



Research Article

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The fibering method approach for a Schrödinger-Poisson system with p -Laplacian in bounded domains

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Abstract: In this article, we study a p -Laplacian Schrödinger-Poisson system involving a parameter $q \neq 0$ in bounded domains. By using the Nehari manifold and the fibering method, we obtain the non-existence and multiplicity of nontrivial solutions. On one hand, there exists $q^* > 0$ such that only the trivial solution is admitted for $q \in (q^*, +\infty)$. On the other hand, there are two positive solutions existing for $q \in (0, q_0^* + \varepsilon)$, where $\varepsilon > 0$ and $q_0^* + \varepsilon < q^*$. In particular, q^* and q_0^* correspond to the supremum for the nonlinear generalized Rayleigh quotients, respectively. The specific form of the nonlinear generalized Rayleigh quotients is calculated. Moreover, it is worth mentioning that we also obtain the qualitative properties associated with the energy level of the solutions.

Keywords: Schrödinger-Poisson system, variational methods, p -Laplacian, fibering method

MSC 2020: 35J20, 35J47, 35J66

1 Introduction

This article studies the following p -Laplacian Schrödinger-Poisson system:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + q^2\phi u = |u|^{\gamma-2}u & \text{in } \Omega, \\ -\Delta\phi = 4\pi u^2 & \text{in } \Omega, \\ u = 0, \phi = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

which can be regarded as an extension of the classical Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + q^2\phi u = |u|^{\gamma-2}u & \text{in } \Omega, \\ -\Delta\phi = 4\pi u^2 & \text{in } \Omega, \\ u = 0, \phi = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, and $2 \leq p < \gamma < 3$.

As we all know, systems similar to (1.2) have been investigated extensively, see [1–5]. In particular, the non-existence and multiplicity of nontrivial solutions of system (1.2) have been proved in [2]. From a physical point of view, systems like (1.2) describe a quantum particle's interaction with an electromagnetic field in quantum mechanics models [6–8] and semiconductor theory [9,10].

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Recently, the p -Laplacian Schrödinger-Poisson system has received the attention of some authors, see [11,12]. The existence of nontrivial solutions of the p -Laplacian Schrödinger-Poisson system was obtained by using the mountain pass theorem in [11,12]. Moreover, we find that a type of p -Laplacian Kirchhoff-Schrödinger-Poisson system is studied on the Heisenberg group in [13], and the existence and multiplicity results for the above systems are established by using the concentration-compactness principle in bounded domains. Hence, inspired by [14], we concentrate on the geometric properties of the solutions by fibering map and Nehari manifold, which is helpful to study the non-existence and multiplicity of nontrivial solutions of system (1.1). As a result, we obtain two solutions which are positive by the maximum principle, when q is taken in an appropriate range. Meanwhile, we point out that similar results have been obtained in some nonlinear problems in [15–17] as well.

Now, we give the norm of ϕ and u here and throughout the article: $\phi \in H_0^1(\Omega)$ with the norm

$$\|\phi\|_H = \left(\int_{\Omega} |\nabla \phi|^2 dx \right)^{\frac{1}{2}}$$

and $u \in W_0^{1,p}(\Omega)$ with the norm

$$\|u\| = (\|\nabla u\|_p^p + \|u\|_p^p)^{\frac{1}{2}}.$$

In particular, we denote L^p -norm by $\|u\|_p$. Furthermore, denote the unique solution of the second equation in system (1.1) by $\phi_u \in H_0^1(\Omega)$ for every $u \in W_0^{1,p}(\Omega)$, which can be obtained by the Lax-Milgram theorem. Since ϕ_u satisfies

$$-\Delta \phi_u = 4\pi u^2, \quad (1.3)$$

for every $u \in W_0^{1,p}(\Omega)$, from (1.3) we have

$$4\pi \int_{\Omega} \phi_u u^2 = \|\phi_u\|_H^2. \quad (1.4)$$

In particular, the integral of the whole article is integrated on Ω , and we will not specifically speak later.

We represent the unique solution of the second equation of system (1.1) with ϕ_u and bring it into the first equation so that the system reduces to a single equation. Then to solve system (1.1) is equivalent to study the single equation

$$-\Delta_p u + |u|^{p-2}u + q^2 \phi_u u = |u|^{p-2}u \quad \text{in } \Omega, \quad (1.5)$$

containing the nonlocal term ϕ_u . For these reasons, from now on, whenever we refer to the solution of system (1.1), we merely mean the solution u of the above equation since $\phi = \phi_u$ is explicitly determined.

The energy functional

$$\mathcal{J}_q(u) = \frac{1}{p} \|u\|_p^p + \frac{q^2}{4} \int_{\Omega} \phi_u u^2 - \frac{1}{\gamma} \|u\|_{\gamma}^{\gamma}, \quad u \in W_0^{1,p}(\Omega),$$

which is related to equation (1.5), is well defined and C^1 . Motivated by the above works, we simplify to find critical points of \mathcal{J}_q .

Before presenting theorems explicitly, for simplicity, the results are given and demonstrated only for $q > 0$, because no confusion exists here when q is replaced by $|q|$. More precisely, the most important result in this article can be stated as follows under the foundation of $2 \leq p < \gamma < 3$ in bounded domains.

Theorem 1.1. *Let $2 \leq p < \gamma < 3$, there exist positive constants ε, q_0^*, q^* satisfying $q_0^* + \varepsilon < q^*$ such that:*

