DE GRUYTER Open Mathematics 2024; 22: 20240013

Research Article

Jing Wang and Ceyu Lei*

Complex dynamics of a nonlinear discrete predator-prey system with Allee effect

https://doi.org/10.1515/math-2024-0013
received February 5, 2024; accepted April 11, 2024

Abstract: The transition between strong and weak Allee effects in prey provides a simple regime shift in ecology. In this article, we study a discrete predator-prey system with Holling type II functional response and Allee effect. First, the number of fixed points of the system, local stability, and global stability is discussed. The population changes of predator and prey under strong or weak Allee effects are proved using the nullclines and direction field, respectively. Second, using the bifurcation theory, the bifurcation conditions for the system to undergo transcritical bifurcation and Neimark-Sacker bifurcation at the equilibrium point are obtained. Finally, the dynamic behavior of the system is analyzed by numerical simulation of bifurcation diagram, phase diagram, and maximum Lyapunov exponent diagram. The results show that the system will produce complex dynamic phenomena such as periodic state, quasi-periodic state, and chaos.

Keywords: discrete system, Allee effect, stability, transcritical bifurcation, Neimark-Sacker bifurcation

MSC 2020: 35K57, 37N25, 39A30, 92D25

1 Introduction

Population refers to a collection of the same kind of individuals living in a certain area at the same time. In nature, each population belongs to a certain level, and often, there are both predators at the upper level and preys at the lower level. As a result, in order to grasp the link between populations at all levels, assess and predict population persistence and extinction, a population model to explain the population system's evolution law must be established.

The general population model can be divided into two categories: differential equation model and difference equation model. In general, when the population number is relatively large or generations overlap, it can be described by differential equation. The difference equation is suitable for describing the population with long life cycle, few population, and non-overlapping generations. For example, herbs often flower in their first year and then die, roots and all, after setting seed, and a small fraction (<1%) of the 22,000 species of teleost fish are semelparous and die soon after spawning (see [1] for more examples). In 1976, May [2] showed that although the first-order difference equation model is simple, it can also show a series of surprising dynamic behaviors, such as from stable point to a bifurcation hierarchy of stable cycles, and finally produce chaos. However, in a continuous-time model, a minimum of three species are needed for exhibiting chaos [3]. It can be seen that the discrete model will show more interesting dynamic behavior. Therefore, the discrete dynamic system model has attracted the attention of many scholars [4–9]. AlSharawi et al. [10] considered the influence of vigilance of prey on dynamics of a discrete-time predator-prey system. They studied the stability, persistence, flip bifurcation, and Neimark-Sacker bifurcation of the discrete system. The results show that with the increase of prey vigilance,

*** Corresponding author: Ceyu Lei,** Department of Mathematics, Northwest Normal University, Lanzhou, 730070, P.R. China, e-mail: leiceyu@126.com
Jing Wang: School of Education, Lanzhou University of Arts and Science, Lanzhou 730010, P.R. China, e-mail: wangjing7723@163.com

the density of predator population continuously decreases and high vigilance will have a detrimental role for the prey population. Streipert et al. [11] deduced a discrete predator-prey model through the first principles in economics. They extended standard phase plane analysis by introducing the next iterate root-curve associated with the nontrivial prey nullcline. The stability of the system is proved by combining the curve with the nullclines and direction field. Finally, it is proved that the system will have a transcritical bifurcation at the boundary equilibrium point and a Neimark-Sacker bifurcation at the internal equilibrium point.

The Allee effect, a reduction of the per capita growth rate of a population of biological species at densities smaller than a critical value, was first introduced by Allee in 1931 [12]. Allee effects are mainly classified into two ways: strong and weak Allee effects. There is a critical value for the strong Allee effect. When the population density is less than this critical value, the population shows a negative growth trend, and then, the population will become extinct. When the population density is higher than this critical value, the population growth is positive, and then, it can develop. The weak Allee effect does not have a critical threshold. Its characteristic is that the individual growth rate at low density is lower than that at high density and always keeps a positive value, so the two populations can coexist permanently. Empirical evidence of Allee effects has been observed in many natural species, for example, plants [13,14], insects [15], marine invertebrates [16], and birds and mammals [17], etc. Therefore, many researchers have studied the bifurcation and stability analysis for discrete-time predator-prey system with the Allee effect [7,18–21]. However, in these articles, they did not consider the impact of strong and weak Allee effects on the local stability of the equilibrium point of the predation system, nor did they prove the global stability of the equilibrium point. So, in this article, we consider a discrete-time predator-prey model with Holling type II functional response and Allee effect in preys, which is given by

