac{n}{2}\right]-1} \left(\frac{n}{2} - j \right)^p.$$

Hence, we obtain

$$\sum_{i=1}^{n-1} \left| i - \frac{n}{2} \right|^p = \begin{cases} 2 \displaystyle\sum_{i=1}^{\frac{n}{2}} \left(\frac{n}{2} - i \right)^p & \text{if } n \text{ is even}, \\ 2 \displaystyle\sum_{i=1}^{\frac{n-1}{2}} \left(\frac{n}{2} - i \right)^p & \text{if } n \text{ is odd}. \end{cases}$$

Thus, from Theorem 2.4, we deduce the following results.

Corollary 2.5. *Let $n \geq 4$ be even, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $a = (a_1, \dots, a_n) \in A_{n,q}$, then*

$$\left|\frac{a_1 + a_n}{2} - \frac{1}{n}\sum_{i=1}^{n} a_i\right| \leq \frac{1}{n}\left(\frac{n-1}{2}\right)^{\frac{1}{q}} 2^{\frac{1}{p}} \left(\sum_{i=1}^{\frac{n}{2}} \left(\frac{n}{2} - i\right)^p\right)^{\frac{1}{p}} (a_1'^q + a_{n-1}'^q)^{\frac{1}{q}}.$$

Corollary 2.6. *Let $n \geq 4$ be odd, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $a = (a_1, \dots, a_n) \in A_{n,q}$, then*

$$\left|\frac{a_1 + a_n}{2} - \frac{1}{n}\sum_{i=1}^{n} a_i\right| \leq \frac{1}{n}\left(\frac{n-1}{2}\right)^{\frac{1}{q}} 2^{\frac{1}{p}} \left(\sum_{i=1}^{\frac{n-1}{2}} \left(\frac{n}{2} - i\right)^p\right)^{\frac{1}{p}} (a_1'^q + a_{n-1}'^q)^{\frac{1}{q}}.$$

2.3 Discrete version of inequality (1.7)

The following result provides a discrete version of inequality (1.7), which was established in [27].

Theorem 2.7. *Let $n \geq 2$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. We have*

$$\left|\frac{a_1 + a_n}{2} - \frac{1}{n}\sum_{i=1}^{n} a_i\right| \leq \frac{(n-2)(n-1)}{n} L, \tag{2.8}$$

where $L = \max\left\{\frac{|a_i - a_j|}{|i-j|} : i, j \in \{1, \dots, n\},\ i \neq j\right\}$.

Proof. For $n = 2$, (2.8) is obvious. So, we may suppose that $n \geq 3$. For all natural number $k \geq 2$, we can write a_k as

$$a_k = (a_k - a_{k-1}) + (a_{k-1} - a_{k-2}) + \dots + (a_2 - a_1) + a_1,$$

i.e.,

$$a_k = a_1 + \sum_{j=2}^{k} (a_j - a_{j-1}), \quad k = 2, 3, \dots \tag{2.9}$$

Making use of (2.9), we obtain

$$\begin{aligned}
\sum_{i=1}^{n} a_i &= a_1 + \sum_{k=2}^{n} a_k \\
&= a_1 + \sum_{k=2}^{n} \left(a_1 + \sum_{j=2}^{k} (a_j - a_{j-1})\right) \\
&= a_1 + (n-1)a_1 + \sum_{k=2}^{n} \sum_{j=2}^{k} (a_j - a_{j-1}) \\
&= na_1 + \sum_{k=2}^{n-1} \sum_{j=2}^{k} (a_j - a_{j-1}) + \sum_{j=2}^{n} (a_j - a_{j-1}) \\
&= na_1 + \sum_{k=2}^{n-1} \sum_{j=2}^{k} (a_j - a_{j-1}) + (n-1)(a_n - a_1),
\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}a_i-\frac{a_1+a_n}{2}&=a_1+\frac{1}{n}\sum_{k=2}^{n-1}\sum_{j=2}^{k}(a_j-a_{j-1})+\frac{n-1}{n}(a_n-a_1)-\frac{a_1+a_n}{2}\\&=a_1\left(1-\frac{n-1}{n}-\frac{1}{2}\right)+a_n\left(\frac{n-1}{n}-\frac{1}{2}\right)+\frac{1}{n}\sum_{k=2}^{n-1}\sum_{j=2}^{k}(a_j-a_{j-1})\\&=\frac{n-2}{2n}(a_n-a_1)+\frac{1}{n}\sum_{k=2}^{n-1}\sum_{j=2}^{k}(a_j-a_{j-1}).\end{aligned}$$