- (1) *For $q > q^*$, the functional \mathcal{J}_q has no critical points except the zero function in $W_0^{1,p}(\Omega)$.*
- (2) *For $q \in (0, q_0^* + \varepsilon)$, the functional \mathcal{J}_q has two nontrivial critical points $u_q, w_q \in W_0^{1,p}(\Omega)$. More specifically,*
 - (i) *If $q \in (0, q_0^*)$, one is a mountain pass critical point w_q with*

$$\mathcal{J}_q(w_q) > 0.$$

The other is a global minimum point u_q with

$$\mathcal{J}_q(u_q) < 0.$$

(ii) If $q = q_0^*$, one is a mountain pass critical point $w_{q_0^*}$ with

$$\mathcal{J}_{q_0^*}(w_{q_0^*}) > 0.$$

The other is a global minimum point $u_{q_0^*}$ with

$$\mathcal{J}_{q_0^*}(u_{q_0^*}) = 0.$$

(iii) If $q \in (q_0^*, q_0^* + \varepsilon)$, one is a mountain pass critical point w_q with

$$\mathcal{J}_q(w_q) > \mathcal{J}_q(u_q).$$

The other is a local minimum point u_q with

$$\mathcal{J}_q(u_q) > 0.$$

In Theorem 1.1, q^* and q_0^* are the extreme values of the Nehari manifold method and correspond to the supremum for the nonlinear generalized Rayleigh quotients, respectively (more details for the notion of extremal parameters, see [18]). As Theorem 1.1 shows, we do not state what happens to the solutions when q between $q_0^* + \varepsilon$ and q^* . Similar to [19], it seems plausible that there exist two positive solutions for q less than q^* and close to it, although we have not proved it yet. Furthermore, it is easy to see that whenever $q = q_0^*$, $\mathcal{J}_{q_0^*}$ is non-negative and we obtain a global minimizer at an energy level of zero. However, extra investigation is required to prove why it does not correspond to the zero function.

Motivated by Ilyasov [18] concerning the so-called extremal value for the application of the Nehari manifold method, this article studies the non-existence and multiplicity of the nontrivial solutions of system (1.1). For this purpose, we introduce the fibering method and Nehari manifolds to find critical points of functional \mathcal{J}_q . By presenting the possible situations of the fiber map $\psi_{q,u}(t)$, the geometry of the functional \mathcal{J}_q is directly determined under different values of parameter q . In particular, we calculate the specific form of the nonlinear generalized Rayleigh quotients introduced in [18], which, to the best of our knowledge, has not yet appeared in other articles for this p -Laplacian Schrödinger-Poisson system. Moreover, we apply the Mountain pass theorem and Ekeland's variational principle, and verify that system (1.1) has two nontrivial solutions when $q \in (0, q_0^* + \varepsilon)$ in bounded domains.

This article is organized as follows. In Section 2, we give some preliminaries and technical results. In particular, Theorem 1.1 (1) is proved here. Sections 3–5 are devoted to prove Theorem 1.1 (2) when q is in different values.

2 Preliminaries and technical results

In this section, we give some necessary preliminaries and technical results.

Lemma 2.1. *For fixed $u \in W_0^{1,p}(\Omega)$, there exists a unique nonnegative solution $\phi_u \in H_0^1(\Omega)$ solving $-\Delta\phi = 4\pi u^2$ weakly in Ω . Moreover, $\|\phi_u\|_H \leq C\|u\|^2$.*

Proof. For each $u \in W_0^{1,p}(\Omega)$, we define a linear functional $T : H_0^1(\Omega) \rightarrow \mathbb{R}$ by $T(v) = 4\pi \int u^2 v$. Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\lambda(\Omega)$, $\forall \lambda \in [1, 6)$ is compact, by using Hölder's inequality and the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, we deduce that

$$|T(v)| \leq 4\pi \left(\int |u|^{\frac{12}{5}} \right)^{\frac{5}{6}} \left(\int |v|^6 \right)^{\frac{1}{6}} \leq C \|u\|^2 \|v\|_H, \quad (2.1)$$

from which the boundedness of T on $H_0^1(\Omega)$ follows. According to the definition of weak solution of (1.3) and the Lax-Milgram theorem, there exists a unique $\phi_u \in H_0^1(\Omega)$ such that

$$\int \nabla \phi_u \nabla v = 4\pi \int u^2 v, \quad \forall v \in H_0^1(\Omega). \quad (2.2)$$

Therefore, combining (2.1) with (2.2) and choosing $v = \phi_u$, we obtain $\|\phi_u\|_H \leq C\|u\|^2$. \square

Proposition 2.2. For all $u \in W_0^{1,p}(\Omega)$,

$$\int |u|^3 \leq \frac{1}{4\pi} \|\phi_u\|_H \|\nabla u\|_2.$$

Proof. For each fixed $u \in W_0^{1,p}(\Omega)$, multiplying equation (1.3) by $|u| \in W_0^{1,p}(\Omega)$ and integrating on Ω , using Hölder's inequality and the embedding $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$, we obtain

$$4\pi \int |u|^3 = \int \nabla \phi_u \nabla |u| \leq \|\nabla \phi_u\|_2 \|\nabla |u|\|_2.$$

It follows that

$$\int |u|^3 \leq \frac{1}{4\pi} \|\phi_u\|_H \|\nabla |u|\|_2. \quad \square$$

Proposition 2.3. There exists $\rho > 0$ and $M > 0$ satisfying

$$\mathcal{J}_q(u) \geq M \quad \text{for all } q > 0 \quad \text{and} \quad u \in W_0^{1,p}(\Omega) \text{ with } \|u\| = \rho.$$

Proof. The conclusion easily follows from

$$\mathcal{J}_q(u) \geq \frac{1}{p} \|u\|^p - \frac{1}{\gamma} \|u\|^\gamma \geq \frac{1}{p} \|u\|^p - C \|u\|^\gamma, \quad (2.3)$$

where $C > 0$ is the Sobolev embedding constant. \square

Now, we study the geometry of the functional \mathcal{J}_q . Observing $\phi_{tu} = t^2 \phi_u$ and defining $\psi_{q,u}(t) = \mathcal{J}_q(tu)$ with $\psi_{q,u} : [0, \infty) \rightarrow \mathbb{R}$, we have

$$\psi_{q,u}(t) = \frac{t^p}{p} \|u\|^p + \frac{q^2 t^4}{4} \int \phi_u u^2 - \frac{t^\gamma}{\gamma} \|u\|^\gamma.$$

For brevity, we will also use the notation $\psi = \psi_{q,u}$ whenever q and u are fixed. A simple analysis is as follows.

Proposition 2.4. For each $q > 0$ and $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, the graph of ψ has three possibilities on $(0, +\infty)$:

- (i) The function ψ has two critical points, i.e., $0 < t_q^-(u) < t_q^+(u)$. Furthermore, $t_q^-(u)$ is a local maximum point with $\psi''(t_q^-(u)) < 0$ and $t_q^+(u)$ is a local minimum point with $\psi''(t_q^+(u)) > 0$.
- (ii) The function ψ has one critical point, i.e., $0 < t_q(u)$. Furthermore, $\psi''(t_q(u)) = 0$ and ψ is increasing.
- (iii) The function ψ has no critical points and is increasing.