$$\begin{cases} x_{n+1} = x_n \exp\left[r(K - x_n)(x_n - A) - \dfrac{\beta y_n}{a + cx_n}\right], \\ y_{n+1} = y_n \exp\left[\dfrac{\alpha\beta x_n}{a + cx_n} - d\right], \end{cases} \tag{1}$$

where x and y represent the densities of prey and predator population, respectively, and $f(x) = \frac{\beta x}{a + cx}$ is the generalized Holling type II functional response. In System (1), r denotes the intrinsic growth rate of the prey population, K denotes the environmental carrying capacity of the prey, d is the predator's mortality rate, $0 < \alpha < 1$ represents the conversion efficiency of intake of prey into new predators, β denotes the maximal predator capita consumption rate, a represents the half-saturation constant, and c denotes the handling time. The parameters r, K, β, c, and d are assumed to be positive. The parameter A represents the threshold of multiplicative Allee effect. When $0 < A < K$, it indicates the the strong Allee effect; when $-K < A < 0$, it indicates the weak Allee effect. Note that $A = 0$ is the transition between the weak and strong Allee effects.

In this article, we study a discrete predator-prey system with Holling type-II functional response and Allee effect. There are several highlights in our analysis: (a) the influence of Allee effect parameter A on the local stability of equilibrium point is analyzed; (b) the conditions of global stability of discrete System (1) at the positive equilibrium point are obtained; (c) the population changes of predator and prey under strong or weak Allee effect are proved using the nullclines and direction field, respectively; and (d) the transcritical bifurcation and Neimark-Sacker bifurcation of System (1) are completely and rigorously analyzed.

This article is organized as follows. In Section 2, we analyze the dynamics of System (1), including the local stability of the equilibrium points, the global stability of the positive equilibrium point, and the bifurcation analysis at the equilibrium point. In Section 3, we verify our analytical results through numerical simulations. In Section 4, this article is ended with a brief conclusion.

2 Model dynamics

Lemma 1. *Solutions of System* (1) *with nonnegative initial conditions remain nonnegative. If* $x_0 = 0$, *then* $x_n = 0$ *for all* $n \geq 0$. *If* $y_0 = 0$, *then* $y_n = 0$ *for all* $n \geq 0$. *If* $x_0 > 0$ *and* $y_0 \geq 0$, *then* $x_n > 0$ *for all* $n \geq 0$. *If* $x_0 \geq 0$ *and* $y_0 > 0$, *then* $y_n > 0$ *for all* $n \geq 0$.

Proof. It can be directly proved by the model structure. □

Lemma 2.
(I) *System* (1) *always has a trivial equilibrium point* $E_0(0, 0)$.
(II) *System* (1) *always has a positive semi-trivial equilibrium point* $E_K(K, 0)$.
(III) *System* (1) *always has a semi-trivial equilibrium point* $E_A(A, 0)$.
(IV) *Under the strong Allee effect, if* $0 < A < \frac{ad}{\alpha\beta - cd} < K$, *then System* (1) *has a positive nontrivial equilibrium point* $E^*(x^*, y^*)$. *Under the weak Allee effect, if* $0 < \frac{ad}{\alpha\beta - cd} < K$, *then System* (1) *has a positive nontrivial equilibrium point* $E^*(x^*, y^*)$, *where* $x^* = \frac{ad}{\alpha\beta - cd}$ *and* $y^* = \frac{r}{\beta}(K - x^*)(x^* - A)(a + cx^*)$.

Proof. Direct computation. □

Lemma 3. *Consider System* (1) *with initial conditions* (x_0, y_0). *If* $x_0 = 0$ *and* $y_0 \geq 0$, *then* (x_n, y_n) *converges to* E_0.

Proof. Since $x_n = 0$ for all $n \geq 0$ if $x_0 = 0$. In this case, $y_{n+1} = \frac{1}{e^{dn}} y_0$. This converges to zero for $d > 0$. This completes the proof. □

The linearized form of System (1) is then

$$\begin{cases} x_{t+1} = f_x x_t + f_y y_t, \\ y_{t+1} = g_x x_t + g_y y_t, \end{cases} \tag{2}$$

which has the Jacobian matrix

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x,y)}, \tag{3}$$

where (x, y) denotes the fixed points of System (1).