Hence, it holds that

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n}a_i-\frac{a_1+a_n}{2}\right|&\le\frac{n-2}{2n}|a_n-a_1|+\frac{1}{n}\sum_{k=2}^{n-1}\sum_{j=2}^{k}|a_j-a_{j-1}|\\&\le\frac{(n-2)(n-1)}{2n}L+\frac{L}{n}\sum_{k=1}^{n-1}(k-1)\\&=\frac{(n-2)(n-1)}{n}L,\end{aligned}$$

which proves (2.8). □

We also have the following result.

Theorem 2.8. *Let* $n\ge 1$ *and* $a=(a_1,\dots,a_n)\in\mathbb{R}^n$. *We have*

$$\left|\frac{a_1+a_n}{2}-\frac{1}{n}\sum_{i=1}^{n}a_i\right|\le\frac{n-2}{2}M,\tag{2.10}$$

where $M=\max\{|a_i-a_j|:i,j\in\{1,\dots,n\}\}$.

Proof. For $n=1$ or $n=2$, (2.10) is obvious. For $n\ge 3$, from the proof of Theorem 2.7, we have

$$\frac{1}{n}\sum_{i=1}^{n}a_i-\frac{a_1+a_n}{2}=\frac{n-2}{2n}(a_n-a_1)+\frac{1}{n}\sum_{k=2}^{n-1}\sum_{j=2}^{k}(a_j-a_{j-1}),$$

which implies that

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n}a_i-\frac{a_1+a_n}{2}\right|&\le\frac{n-2}{2n}|a_n-a_1|+\frac{1}{n}\sum_{k=2}^{n-1}\sum_{j=2}^{k}|a_j-a_{j-1}|\\&\le\frac{n-2}{2n}M+\frac{M}{n}\sum_{k=1}^{n-1}(k-1)\\&=\frac{n-2}{2n}M+\frac{M}{n}\sum_{i=1}^{n-2}j\\&=\frac{n-2}{2n}M+\frac{(n-2)(n-1)}{2n}M\\&=\frac{n-2}{2}M,\end{aligned}$$

which proves (2.10). □

The following examples show that the upper bounds of $\left|\frac{a_1+a_n}{2}-\frac{1}{n}\sum_{i=1}^{n}a_i\right|$ provided by Theorems 2.7 and 2.8 are not comparable in general.

Example 2.9. Let $n \geq 2$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, where

$$a_i = \frac{1}{i}, \quad i = 1, \dots, n.$$

In this case, we have

$$M = \max\left\{\left|\frac{1}{i} - \frac{1}{j}\right| : i, j \in \{1, \dots, n\}\right\} = 1 - \frac{1}{n},$$

$$L = \max\left\{\frac{\left|\frac{1}{i} - \frac{1}{j}\right|}{|i-j|} : i, j \in \{1, \dots, n\}, i \neq j\right\} = \max\left\{\frac{1}{ij} : i, j \in \{1, \dots, n\}, i \neq j\right\} = \frac{1}{2},$$

$$\frac{n-2}{2}M = \frac{(n-2)(n-1)}{n}L = \frac{(n-2)(n-1)}{2n}.$$

Hence, inequalities (2.8) and (2.10) are the same.

Example 2.10. Let $n \geq 2$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, where

$$a_i = i, \quad i = 1, \dots, n.$$

In this case, we have

$$M = \max\{|i-j| : i, j \in \{1, \dots, n\}\} = n - 1,$$

$$L = \max\left\{\frac{|i-j|}{|i-j|} : i, j \in \{1, \dots, n\}, i \neq j\right\} = 1,$$

$$\frac{n-2}{2}M = \frac{(n-2)(n-1)}{2},$$

$$\frac{(n-2)(n-1)}{n}L = \frac{(n-2)(n-1)}{n}.$$

Observe that

$$\frac{(n-2)(n-1)}{n}L \leq \frac{n-2}{2}M,$$

which shows that (2.8) is more sharp than (2.10).