Note that (i) occurs for q small and (iii) for q large (Figure 1).

Now, we introduce the Nehari manifolds associated with the functional \mathcal{J}_q , namely,

$$\mathcal{N}_q = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} : \psi'(1) = 0\}.$$

For $u \in \mathcal{N}_q$,

$$\|u\|^p \leq \|u\|^p + q^2 \int \phi_u u^2 \leq C \|u\|^\gamma. \quad (2.4)$$

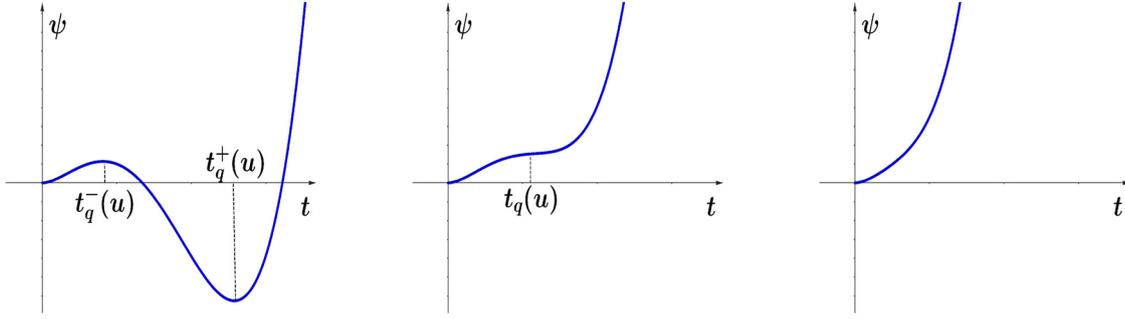


Figure 1: Fiber map graphs for Proposition 2.4.

Then, we know that

$$\exists \tilde{C} > 0 \text{ such that for all } q > 0, u \in \mathcal{N}_q \text{ one has } \|u\| \geq \tilde{C}, \quad (2.5)$$

that is, all the Nehari manifolds are uniformly bounded away from zero in q .

Moreover, the Nehari set can be divided into three disjoint sets:

$$\mathcal{N}_q = \mathcal{N}_q^+ \cup \mathcal{N}_q^0 \cup \mathcal{N}_q^-,$$

where

$$\mathcal{N}_q^+ = \{u \in \mathcal{N}_q : \psi''(1) > 0\},$$

$$\mathcal{N}_q^0 = \{u \in \mathcal{N}_q : \psi''(1) = 0\},$$

$$\mathcal{N}_q^- = \{u \in \mathcal{N}_q : \psi''(1) < 0\}.$$

As a result of the implicit function theorem, one obtains:

Proposition 2.5. Consider the disjoint sets:

- (i) $\mathcal{N}_q^+, \mathcal{N}_q^-$ are C^1 manifolds of codimension 1 in $W_0^{1,p}(\Omega)$ if $\mathcal{N}_q^+, \mathcal{N}_q^-$ are non-empty;
- (ii) $u \in \mathcal{N}_q^+ \cup \mathcal{N}_q^-$ is a critical point for the functional \mathcal{J}_q , equivalently, u is a critical point of the constrained functional $(\mathcal{J}_q)|_{\mathcal{N}_q^+ \cup \mathcal{N}_q^-} : \mathcal{N}_q^+ \cup \mathcal{N}_q^- \rightarrow \mathbb{R}$.

From the previous discussion, we have $tu \in \mathcal{N}_q^0$ for a fixed $u \in W_0^{1,p}(\Omega)$ if and only if $\psi'_{q,tu}(1) = \psi''_{q,tu}(1) = 0$. Indeed, this is equivalent to the system

$$\begin{cases} t^{p-1}\|u\|^p + q^2 t^3 \int \phi_u u^2 - t^{\gamma-1} \|u\|_\gamma^\gamma = 0, \\ (p-1)t^{p-2}\|u\|^p + 3q^2 t^2 \int \phi_u u^2 - (\gamma-1)t^{\gamma-2} \|u\|_\gamma^\gamma = 0. \end{cases} \quad (2.6)$$

After direct calculation, we obtain the unique solution with respect to the variables q and t , which are

$$t(u) = \left(\frac{(4-p)\|u\|^p}{(4-\gamma)\|u\|_\gamma^\gamma} \right)^{\frac{1}{\gamma-p}}$$

and

$$q(u) = C_p \frac{\|u\|_\gamma^{\frac{\gamma(4-p)}{2(\gamma-p)}}}{\|u\|^{\frac{p(4-\gamma)}{2(\gamma-p)}} \|\phi_u\|_H}, \quad (2.7)$$

where

$$C_p = \left[\left[\left(\frac{4-p}{4-\gamma} \right)^{\frac{\gamma-4}{\gamma-p}} - \left(\frac{4-p}{4-\gamma} \right)^{\frac{p-4}{\gamma-p}} \right] 4\pi \right]^{\frac{1}{2}}. \quad (2.8)$$

In particular, there exists a connection between q and t ,

$$t(u) = \left(\frac{q^2(u) \|\phi_u\|_H^2}{C \|u\|_Y^p} \right)^{\frac{1}{\gamma-4}},$$

where C is a positive constant. Now, we define the extremal value

$$q^* = \sup\{q(u) : u \in W_0^{1,p}(\Omega) \setminus \{0\}\}.$$

Lemma 2.6. *The function $W_0^{1,p}(\Omega) \setminus \{0\} \ni u \mapsto q(u)$ defined in (2.7) is 0-homogeneous. Moreover, $q^* < \infty$.*