To obtain the stable results of (2), we consider the algebraic equation

$$\lambda^2 - T\lambda + D = 0, \tag{4}$$

where

$$T = \mathrm{Tr}J = f_x + g_y \quad \text{and} \quad D = \mathrm{Det}J = f_x g_y - f_y g_x.$$

It is well-known that it has two roots of the form

$$\lambda_{1,2} = (T \pm \sqrt{T^2 - 4D})/2.$$

Lemma 4. [22] *The equilibrium point* (x, y) *is called*
(I) *sink* (*locally asymptotically stable*) *if* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$;
(II) *source* (*locally unstable*) *if* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$;
(III) *saddle if* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ (*or* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$);
(IV) *non-hyperbolic if* $|\lambda_1| = 1$ *or* $|\lambda_2| = 1$.

Lemma 5. [23] *Let* $E^*(x^*, y^*)$ *be the unique positive equilibrium point of System* (1), *then the following propositions hold*:
(1) *It is a sink if*

$$D < 1 \quad \textit{and} \quad |T| < D + 1.$$

(2) *It is a source if*

$$D > 1 \quad \textit{and} \quad |T| < D + 1, \quad \textit{or} \quad |T| > D + 1.$$

(3) *It is a saddle if*

$$0 < |T| + D + 1 < 2|T|.$$

(4) *It is non-hyperbolic if*

$$|T| = |D + 1|, \quad or \quad D = 1 \quad and \quad |T| \leq 2.$$

Jacobian matrix can be evaluated at $E_0(0, 0)$ as

$$J(E_0) = \begin{pmatrix} e^{-rAK} & 0 \\ 0 & e^{-d} \end{pmatrix}. \tag{5}$$

The eigenvalues of the Jacobian are $\lambda_1 = e^{-d}$ and $\lambda_2 = e^{-rAK}$ at trivial equilibrium point $E_0(0, 0)$. The results regarding dynamical behaviors are listed in Table 1.

From Table 1, we can obtain the following theorem.

Theorem 1. *$E_0(0, 0)$ is always locally asymptotically stable under strong Allee effect, while it is always unstable under weak Allee effect.*

The Jacobian matrix computed at $E_K(K, 0)$ is

$$J(E_K) = \begin{pmatrix} 1 + rAK - rK^2 & -\dfrac{\beta K}{a + cK} \\ 0 & \exp\left[\dfrac{\alpha\beta K}{a + cK} - d\right] \end{pmatrix}. \tag{6}$$

The eigenvalues of the Jacobian are $\lambda_1 = 1 + rAK - rK^2$ and $\lambda_2 = \exp[\frac{\alpha\beta K}{a + cK} - d]$. The properties of semi-trivial equilibrium point $E_K(K, 0)$ are summarized in Table 2.

Table 1: Properties of origin equilibrium point $E_0(0, 0)$

Conditions		**Eigenvalues**		**Properties**
		$\lambda_1 = e^{-d}$	$\lambda_2 = e^{-rAK}$	
$d > 0$	$0 < A < K$	$\lvert\lambda_1\rvert < 1$	$\lvert\lambda_2\rvert < 1$	Sink
	$-K < A < 0$		$\lvert\lambda_2\rvert > 1$	Saddle
	$A = 0$		$\lvert\lambda_2\rvert = 1$	Non-hyperbolic

Table 2: Properties of semi-trivial equilibrium point $E_K(K, 0)$

Conditions		**Eigenvalues**		**Properties**
		$\lambda_1 = 1 + rAK - rK^2$	$\lambda_2 = \exp[\frac{\alpha\beta K}{a + cK} - d]$	
$0 < rK(K-A) < 2$	$(\alpha\beta - cd)K = ad$	$\lvert\lambda_1\rvert < 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$(\alpha\beta - cd)K < ad$		$\lvert\lambda_2\rvert < 1$	Sink
	$(\alpha\beta - cd)K > ad$		$\lvert\lambda_2\rvert > 1$	Saddle
$rK(K-A) > 2$	$(\alpha\beta - cd)K = ad$	$\lvert\lambda_1\rvert > 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$(\alpha\beta - cd)K < ad$		$\lvert\lambda_2\rvert < 1$	Saddle
	$(\alpha\beta - cd)K > ad$		$\lvert\lambda_2\rvert > 1$	Source
$rK(K-A) = 2$	$(\alpha\beta - cd)K = ad$	$\lvert\lambda_1\rvert = 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$(\alpha\beta - cd)K < ad$		$\lvert\lambda_2\rvert < 1$	Non-hyperbolic
	$(\alpha\beta - cd)K > ad$		$\lvert\lambda_2\rvert > 1$	Non-hyperbolic

From Table 2, we can obtain the following theorem.