Example 2.11. Let $n \geq 3$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, where

$$a_1 = 1, \quad a_2 = 2, \quad a_i = \frac{1}{i} + 1, \quad i = 3, \dots, n.$$

In this case, we have

$$\begin{aligned} M &= \max\{|a_i - a_j| : i, j \in \{1, \dots, n\}\} \\ &= \max\left\{1, \frac{1}{i}, 1 - \frac{1}{i}, \left|\frac{1}{i} - \frac{1}{j}\right| : i, j \in \{3, \dots, n\}\right\} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} L &= \max\left\{\frac{|a_i - a_j|}{|i-j|} : i, j \in \{1, \dots, n\}, i \neq j\right\} \\ &= \max\left\{1, \frac{1}{i(i-1)}, \frac{i-1}{i(i-2)}, \frac{1}{ij} : i, j \in \{3, \dots, n\}, i \neq j\right\} \\ &= 1. \end{aligned}$$

Hence,

$$\frac{n-2}{2}M = \frac{n-2}{2} \le \frac{(n-2)(n-1)}{n} = \frac{(n-2)(n-1)}{n}L,$$

which shows that (2.10) is more sharp than (2.8).

2.4 Discrete versions of inequalities (1.8)

Let $A = (a_{i,j})_{1\le i\le n, 1\le j\le m}$ be a real matrix of size $n \times m$, where $n, m \ge 3$. We first introduce the following definition.

Definition 2.12. We say that A is a convex matrix, if

(i) for all $i = 1,\dots,n$, the sequence $A(i,\cdot) = (a_{i1},\dots,a_{im}) \in \mathbb{R}^m$ is convex;

(ii) for all $j = 1,\dots,m$, the sequence $A(\cdot,j) = (a_{1j},\dots,a_{nj}) \in \mathbb{R}^n$ is convex.

The following result provides discrete versions of inequalities (1.8) established in [25].

Theorem 2.13. *Let $n, m \ge 3$ and $A = (a_{i,j})_{1\le i\le n, 1\le j\le m}$ be a real convex matrix. We have*

$$\begin{aligned}
&\frac{n+m}{4}\left(a_{N_n,N_m} + a_{N_n,m+1-N_m} + a_{n+1-N_n,N_m} + a_{n+1-N_n,m+1-N_m}\right)\\
&\le \frac{1}{2}\left(\sum_{i=1}^{n}\left(a_{i,N_m} + a_{i,m+1-N_m}\right) + \sum_{j=1}^{m}\left(a_{N_n,j} + a_{n+1-N_n,j}\right)\right)\\
&\le \left(\frac{1}{m} + \frac{1}{n}\right)\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\\
&\le \frac{1}{2}\left(\sum_{i=1}^{n}\left(a_{i1} + a_{im}\right) + \sum_{j=1}^{m}\left(a_{1j} + a_{nj}\right)\right)\\
&\le \frac{n+m}{4}\left(a_{11} + a_{1m} + a_{n1} + a_{nm}\right),
\end{aligned} \tag{2.11}$$

where $N_m = \left[\frac{m+1}{2}\right]$ and $N_n = \left[\frac{n+1}{2}\right]$.

Proof. Let $i \in \{1,\dots,n\}$. Since $A(i,\cdot) \in \mathbb{R}^m$ is a convex sequence, by the double discrete Hermite-Hadamard inequality (1.4), we have

$$\frac{a_{i,N_m} + a_{i,m+1-N_m}}{2} \le \frac{1}{m}\sum_{j=1}^{m} a_{ij} \le \frac{a_{i1} + a_{im}}{2}.$$

If we sum the above inequality over $1 \le i \le n$, we obtain

$$\frac{1}{2}\sum_{i=1}^{n}\left(a_{i,N_m} + a_{i,m+1-N_m}\right) \le \frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} \le \frac{1}{2}\sum_{i=1}^{n}\left(a_{i1} + a_{im}\right). \tag{2.12}$$

Similarly, let $j \in \{1,\dots,m\}$. Since $A(\cdot,j) \in \mathbb{R}^n$ is a convex sequence, by the double discrete Hermite-Hadamard inequality (1.4), we have

$$\frac{a_{N_n,j} + a_{n+1-N_n,j}}{2} \le \frac{1}{n}\sum_{i=1}^{n} a_{ij} \le \frac{a_{1j} + a_{nj}}{2}.$$

If we sum the above inequality over $1 \le j \le m$, we obtain

$$\frac{1}{2}\sum_{j=1}^{m}(a_{N_n,j} + a_{n+1-N_n,j}) \leq \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij} \leq \frac{1}{2}\sum_{j=1}^{m}(a_{1j} + a_{nj}). \tag{2.13}$$