Proof. It is obvious that $q(u)$ is 0-homogeneous. Next, we prove $q^* < \infty$. In fact, from the interpolation inequality for $\gamma \in (2, 3)$, we obtain

$$\|u\|_Y^p \leq \|u\|_2^{6-2\gamma} \|u\|_3^{3\gamma-6} \quad (2.9)$$

for all $u \in W_0^{1,p}(\Omega)$. From Proposition 2.2 and $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$, $L^p(\Omega) \hookrightarrow L^2(\Omega)$, we have

$$\begin{aligned} \|u\|_Y^{p(4-p)} &\leq \|u\|_2^{(6-2\gamma)(4-p)} \|u\|_3^{(3\gamma-6)(4-p)} \\ &\leq C \|u\|_2^{(6-2\gamma)(4-p)} \|\nabla u\|_2^{(y-2)(4-p)} \|\phi_u\|_H^{(y-2)(4-p)} \\ &\leq C \|u\|_p^{(6-2\gamma)(4-p)} \|\nabla u\|_p^{(y-2)(4-p)} \|\phi_u\|_H^{(y-2)(4-p)} \\ &\leq C \|u\|^{(4-\gamma)(4-p)} \|\phi_u\|_H^{(y-2)(4-p)}, \end{aligned} \quad (2.10)$$

for some constant $C > 0$. Consequently, according to Lemma 2.1 and the definition of $q(u)$, we conclude that

$$\begin{aligned} q(u) &\leq C \frac{\|u\|^{\frac{(4-\gamma)(4-p)}{2(\gamma-p)}} \|\phi_u\|_H^{\frac{(y-2)(4-p)}{2(\gamma-p)}}}{\|u\|^{\frac{(4-\gamma)p}{2(\gamma-p)}} \|\phi_u\|_H} \\ &\leq C \|u\|^{\frac{(4-\gamma)(2-p)}{\gamma-p}} \|\phi_u\|_H^{\frac{(y-4)(2-p)}{2(\gamma-p)}} \\ &\leq C \|u\|^{\frac{(4-\gamma)(2-p)}{\gamma-p}} \|u\|^{\frac{(y-4)(2-p)}{\gamma-p}} \\ &= C. \end{aligned} \quad \square$$

On the contrary, we have another extremal value which is crucial in the following proof. Under this circumstance, the functional \mathcal{J}_q is always non-negative. Let us begin by fixing $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and taking the following system into consideration:

$$\begin{cases} \psi_{q_0, u}(t_0) = \frac{t_0^p}{p} \|u\|^p + \frac{q_0^2 t_0^4}{4} \int \phi_u u^2 - \frac{t_0^y}{\gamma} \|u\|_Y^p = 0, \\ \psi'_{q_0, u}(t_0) = t_0^{p-1} \|u\|^p + q_0^2 t_0^3 \int \phi_u u^2 - t_0^{y-1} \|u\|_Y^p = 0. \end{cases} \quad (2.11)$$

With regard to the variables t_0 and q_0 , we give the unique solution

$$t_0(u) = \left(\frac{\gamma(4-p)\|u\|^p}{p(4-\gamma)\|u\|_Y^p} \right)^{\frac{1}{\gamma-4}}$$

and

$$q_0(u) = C_{0,p} \frac{\|u\|_Y^{\frac{\gamma(4-p)}{2(\gamma-p)}}}{\|u\|^{\frac{p(4-\gamma)}{2(\gamma-p)}} \|\phi_u\|_H}, \quad (2.12)$$

where

$$C_{0,p} = \left[\left[\left(\frac{\gamma(4-p)}{p(4-\gamma)} \right)^{\frac{\gamma-4}{\gamma-p}} - \left(\frac{\gamma(4-p)}{p(4-\gamma)} \right)^{\frac{p-4}{\gamma-p}} \right] 4\pi \right]^{\frac{1}{2}}. \quad (2.13)$$

In particular, there exists a connection between q_0 and t_0 ,

$$t_0(u) = \left(\frac{q_0^2(u) \|\phi_u\|_H^2}{C \|u\|_Y^\gamma} \right)^{\frac{1}{\gamma-4}},$$

where C is a positive constant. Observe that $C_{0,p} < C_p$, where C_p is the one shown in (2.8). After that, $q_0(u) < q(u)$. The extremal value was described as

$$q_0^* = \sup\{q_0(u) : u \in W_0^{1,p}(\Omega) \setminus \{0\}\}.$$

Remark 1. Since $q_0(u)$ is a multiple of $q(u)$, Lemma 2.6 also holds true for the function $q_0(u)$.

Proposition 2.7. *There exists a positive constant m such that*

$$\mathcal{J}_q(u) \geq m \quad \text{for all } q > 0, u \in \mathcal{N}_q^0.$$

Proof. Using equation (2.6) and taking $t = 1$, we can deduce that

$$\begin{cases} \|u\|^p + q^2 \int \phi_u u^2 - \|u\|_Y^\gamma = 0, \\ (p-1)\|u\|^p + 3q^2 \int \phi_u u^2 - (\gamma-1)\|u\|_Y^\gamma = 0. \end{cases} \quad (2.14)$$

It follows that

$$q^2 \int \phi_u u^2 = \|u\|_Y^\gamma - \|u\|^p \quad \text{and} \quad \|u\|_Y^\gamma = \frac{p-4}{\gamma-4} \|u\|^p.$$

Then, we have $\mathcal{J}_q(u) = \frac{(4-p)(\gamma-p)}{4p\gamma} \|u\|^p$ for each $u \in \mathcal{N}_q^0$. Hence, we complete the proof by (2.5). \square

Lemma 2.8. *Consider the operator $\Phi : W_0^{1,p}(\Omega) \rightarrow H_0^1(\Omega)$, $\Phi(u) = \phi_u$, that is, the solution of the problem $-\Delta \phi_u = 4\pi u^2$ in $H_0^1(\Omega)$. Let u_n be a sequence satisfying $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$. Then, $\phi_{u_n} \rightarrow \phi_u$ in $H_0^1(\Omega)$ and, as a consequence,*

$$\int \phi_{u_n} u_n^2 \rightarrow \int \phi_u u^2.$$

Proof. Define the linear operators $T_n, T : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$T_n(v) = 4\pi \int u_n^2 v, \quad T(v) = 4\pi \int u^2 v.$$

Recall that the embedding $W_0^{1,p}(\Omega)$ into $L^\lambda(\Omega)$ is compact for $1 \leq \lambda < 6$. Then

$$\int |u_n^2 - u^2|^{\frac{6}{5}} = \int |u_n + u|^{\frac{6}{5}} |u_n - u|^{\frac{6}{5}} \leq \left(\int |u_n + u|^{\frac{12}{5}} \right)^{\frac{1}{2}} \left(\int |u_n - u|^{\frac{12}{5}} \right)^{\frac{1}{2}},$$

hence $u_n^2 \rightarrow u^2$ in $L^{\frac{6}{5}}$. Note that

$$|T_n(v) - T(v)| \leq 4\pi \left(\int |u_n^2 - u^2|^{\frac{6}{5}} \right)^{\frac{5}{6}} \left(\int |v|^6 \right)^{\frac{1}{6}} \leq 4\pi \|u_n^2 - u^2\|_{\frac{6}{5}} \|v\|_H,$$

which means T_n converges strongly to T .