Theorem 2. *When* $0 < rK(K - A) < 2$ *and* $(\alpha\beta - cd)K < ad$ *are satisfied, the semi-trivial equilibrium point* $E_K(K, 0)$ *is locally asymptotically stable.*

Jacobian matrix can be evaluated at $E_A(A, 0)$ as

$$J(E_2) = \begin{pmatrix} 1 + rAK - rA^2 & -\dfrac{\beta A}{a + cA} \\ 0 & \exp\left[\dfrac{\alpha\beta A}{a + cA} - d\right] \end{pmatrix}. \tag{7}$$

The eigenvalues of the Jacobian are $\lambda_1 = 1 + rAK - rA^2$ and $\lambda_2 = \exp[\frac{\alpha\beta A}{a + cA} - d]$ at the semi-trivial equilibrium point $E_A(A, 0)$. The results regarding dynamical behaviors are listed in Table 3.

From Table 3, we can obtain the following theorem.

Theorem 3. *The semi-trivial equilibrium point* $E_A(A, 0)$ *is always unstable under the strong Allee effect. Under the weak Allee effect, when* $0 < rA(A - K) < 2$ *and* $\frac{\alpha\beta A}{a + cA} < d$ *are satisfied, it is locally asymptotically stable.*

$J(x, y)$ evaluated at the positive equilibrium point $E^*(x^*, y^*)$ is

$$J(E^*) = \begin{pmatrix} f_{x^*} & f_{y^*} \\ g_{x^*} & g_{y^*} \end{pmatrix} = \begin{pmatrix} 1 + rx^*(K + A) - 2rx^{*2} + \dfrac{c\beta x^* y^*}{(a + cx^*)^2} & -\dfrac{\beta x^*}{a + cx^*} \\ \dfrac{a\alpha\beta y^*}{(a + cx^*)^2} & 1 \end{pmatrix}. \tag{8}$$

Then characteristic equation of $J(E^*)$ is given by

$$\lambda^2 - \mathrm{Tr}J\lambda + \mathrm{Det}J = 0, \tag{9}$$

Table 3: Properties of semi-trivial equilibrium point $E_A(A, 0)$

Conditions		Eigenvalues		Properties
		$\lambda_1 = 1 + rAK - rA^2$	$\lambda_2 = \exp[\frac{\alpha\beta A}{a + cA} - d]$	
Strong Allee effect: $0 < A < K$				
$0 < A < K$	$\frac{\alpha\beta A}{a + cA} = d$	$\lvert\lambda_1\rvert > 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$\frac{\alpha\beta A}{a + cA} < d$		$\lvert\lambda_2\rvert < 1$	Saddle
	$\frac{\alpha\beta A}{a + cA} > d$		$\lvert\lambda_2\rvert > 1$	Source
Weak Allee effect: $-K < A < 0$				
$0 < rA(A - K) < 2$	$\frac{\alpha\beta A}{a + cA} = d$	$\lvert\lambda_1\rvert < 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$\frac{\alpha\beta A}{a + cA} < d$		$\lvert\lambda_2\rvert < 1$	Sink
	$\frac{\alpha\beta A}{a + cA} > d$		$\lvert\lambda_2\rvert > 1$	Saddle
$rA(A - K) > 2$	$\frac{\alpha\beta A}{a + cA} = d$	$\lvert\lambda_1\rvert > 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$\frac{\alpha\beta A}{a + cA} < d$		$\lvert\lambda_2\rvert < 1$	Saddle
	$\frac{\alpha\beta A}{a + cA} > d$		$\lvert\lambda_2\rvert > 1$	Source

where

$$T = \mathrm{Tr}J = 2 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2},$$

$$D = \mathrm{Det}J = 1 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} + \frac{a\alpha\beta^2 x^* y^*}{(a + cx^*)^3}.$$

In summary, we have the following theorem.

Theorem 4. *System* (1) *at the positive equilibrium point* $E^*(x^*, y^*)$ *is local asymptotically stable when the conditions*

$$4 + 2rx^*(K + A) - 4rx^{*2} + \frac{2c\beta x^* y^*}{(a + cx^*)^2} + \frac{a\alpha\beta^2 x^* y^*}{(a + cx^*)^3} > 0$$

and

$$rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} + \frac{a\alpha\beta^2 x^* y^*}{(a + cx^*)^3} < 0$$

hold.

Proof. It can be proven by Lemma 5. □

Theorem 5. *The positive equilibrium point* $E^*(x^*, y^*)$ *of System* (1) *is a global attractor if* $1 + r(K + A)x^* - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} \geq 0$.