Summing inequalities (2.12) and (2.13), we obtain

$$\frac{1}{2}\left(\sum_{i=1}^{n}(a_{i,N_m} + a_{i,m+1-N_m}) + \sum_{j=1}^{m}(a_{N_n,j} + a_{n+1-N_n,j})\right) \leq \left(\frac{1}{m} + \frac{1}{n}\right)\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij} \leq \frac{1}{2}\left(\sum_{i=1}^{n}(a_{i1} + a_{im}) + \sum_{j=1}^{m}(a_{1j} + a_{nj})\right). \tag{2.14}$$

Using the left-hand side inequality in (1.4), we obtain

$$\frac{a_{N_n,N_m} + a_{n+1-N_n,N_m}}{2} \leq \frac{1}{n}\sum_{i=1}^{n}a_{i,N_m}$$

and

$$\frac{a_{N_n,m+1-N_m} + a_{n+1-N_n,m+1-N_m}}{2} \leq \frac{1}{n}\sum_{i=1}^{n}a_{i,m+1-N_m}.$$

Summing the above two inequalities, we obtain

$$\frac{n}{2}(a_{N_n,N_m} + a_{n+1-N_n,N_m} + a_{N_n,m+1-N_m} + a_{n+1-N_n,m+1-N_m}) \leq \sum_{i=1}^{n}(a_{i,N_m} + a_{i,m+1-N_m}). \tag{2.15}$$

Similarly, we have

$$\frac{a_{N_n,N_m} + a_{N_n,m+1-N_m}}{2} \leq \frac{1}{m}\sum_{j=1}^{m}a_{N_n,j}$$

and

$$\frac{a_{n+1-N_n,N_m} + a_{n+1-N_n,m+1-N_m}}{2} \leq \frac{1}{m}\sum_{j=1}^{m}a_{n+1-N_n,j}.$$

Summing the above two inequalities, we obtain

$$\frac{m}{2}(a_{N_n,N_m} + a_{N_n,m+1-N_m} + a_{n+1-N_n,N_m} + a_{n+1-N_n,m+1-N_m}) \leq \sum_{j=1}^{m}(a_{N_n,j} + a_{n+1-N_n,j}). \tag{2.16}$$

Summing (2.15) and (2.16), we obtain

$$\begin{aligned}&\frac{n+m}{4}(a_{N_n,N_m} + a_{N_n,m+1-N_m} + a_{n+1-N_n,N_m} + a_{n+1-N_n,m+1-N_m})\\ &\quad\leq \frac{1}{2}\left(\sum_{i=1}^{n}(a_{i,N_m} + a_{i,m+1-N_m}) + \sum_{j=1}^{m}(a_{N_n,j} + a_{n+1-N_n,j})\right).\end{aligned} \tag{2.17}$$

Similarly, using the right-hand side inequality in (1.4), we obtain

$$\frac{1}{n}\sum_{i=1}^{n}a_{i1} \leq \frac{a_{11} + an1}{2}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}a_{im} \leq \frac{a_{1m} + anm}{2},$$

which implies that

$$\sum_{i=1}^{n}(a_{i1} + a_{im}) \leq \frac{n}{2}(a_{11} + a_{n1} + a_{1m} + a_{nm}). \tag{2.18}$$

Proceeding as above, we obtain

$$\sum_{j=1}^{m}(a_{1j}+a_{nj}) \le \frac{m}{2}(a_{11}+a_{1m}+a_{n1}+a_{nm}). \tag{2.19}$$

Summing (2.18) and (2.19), we obtain

$$\frac{1}{2}\left(\sum_{i=1}^{n}(a_{i1}+a_{im})+\sum_{j=1}^{m}(a_{1j}+a_{nj})\right) \le \frac{n+m}{4}(a_{11}+a_{1m}+a_{n1}+a_{nm}). \tag{2.20}$$

Finally, (2.11) follows from (2.14), (2.17), and (2.20). □

We provide below an example to check the validity of Theorem 2.13.