From (1.3), we obtain that

$$\int \nabla \phi_u \nabla v = 4\pi \int u^2 v, \quad \forall v \in H_0^1(\Omega), \quad (2.15)$$

$$\int \nabla \phi_{u_n} \nabla v = 4\pi \int u_n^2 v, \quad \forall v \in H_0^1(\Omega). \quad (2.16)$$

Subtracting (2.15) from (2.16) and choosing $v = \phi_{u_n} - \phi_u$, we obtain

$$\begin{aligned} \|\phi_{u_n} - \phi_u\|_H^2 &= 4\pi \int (u_n^2 - u^2)(\phi_{u_n} - \phi_u) \\ &\leq 4\pi \left(\int |u_n^2 - u^2|^{\frac{6}{5}} \right)^{\frac{5}{6}} \left(\int |\phi_{u_n} - \phi_u|^6 \right)^{\frac{1}{6}} \\ &\leq 4\pi \|u_n^2 - u^2\|_{\frac{6}{5}} \|\phi_{u_n} - \phi_u\|_H. \end{aligned}$$

Hence, $\phi_{u_n} \rightarrow \phi_u$ in $H_0^1(\Omega)$.

Conversely, observing that $\phi_{u_n} \rightarrow \phi_u$ in $L^6(\Omega)$ and $u_n^2 \rightarrow u^2$ in $L^{\frac{6}{5}}$, we have

$$\begin{aligned} \left| \int \phi_{u_n} u_n^2 - \int \phi_u u^2 \right| &= \left| \int \phi_{u_n} u_n^2 - \int \phi_{u_n} u^2 + \int \phi_{u_n} u^2 - \int \phi_u u^2 \right| \\ &\leq \left| \int \phi_{u_n} u_n^2 - \int \phi_{u_n} u^2 \right| + \left| \int \phi_{u_n} u^2 - \int \phi_u u^2 \right| \\ &\leq \left(\int |u_n^2 - u^2|^{\frac{6}{5}} \right)^{\frac{5}{6}} \left(\int |\phi_{u_n}|^6 \right)^{\frac{1}{6}} + \left(\int |u^2|^{\frac{6}{5}} \right)^{\frac{5}{6}} \left(\int |\phi_{u_n} - \phi_u|^6 \right)^{\frac{1}{6}} \\ &\leq C \|u_n^2 - u^2\|_{\frac{6}{5}} \|\phi_{u_n}\|_H + C \|u\|_{\frac{12}{5}}^2 \|\phi_{u_n} - \phi_u\|_H \rightarrow 0, \end{aligned} \quad (2.17)$$

then we conclude $\int \phi_{u_n} u_n^2 \rightarrow \int \phi_u u^2$. \square

Lemma 2.9. *The functional $\mathcal{J}_q(u)$ satisfies the following properties:*

- (1) $\mathcal{J}_q(u)$ is weakly lower semi-continuous and coercive.
- (2) $\mathcal{J}_q(u)$ satisfies the Palais-Smale condition.

Proof. To prove (1), we take $u_n \rightharpoonup u$ and then by using the compactness of the Sobolev embedding, Brezis-Lieb's lemma, and Lemma 2.8, we obtain

$$\int |u_n|^\gamma \rightarrow \int |u|^\gamma, \quad \int \phi_{u_n} u_n^2 \rightarrow \int \phi_u u^2.$$

It follows that $\mathcal{J}_q(u)$ is w.l.s.c.

Conversely, we begin to prove that $\mathcal{J}_q(u)$ is coercive. By using Hölder's inequality, Young's inequality, and Proposition 2.2, we obtain

$$\begin{aligned} \|u\|_3^3 &\leq \frac{1}{4\pi} |\text{meas}(\Omega)|^{\frac{1}{\tau}} \|\nabla \phi_u\|_2 \|\nabla u\|_p \\ &\leq \frac{1}{4\pi} \left(\frac{1}{\tau} |\text{meas}(\Omega)| + \frac{1}{p\varepsilon^p} \|\nabla u\|_p^p + \frac{1}{2} \varepsilon^2 \|\nabla \phi_u\|_2^2 \right) \\ &= \frac{1}{4\pi} \left(\frac{1}{\tau} |\text{meas}(\Omega)| + \frac{1}{p\varepsilon^p} \|\nabla u\|_p^p + \frac{1}{2} \varepsilon^2 \|\phi_u\|_H^2 \right), \end{aligned} \quad (2.18)$$

where

$$\frac{1}{\tau} + \frac{1}{p} + \frac{1}{2} = 1,$$

$\tau = \frac{2p}{p-2} \in (6, +\infty)$ because of $p \in (2, 3)$.

We take $D_q = \frac{q^2}{16\pi} - \frac{\varepsilon^{p+2}}{4} > 0$ such that

$$\begin{aligned} \mathcal{J}_q(u) &= \frac{1}{2p} \|\nabla u\|_p^p + \frac{1}{2p} \|\nabla u\|_p^p + \frac{1}{p} \|u\|_p^p + \frac{q^2}{4} \int \phi_u u^2 - \frac{1}{\gamma} \|u\|_\gamma^\gamma \\ &\geq \frac{1}{2p} \|\nabla u\|_p^p + \left(\frac{q^2}{16\pi} - \frac{\varepsilon^{p+2}}{4} \right) \|\phi_u\|_H^2 + \frac{1}{p} \|u\|_p^p + 2\pi\varepsilon^p \|u\|_3^3 - \frac{1}{\gamma} \|u\|_\gamma^\gamma - \frac{\varepsilon^p |\Omega|}{2\tau} \\ &= \frac{1}{2p} \|u\|^p - \frac{\varepsilon^p |\Omega|}{2\tau} + D_q \|\phi_u\|_H^2 + \int f(|u|), \end{aligned} \quad (2.19)$$

where

$$f(t) = \frac{1}{2p} t^p + 2\pi\varepsilon^p t^3 - \frac{1}{\gamma} t^\gamma \quad \text{for all } t > 0.$$

A simple analysis shows that $I = \inf_{t>0} f(t) > -\infty$ and if $f(t) < 0$ for some $t > 0$, then $f^{-1}((-\infty, 0)) = (\alpha, \beta)$, where $0 < \alpha < \beta < \infty$.