Proof. Let $f(x, y) = x\exp[r(K - x)(x - A) - \frac{\beta y}{a + cx}]$, $g(x, y) = y\exp[\frac{\alpha\beta x}{a + cx} - d]$. Then, $f(x, y)$ is non-decreasing in x if $1 + r(K + A)x^* - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} \geq 0$ and non-increasing in y for all (x, y). Moreover, $g(x, y)$ is non-decreasing in both arguments x and y for all (x, y). Let (m_1, M_1, m_2, M_2) be a positive solution of the system

$$m_1 = f(m_1, M_2), \quad M_1 = f(M_1, m_2),$$

$$m_2 = g(m_1, m_2), \quad M_2 = g(M_1, M_2).$$

Then, we have

$$m_1 = m_1\exp\left[r(K - m_1)(m_1 - A) - \frac{\beta M_2}{a + cm_1}\right],$$

$$M_1 = M_1\exp\left[r(K - M_1)(M_1 - A) - \frac{\beta m_2}{a + cM_1}\right],$$

and

$$m_2 = m_2\exp\left[\frac{\alpha\beta m_1}{a + cm_1} - d\right], \quad M_2 = M_2\exp\left[\frac{\alpha\beta M_1}{a + cM_1} - d\right].$$

Through calculation, we can obtain

$$m_1 = M_1, m_2 = M_2.$$

Hence, the unique positive equilibrium point $E^*(x^*, y^*)$ is a global attractor if $1 + r(K + A)x^* - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} \geq 0$. □

From Theorems 4 and 5, we can obtain the following conclusions.

Theorem 6. *Assume that* $1 + r(K + A)x^* - 2rx^{*2} + \frac{c\beta x^* y^*}{(a+cx^*)^2} \geq 0$, $D < 1$, *and* $|T| < D + 1$. *Then, the unique positive equilibrium point* $E^*(x^*, y^*)$ *of System* (1) *is globally asymptotically stable.*

In fact, x_n is increasing if the forward operator, $\triangle x_n = x_{n+1} - x_n$, is positive. For System (1), the forward operators are

$$\triangle x_n = x_{n+1} - x_n = x_n \left\{ \exp\left[r(K - x_n)(x_n - A) - \frac{\beta y_n}{a + cx_n} \right] - 1 \right\} \tag{10}$$

and

$$\triangle y_n = y_{n+1} - y_n = y_n \left\{ \exp\left[\frac{\alpha\beta x_n}{a + cx_n} - d \right] - 1 \right\}. \tag{11}$$

From (10), we see that $x_n > 0$ is increasing if and only if $y_n < \frac{r}{\beta}(K - x_n)(x_n - A)(a + cx_n)$, since $x_{n+1} > 0$ for $x_n > 0$. Thus, if (x_n, y_n) is above the line

$$y = l(x) := \frac{r}{\beta}(K - x)(x - A)(a + cx), \tag{12}$$

$y_n > l(x_n)$, then the sequence of iterates, x_n is decreasing, and if $y_n < l(x_n)$, then x_n is increasing. Similarly, from (11), it follows that y_n is increasing as long as $x_n > \frac{ad}{\alpha\beta - cd}$ and decreasing if $x_n < \frac{ad}{\alpha\beta - cd}$.

By Lemma 1, solutions remain in the first quadrant for all nonnegative initial conditions. We divide the first quadrant into regions based on the component-wise monotonicity obtained by solving $x_{n+1} = x_n$ and $y_{n+1} = y_n$. The curves along which (1) satisfying $\triangle x_n = 0$ and $\triangle y_n = 0$ are given by the line

$$y_n = l(x_n) = \frac{r}{\beta}(K - x_n)(x_n - A)(a + cx_n) \quad \text{and} \quad x_n = \frac{ad}{\alpha\beta - cd}, \tag{13}$$

respectively. We refer to these curves as nullclines. These two curves divide the first quadrant into three or four regions R_i $(i = 1, 2, 3, 4)$.