Example 2.14. Let $n=m=3$. In this case, we have $N_m=N_n=2$. Furthermore, we obtain

$$\frac{n+m}{4}(a_{N_n,N_m}+a_{N_n,m+1-N_m}+a_{n+1-N_n,N_m}+a_{n+1-N_n,m+1-N_m}) = \frac{6}{4}(a_{2,2}+a_{2,2}+a_{2,2}+a_{2,2}) = 6a_{22},$$

$$\frac{1}{2}\left(\sum_{i=1}^{n}(a_{i,N_m}+a_{i,m+1-N_m})+\sum_{j=1}^{m}(a_{N_n,j}+a_{n+1-N_n,j})\right) = \frac{1}{2}\left(\sum_{i=1}^{3}(a_{i,2}+a_{i,2})+\sum_{j=1}^{3}(a_{2,j}+a_{2,j})\right) = \sum_{i=1}^{3}a_{i,2}+\sum_{j=1}^{3}a_{2,j},$$

$$\left(\frac{1}{m}+\frac{1}{n}\right)\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij} = \frac{2}{3}\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij},$$

$$\frac{1}{2}\left(\sum_{i=1}^{n}(a_{i1}+a_{im})+\sum_{j=1}^{m}(a_{1j}+a_{nj})\right) = \frac{1}{2}\left(\sum_{i=1}^{3}(a_{i1}+a_{i3})+\sum_{j=1}^{3}(a_{1j}+a_{3j})\right),$$

and

$$\frac{n+m}{4}(a_{11}+a_{1m}+a_{n1}+a_{nm}) = \frac{3}{2}(a_{11}+a_{13}+a_{31}+a_{33}).$$

Hence, (2.11) reduces to

$$6a_{22} \le \sum_{i=1}^{3}a_{i,2}+\sum_{j=1}^{3}a_{2,j} \le \frac{2}{3}\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij} \le \frac{1}{2}\left(\sum_{i=1}^{3}(a_{i1}+a_{i3})+\sum_{j=1}^{3}(a_{1j}+a_{3j})\right) \le \frac{3}{2}(a_{11}+a_{13}+a_{31}+a_{33}). \tag{2.21}$$

Consider now the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 3 & 6 \end{pmatrix}.$$

It can be easily seen that A is a convex matrix in the sense of Definition 2.12. On the other hand, we have

$$6a_{22} = 6,$$

$$\sum_{i=1}^{3}a_{i,2}+\sum_{j=1}^{3}a_{2,j} = 11,$$

$$\frac{2}{3}\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij} = \frac{50}{3},$$

$$\frac{1}{2}\left(\sum_{i=1}^{3}(a_{i1}+a_{i3})+\sum_{j=1}^{3}(a_{1j}+a_{3j})\right) = \frac{39}{2},$$

$$\frac{3}{2}(a_{11}+a_{13}+a_{31}+a_{33}) = \frac{45}{2},$$

which confirms the validity of (2.21).

3 Conclusion

For a given sequence $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, some upper bounds of the term $E_n \coloneqq \left|\frac{a_1 + a_n}{2} - \frac{1}{n}\sum_{i=1}^{n} a_i\right|$ are obtained. Namely, we first considered the class of sequences $a \in \mathbb{R}^n$ such that

$$a' = (|a_2 - a_1|, \dots, |a_n - a_{n-1}|) \in \mathbb{R}^{n-1}$$

is convex. For this class of sequences, a discrete version of inequality (1.5) [26] is established (Theorem 2.1). We next considered the class of sequences $a \in \mathbb{R}^n$ such that $a'^q = (|a_2 - a_1|^q, \dots, |a_n - a_{n-1}|^q) \in \mathbb{R}^{n-1}$ is convex for some real number $q > 1$. For this class of sequences, a discrete version of inequality (1.6)[26] is proved (Theorem 2.4). We also derived two upper bounds of E_n for an arbitrary sequence $a \in \mathbb{R}^n$. The first one (Theorem 2.7) involves the real number $L = \max\left\{\frac{|a_i - a_j|}{|i - j|} : i, j \in \{1, \dots, n\}, i \neq j\right\}$. The obtained inequality is a discrete version of inequality (1.7) [27]. The second upper bound (Theorem 2.8) involves the real number $M = \max\{|a_i - a_j| : i, j \in \{1, \dots, n\}\}$. We finally introduced the notion of convex matrices $A = (a_{i,j})_{1 \le i \le n, 1 \le j \le m}$, which is a discrete version of the notion of convex functions on the coordinates considered in [25]. For this set of matrices, discrete versions of inequalities (1.8) [25] are proved (Theorem 2.13).

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