If $I \geq 0$, since $D_q > 0$ we conclude from (2.19) that

$$\mathcal{J}_q(u) \geq \frac{1}{2p} \|u\|^p - \frac{\varepsilon^p |\Omega|}{2\tau} + D_q \|\phi_u\|_H^2 \geq \frac{1}{2p} \|u\|^p - C. \quad (2.20)$$

If $I < 0$, then

$$\mathcal{J}_q(u) \geq \frac{1}{2p} \|u\|^p - \frac{\varepsilon^p |\Omega|}{2\tau} + D_q \|\phi_u\|_H^2 + I \text{meas}(A) \geq \frac{1}{2p} \|u\|^p - C, \quad (2.21)$$

where $A = \{x \in \Omega : u(x) \in (\alpha, \beta)\}$. Hence, $\mathcal{J}_q(u)$ is coercive.

Now, we begin to prove that u_n admits a converging subsequence. By the boundedness and coercivity of $\mathcal{J}_q(u)$, it is clear that u_n is also bounded. Thanks to the boundedness of $\{u_n\}$, there exists $u_0 \in W_0^{1,p}(\Omega)$ such that, up to subsequences, $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$, then $u_n \rightarrow u_0$ in $L^\lambda(\Omega)$, $1 \leq \lambda < 6$ because of the previous compact embedding. We evaluate

$$\langle \mathcal{J}'_q(u_n), u_n - u \rangle = \int |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) + \int |u_n|^{p-2} u_n (u_n - u) + q^2 \int \phi_{u_n} u_n (u_n - u) - \int |u_n|^{p-2} u_n (u_n - u).$$

By using Hölder's inequality, we have:

$$\int \phi_{u_n} u_n (u_n - u) \leq \|\phi_{u_n}\|_3 \|u_n\|_3 \|u_n - u\|_3.$$

By Lemma 2.1, the above expression tends to zero as $n \rightarrow \infty$. Moreover,

$$\int |u_n|^{p-2} u_n (u_n - u) \rightarrow 0, \quad \int |u_n|^{p-2} u_n (u_n - u) \rightarrow 0.$$

Then

$$\int |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) \rightarrow 0.$$

Finally, we conclude $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$. □

Remark 2. Lemma 2.9 can be generalized as follows: depending on the smooth dependence of \mathcal{J}'_q on q , we obtain the conclusion that if $q_n \rightarrow q$ and $\{u_n\} \subset W_0^{1,p}(\Omega)$ is a sequence such that $\mathcal{J}'_q(u_n) \rightarrow 0$ as $n \rightarrow +\infty$, then $\{u_n\}$ is convergent, up to subsequences.

Finally, we give the following geometrical interpretations of $q(u)$ and $q_0(u)$, which can be easily derived by Proposition 2.4.

Proposition 2.10. *For each $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, $q > 0$, there holds*

- (i) *The fiber map ψ has a critical point with second derivative zero at $t(u)$. Furthermore, if $0 < q < q(u)$, then ψ corresponds to Proposition 2.4 (i) while if $q > q(u)$, ψ corresponds to Proposition 2.4 (iii).*
- (ii) *The fiber map ψ has a critical point with zero energy at $t_0(u)$. Furthermore, if $0 < q < q_0(u)$, $\inf_{t>0} \psi(t) < 0$ while if $q > q_0(u)$, $\inf_{t>0} \psi(t) = 0$.*

In addition, the parameter q_0^* possesses such a geometric interpretation that when $0 < q < q_0^*$, $\mathcal{J}_q(u) < 0$ for at least one $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ while $q \geq q_0^*$, $\mathcal{J}_q(u) \geq 0$ for all $u \in W_0^{1,p}(\Omega)$. It is a consequence that small values q are necessary to prove the existence of a function with a negative value, such that \mathcal{J}_q has a mountain pass geometry.

Corollary 2.11. *If $q \geq q_0^*$, then $\mathcal{J}_q \geq 0$ for all $u \in W_0^{1,p}(\Omega)$. Moreover, if $q < q_0^*$, then there exists $u \in W_0^{1,p}(\Omega)$ such that $\mathcal{J}_q < 0$.*

Proof. In fact, assume that $q \geq q_0^*$. Deriving from Proposition 2.10 (ii), we obtain $\inf_{t>0} \psi(t) = 0$. This means that when $q > q_0(u)$ for each $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we have $\mathcal{J}_q(u) \geq 0$. Conversely, suppose that $q < q_0^*$. According to the definition of q_0^* , there exists $w \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $q < q_0(w) < q_0^*$. Consequently, we deduce from Proposition 2.10 (ii) that $\inf_{t>0} \psi(t) < 0$ and then there exists $t > 0$ such that if $u = tw$, $\mathcal{J}_q < 0$. \square

Corollary 2.12. *If $q > q^*$, the functional \mathcal{J}_q has no critical points except the zero function. Moreover, if $q < q^*$, then $\mathcal{N}_q^- \neq \emptyset$ and $\mathcal{N}_q^+ \neq \emptyset$.*

Proof. It suffices to prove that the function ψ has no critical points when $q > q^*$ for each $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. Actually, due to the inequality $q(u) \leq q^* < q$ and Proposition 2.10 (i), we can obtain the conclusion easily. Now, assume that $q < q^*$. According to the definition of q^* , there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $q < q(u) < q^*$. As a result, we deduce from Proposition 2.10 (i) that $\mathcal{N}_q^- \neq \emptyset$ and $\mathcal{N}_q^+ \neq \emptyset$. \square

In particular, Theorem 1.1 (1) has been proved here.

3 Existence of solutions for $q \in (0, q_0^*)$

In this section, we will prove Theorem 1.1 (2) (i).