We define the regions

$$R_1 := \left\{ (x_n, y_n) \in (0, +\infty)^2 : x_n > \frac{ad}{\alpha\beta - cd} \quad \text{and} \quad y_n \geq l(x_n) \right\},$$

$$R_2 := \left\{ (x_n, y_n) \in (0, +\infty)^2 : x_n \leq \frac{ad}{\alpha\beta - cd} \quad \text{and} \quad y_n > l(x_n) \right\},$$

$$R_3 := \left\{ (x_n, y_n) \in (0, +\infty)^2 : x_n < \frac{ad}{\alpha\beta - cd} \quad \text{and} \quad y_n \leq l(x_n) \right\},$$

$$R_4 := \left\{ (x_n, y_n) \in (0, +\infty)^2 : x_n \geq \frac{ad}{\alpha\beta - cd} \quad \text{and} \quad y_n < l(x_n) \right\}.$$

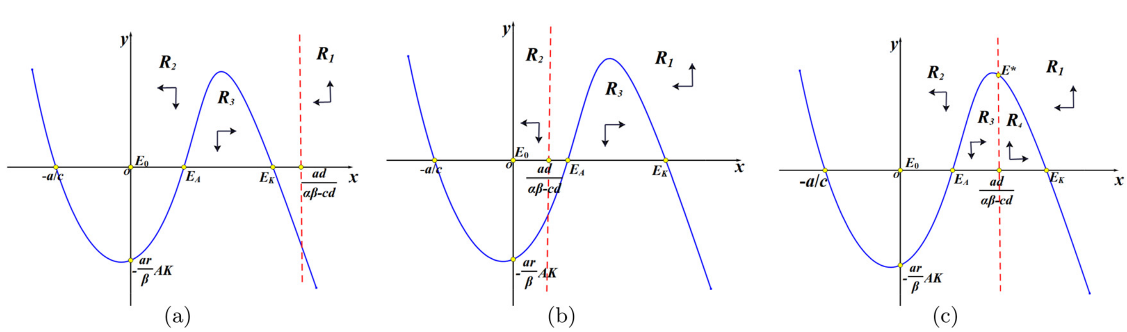

Figure 1: Phase diagram of System (1) when $0 < A < K$. The red dashed line corresponds to the predator nullclines, and the blue curves correspond to the prey nullclines. A horizontal arrow pointing to the right (left) represents $x_{n+1} - x_n > 0 (< 0)$, and a vertical arrow pointing up (down) represents $y_{n+1} - y_n > 0 (< 0)$. Subfigure (a) is a schematic image if $K < \frac{ad}{\alpha\beta - cd}$, (b) is a schematic image if $0 < \frac{ad}{\alpha\beta - cd} < A$, while (c) shows the case when $A < \frac{ad}{\alpha\beta - cd} < K$. Note that in (a) and (b), there is no region R_4 in the first quadrant.

In Figure 1, when $0 < A < K$, for $\frac{ad}{\alpha\beta - cd} > K$ or $0 < \frac{ad}{\alpha\beta - cd} < A$, $R_4 = \varnothing$, and $R_3 = \{(x_n, y_n) \in (0, +\infty)^2 : A < x_n < K \text{ and } y_n \leq l(x_n)\}$. It can be seen from Figure 1 that only when $A < \frac{ad}{\alpha\beta - cd} < K$, System (1) has a unique positive equilibrium point. By observing the arrow direction in the image, we can find that when the number of prey population is small, the prey will be extinct and the predators will also be extinct. This is due to $x_{n+1} - x_n < 0$ and $y_{n+1} - y_n < 0$, i.e., $x_{n+1} < x_n$ and $y_{n+1} < y_n$. Thus, the trivial equilibrium E_0 is stable under the strong Allee effect, which is consistent with the conclusion of Theorem 1.

In Figure 2, when $-K < A < 0$, for $\frac{ad}{\alpha\beta - cd} > K$, $R_4 = \varnothing$ and $R_3 = \{(x_n, y_n) \in (0, +\infty)^2 : 0 < x_n < K \text{ and } y_n \leq l(x_n)\}$. Similarly, we can find that when the prey population is small, under the influence of the weak Allee effect, the prey population will not be extinct, and the population will increase. However, the number of predator population will gradually decrease and become extinct. This is different from the strong Allee effect. And by observing Figures 1 and 2, it can be found that when $x > K$, the growth rate of the prey population shows a negative growth, which is consistent with the biological significance.

Theorem 7. *If $A = 0$, then $E_0 = E_A$ and System* (1) *undergoes a transcritical bifurcation at the trivial equilibrium point E_0.*

Proof. For $A = 0$, the equilibria E_0 and E_A coalesce. The Jacobian evaluated at E_0 given in (5) has eigenvalues $\lambda_1 = e^{-d}$ and $\lambda_2 = 1$, the trivial equilibrium point E_0 is non-hyperbolic. It can be concluded that the central manifold of the map $x \to x\exp[rx(K - x)]$. According to the literature [22], it manifests that System (1) exhibits a transcritical bifurcation at E_0. □

Theorem 8. *If $\frac{ad}{\alpha\beta - cd} = K$, then $E_K = E^*$ and System* (1) *undergoes a transcritical bifurcation at the semi-trivial equilibrium point E_K.*

Proof. For $\frac{ad}{\alpha\beta - cd} = K$, the equilibria E_K and E^* coalesce. The Jacobian evaluated at E_K given in (5) has eigenvalues $\lambda_1 = 1 + rAK - rA^2$ and $\lambda_2 = 1$, and the trivial equilibrium point E_K is non-hyperbolic. Similarly, according to the literature [23], it manifests that System (1) exhibits a transcritical bifurcation at E_K. □

Theorem 9. *If $0 < \frac{ad}{\alpha\beta - cd} = A$, then $E_A = E^*$ and System* (1) *undergoes a transcritical bifurcation at the semi-trivial equilibrium point E_A.*

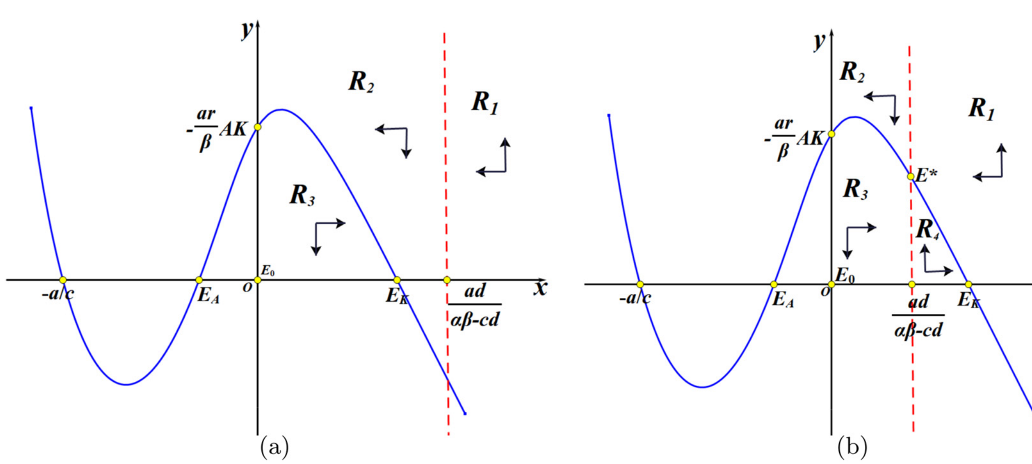

Figure 2: Phase diagram of System (1) when $-K < A < 0$. The red dashed line corresponds to the predator nullclines, and the blue curves correspond to the prey nullclines. A horizontal arrow pointing to the right (left) represents $x_{n+1}-x_n > 0(<0)$, and a vertical arrow pointing up (down) represents $y_{n+1}-y_n > 0(<0)$. Subfigure (a) is a schematic image if $K < \frac{ad}{\alpha\beta - cd}$, while (b) shows the case when $0 < \frac{ad}{\alpha\beta - cd} < K$. Note that in (a), there is no region R_4 in the first quadrant.

Proof. For $\frac{ad}{\alpha\beta - cd} = A$, the equilibria E_A and E^* coalesce. The Jacobian evaluated at E_A given in (5) has eigenvalues $\lambda_1 = 1 + rAK - rK^2$ and $\lambda_2 = 1$, and the trivial equilibrium point E_A is non-hyperbolic. Similarly, according to the literature [23], it manifests that System (1) exhibits a transcritical bifurcation at E_A. □

Theorem 10. *The interior equilibrium point E^* loses its stability via Neimark-Sacker bifurcation if*

$$rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} + \frac{a\alpha\beta^2 x^* y^*}{(a + cx^*)^3} = 0. \tag{14}$$

Proof. Neimark-Sacker bifurcation occurs in the system when a pair of complex eigenvalues with unit modulus [24], i.e.,

$$\mathrm{Det}(J) = 1.$$

It is obtained

$$1 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} + \frac{a\alpha\beta^2 x^* y^*}{(a + cx^*)^3} = 1.$$

This gives the condition for the Neimark-Sacker bifurcation. □

Now, we discuss the Neimark-Sacker bifurcation of the equilibrium point $E^*(x^*, y^*)$. Here, we choose α as a bifurcation parameter. Neimark-Sacker bifurcation in a discrete system is the birth of a closed invariant curve from an equilibrium point. The bifurcation can be supercritical when closed invariant curve is stable and subcritical, when it is unstable.