(i) *The global minimum solution*

Proposition 3.1. *For each $q \in (0, q_0^*)$, $-\infty < \inf_{u \in W_0^{1,p}(\Omega)} \mathcal{J}_q(u) < 0$.*

Proof. From Corollary 2.11, we know that

$$\inf_{u \in W_0^{1,p}(\Omega)} \mathcal{J}_q(u) < 0.$$

Now, we prove $-\infty < \inf_{u \in W_0^{1,p}(\Omega)} \mathcal{J}_q(u)$. By contradiction, we assume that there exists a sequence $\{u_n\}$ such that $\mathcal{J}_q(u_n) \rightarrow -\infty$ as $n \rightarrow +\infty$, then $\|u_n\| \rightarrow +\infty$ because of (2.3). Without loss of generality, we can infer from (2.21) that

$$\frac{1}{2p} \|u_n\|^p \leq \mathcal{J}_q(u_n) + \frac{\varepsilon^p |\Omega|}{2\tau} - I \text{meas}(A_n),$$

where $A_n \subset \Omega$. There is no doubt that it is a contradiction here. \square

Proposition 3.2. For $q \in (0, q_0^*)$, there exists a global minimum u_q such that $\mathcal{J}_q(u_q) < 0$.

Proof. The conclusion can be easily proved by using Proposition 3.1 and Ekeland's variational principle. \square

(ii) *The mountain pass solution*

Define

$$c_q := \inf_{\gamma \in \Gamma_q} \max_{t \in [0,1]} \mathcal{J}_q(\gamma(t)) > 0,$$

where

$$\Gamma_q = \{\gamma \in C([0, 1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \mathcal{J}_q(\gamma(1)) < 0\}.$$

Thanks to Corollary 2.11, Γ_q is non-empty. The following proposition is deduced from Proposition 2.3 and Lemma 2.9.

Proposition 3.3. For each $q \in (0, q_0^*)$, there exists $w_q \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $\mathcal{J}_q(w_q) = c_q$ and $\mathcal{J}'_q(w_q) = 0$. Particularly, $\mathcal{J}_q(w_q) > \mathcal{J}_q(u_q) \in (-\infty, 0)$.

4 Existence of solutions for $q = q_0^*$

In this section, we will prove Theorem 1.1 (2) (ii).

(i) *The global minimum solution*

Corollary 4.1. There exists a global minimum point $u_{q_0^*} \neq 0$ of $\mathcal{J}_{q_0^*}$ such that $\mathcal{J}_{q_0^*}(u_{q_0^*}) = 0$.

Proof. Assume that $q_n \uparrow q_0^*$ as $n \rightarrow \infty$. For each n , there exists $u_n = u_{q_n}$, which is a global minimum for \mathcal{J}_{q_n} and $\mathcal{J}_{q_n}(u_n) < 0$ from Proposition 3.2. Then, we have $\mathcal{J}'_{q_n}(u_n) = 0$ for each n . Conversely, observing all the Nehari manifolds are bounded away from zero uniformly in q , we have $\|u_n\| \geq \tilde{C}$ for each n . Moreover, we conclude that $u_n \rightarrow u \neq 0$ from Remark 2. Since $\mathcal{J}_{q_n}(u_n) < 0$ for each n , passing to the limit as $n \rightarrow \infty$, we obtain $\mathcal{J}_{q_0^*}(u) \leq 0$. As a consequence, we obtain $\mathcal{J}_{q_0^*}(u) = 0$ from Corollary 2.11. Finally, we set $u_{q_0^*} = u$ and the conclusion follows. \square

Remark 3. $q_0(u_{q_0^*}) = q_0^*$ follows from the definition of q_0^* and Corollary 4.1. Moreover, $q_0^* < q(u_{q_0^*})$.

(ii) *The mountain pass solution*

Define

$$c_{q_0^*} := \inf_{\gamma \in \Gamma_{q_0^*}} \max_{t \in [0,1]} \mathcal{J}_{q_0^*}(\gamma(t)) > 0,$$

where

$$\Gamma_{q_0^*} = \{\gamma \in C([0, 1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \mathcal{J}_{q_0^*}(\gamma(1)) = 0\}.$$

The following proposition is deduced from Proposition 2.3 and Lemma 2.9.

Proposition 4.2. *For $q = q_0^*$, there exists $w_{q_0^*} \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $\mathcal{J}_{q_0^*}(w_{q_0^*}) = c_{q_0^*}$ and $\mathcal{J}'_{q_0^*}(w_{q_0^*}) = 0$. Particularly, $\mathcal{J}_{q_0^*}(w_{q_0^*}) > \mathcal{J}_{q_0^*}(u_{q_0^*}) = 0$.*

5 Existence of solutions for $q \in (q_0^*, q_0^* + \varepsilon)$

In this section, we will prove Theorem 1.1 (2) (iii).

Consider the following family of constrained minimization problems: For $q > 0$,

$$\widehat{\mathcal{J}}_q := \inf\{\mathcal{J}_q(u) : u \in \mathcal{N}_q^+ \cup \mathcal{N}_q^0\}.$$

Observe that

$$\widehat{\mathcal{J}}_q = \inf_{u \in W_0^{1,p}(\Omega)} \mathcal{J}_q(u) \quad \text{for all } q \in (0, q_0^*],$$

and from Corollary 2.11, we have $\widehat{\mathcal{J}}_q \geq 0$ for $q \geq q_0^*$.

(i) *The local minimum solution*

Proposition 5.1. *Given $\delta > 0$, there exists $\varepsilon > 0$ such that $\widehat{\mathcal{J}}_q < \delta$ for each $q \in (q_0^*, q_0^* + \varepsilon)$.*

Proof. We obtain $u_{q_0^*} \in \mathcal{N}_q^+$ as in Corollary 4.1. According to Remark 3, there exists $\varepsilon_1 > 0$ such that $q_0^* + \varepsilon_1 < q(u_{q_0^*})$. If $q \downarrow q_0^*$, then $\mathcal{J}_q(u_{q_0^*}) \rightarrow \mathcal{J}_{q_0^*}(u_{q_0^*}) = 0$. Conversely, for $q \in (q_0^*, q_0^* + \varepsilon)$, there exists $t_q^+(u_{q_0^*})$ such that $t_q^+(u_{q_0^*})u_{q_0^*} \in \mathcal{N}_q^+$ by Proposition 2.4 and Proposition 2.10 (ii). When $t_q^+(u_{q_0^*}) \rightarrow 1$ as $q \downarrow q_0^*$, we have

$$0 \leq \mathcal{J}_q(t_q^+(u_{q_0^*})u_{q_0^*}) \leq \mathcal{J}_q(u_{q_0^*}) \rightarrow \mathcal{J}_{q_0^*}(u_{q_0^*}) = 0, \quad q \downarrow q_0^*.$$

The conclusion follows if we choose ε_2 in such a way that $\mathcal{J}_q(t_q^+(u_{q_0^*})u_{q_0^*}) < \delta$ for each $q \in (q_0^*, q_0^* + \varepsilon_2)$. Then, we take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and complete the proof. \square

Recall that by Proposition 2.3, there are positive constants ρ, M satisfying $\mathcal{J}_q(u) \geq M$ for each $\|u\| = \rho$. Without loss of generality, we may assume that $\rho < \widetilde{C}$, where \widetilde{C} is in such a way that

$$\|u\| \geq \widetilde{C} \quad \text{for all } q > 0 \text{ and } u \in \mathcal{N}_q$$

(see (2.5)).

We choose $\delta > 0$ in Proposition 5.1 in such a way that

$$0 < \delta < \min\{M, m\}, \tag{5.1}$$

where m is the positive constant such that, by Proposition 2.7,

$$\mathcal{J}_q(u) \geq m \quad \text{for all } q > 0 \text{ and } u \in \mathcal{N}_q^0$$

Let $\varepsilon > 0$ be as in Proposition 5.1 in correspondence of the above fixed $\delta > 0$.

Proposition 5.2. *For all $q \in (q_0^*, q_0^* + \varepsilon)$,*

$$\inf\{\mathcal{J}_q(u) : \|u\| \geq \rho\} = \widehat{\mathcal{J}}_q.$$

Proof. In fact, we claim that the inequality $\inf\{\mathcal{J}_q(u) : \|u\| \geq \rho\} \leq \widehat{\mathcal{J}}_q$ holds for $\rho < \widetilde{\mathcal{C}}$. According to Proposition 2.4, we distinguish the following three cases. When the fiber map ψ satisfies Proposition 2.4 (i), we have $\inf_{t>\rho}\psi(t) \geq \widehat{\mathcal{J}}_q$. When the fiber map ψ satisfies Proposition 2.4 (ii) or (iii), we have $\inf_{t>\rho}\psi(t) = M$. As the condition $M > \delta > \widehat{\mathcal{J}}_q$ is satisfied, we finish this proof. \square

Corollary 5.3. *For $q \in (q_0^*, q_0^* + \varepsilon)$, there exists $u_q \in \mathcal{N}_q^+$ such that $\mathcal{J}_q(u_q) = \widehat{\mathcal{J}}_q$. Particularly, $\mathcal{J}_q(u_q) > 0$ and $\|u_q\| \geq \widetilde{\mathcal{C}} > \rho$.*

Proof. For fixed $q \in (q_0^*, q_0^* + \varepsilon)$, we take a minimizing sequence $\{u_n\} \in \mathcal{N}_q^+ \cup \mathcal{N}_q^0$ for $\widehat{\mathcal{J}}_q < \delta$ by Proposition 5.1. From Proposition 2.7 that $\mathcal{J}_q(u) \geq m$ on \mathcal{N}_q^0 and $m > \delta$ in (5.1), we assume that $\{u_n\} \subset \mathcal{N}_q^+$. Therefore, by Ekeland's variational principle, we also have $\mathcal{J}'_q(u_n) \rightarrow 0$. Hence, we deduce from Lemma 2.9 that $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$ with $\|u\| \geq \widetilde{\mathcal{C}} > \rho$. Setting $u_q = u$ clearly we obtain that $u_q \in \mathcal{N}_q^+$ and $\mathcal{J}_q(u_q) = \widehat{\mathcal{J}}_q$. On account of the fact that $q > q_0^*$ and the definition of q_0^* , we deduce that $\mathcal{J}_q(u_q) > 0$. \square

(ii) *The mountain pass solution*

Now, for $q \in (q_0^*, q_0^* + \varepsilon)$, we select $\varepsilon > 0$ such that (5.1) holds for fixed $\delta > 0$. Owing to $\mathcal{J}_q(u_q) = \widehat{\mathcal{J}}_q$ for $u_q \in \mathcal{N}_q^+$ (by Corollary 5.3), we define

$$d_q := \inf_{\gamma \in \Gamma_q} \max_{t \in [0,1]} \mathcal{J}(\gamma(t)),$$

where

$$\Gamma_q = \{\gamma \in C([0, 1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_q\}.$$

Proposition 5.4. *For $q \in (q_0^*, q_0^* + \varepsilon)$, there exists $w_q \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $\mathcal{J}_q(w_q) = d_q$ and $\mathcal{J}'_q(w_q) = 0$. Particularly, $\mathcal{J}_q(w_q) > \mathcal{J}_q(u_q)$.*

Proof. From Proposition 2.3 and (5.1), for all $q > 0$, there exists $\rho > 0$ and $M > 0$ satisfying $\mathcal{J}_q(u) \geq M > \delta$ for every $u \in \mathcal{S}_\rho = \{u \in W_0^{1,p}(\Omega) : \|u\| = \rho\}$. Conversely, we know that $0 = \mathcal{J}_q(0) < \delta$ and $\mathcal{J}_q(u_q) = \widehat{\mathcal{J}}_q \leq \delta$. Then, we obtain a mountain pass geometry for the functional \mathcal{J}_q , which can be found in [20]. The proof follows from Lemma 2.9. \square

Now, we conclude the proof of Theorem 1.1 (2). The existence of the minimum critical point u_q is based on Proposition 3.2, Corollary 4.1, and Corollary 5.3. The existence of a mountain pass critical point w_q which satisfies $\mathcal{J}_q(w_q) > \max\{0, \mathcal{J}_q(u_q)\}$ is based on Propositions 3.3, 4.2, and 5.4. Actually, u_q and w_q are critical points of \mathcal{J}_q followed by Proposition 2.5, which correspond to the solutions of system (1.1).

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