The characteristic polynomial (8) of Jacobian matrix of linearized system of (1) about the positive equilibrium point (x^*, y^*) can be rewritten as:

$$\mathbb{F}(\lambda) = \lambda^2 - p(x^*, y^*)\lambda + q(x^*, y^*), \tag{15}$$

where

$$p(x^*, y^*) = 2 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2},$$
$$q(x^*, y^*) = 1 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} + \frac{a\alpha\beta^2 x^* y^*}{(a + cx^*)^3}.$$

Consider (15), then $\mathbb{F}(\lambda) = 0$ has two complex conjugate roots with modulus one if the following conditions are satisfied:

$$1 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2} + \frac{a\alpha\beta^2 x^* y^*}{(a + cx^*)^3} = 1$$

and

$$|T| = |2 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^* y^*}{(a + cx^*)^2}| < 2. \tag{16}$$

Let

$$\Omega_{NS} = \{(r, K, A, a, c, \beta, \alpha, d) \in \mathbb{R}_+^8 : (14) \text{ and } (16) \text{ are satisfied}\}.$$

Then, the dynamic analysis of System (1) is analyzed when the parameters change in the small field of Ω_{NS}. Select parameter $(r, K, A, a, c, \beta, \alpha, d) \in \Omega_{NS}$, and consider the following system:

$$\begin{cases} x_{n+1} = x_n \exp\left[r(K - x_n)(x_n - A) - \dfrac{\beta y_n}{a + cx_n}\right], \\ y_{n+1} = y_n \exp[\dfrac{(\alpha + \alpha^*)\beta x_n}{a + cx_n} - d], \end{cases}$$

where $|\alpha^*| \ll 1$ is a small perturbation parameter.

Let $u = x - x^*$, $v = y - y^*$, then we obtain

$$\begin{cases} u_{n+1} = A_{11}u + A_{12}v + A_{13}u^2 + A_{14}uv + A_{15}v^2 + A_{16}u^3 + A_{17}u^2v + A_{18}uv^2 + A_{19}v^3 + O((|u| + |v|)^4), \\ v_{n+1} = A_{21}u + A_{22}v + A_{23}u^2 + A_{24}uv + A_{25}v^2 + A_{26}u^3 + A_{27}u^2v + A_{28}uv^2 + A_{29}v^3 + O((|u| + |v|)^4), \end{cases} \tag{17}$$

where

$$A_{11} = 1 + x^*\Phi_1, \quad A_{12} = -\frac{\beta x^*}{a + cx^*}, \quad \Phi_1 = r(K + A) - 2rx^* + \frac{c\beta y^*}{(a + cx^*)^2},$$
$$A_{13} = \frac{1}{2}r(K + A) - 2rx^* + \frac{c\beta y^*(a - cx^*)}{2(a + cx^*)^3} + \frac{1}{2}A_{11}\Phi_1, \quad A_{14} = \frac{c\beta x^*}{(a + cx^*)^2} - \frac{\beta A_{11}}{a + cx^*}, \quad A_{15} = \frac{\beta^2 x^*}{2(a + cx^*)^2},$$
$$A_{16} = \frac{1}{2}\left[-4r + \frac{2c^3\beta x^*y^* - 4ac^2\beta y^*}{(a + cx^*)^3} + \Phi_1\left[r(K + A) - 4rx^* + \frac{c\beta y^*(a - cx^*)}{(a + cx^*)^3}\right] - A_{11}\left(2r + \frac{2c^2\beta y^*}{(a + cx^*)^3}\right) + 2A_{13}\Phi_1\right],$$
$$A_{17} = \frac{1}{2}\left[\frac{\beta y^*(a - cx^*)}{(a + cx^*)^3} + \frac{c\beta x^*\Phi_1}{(a + cx^*)^2} + \frac{c\beta A_{11}}{(a + cx^*)} - \frac{2\beta A_{13}}{a + cx^*}\right],$$
$$A_{18} = -\frac{c\beta^2 x^*}{2(a + cx^*)^3} - \frac{\beta A_{14}}{2(a + cx^*)}, \quad A_{19} = -\frac{\beta^3 x^*}{6(a + cx^*)^3}, \quad A_{21} = \frac{a\alpha\beta y^*}{(a + cx^*)^2}, \quad A_{22} = 1,$$
$$A_{23} = \frac{a^2\alpha^2\beta^2 y^*}{2(a + cx^*)^4} - \frac{ac\alpha\beta y^*}{(a + cx^*)^3}, \quad A_{24} = \frac{a\alpha\beta}{(a + cx^*)^2}, \quad A_{25} = A_{28} = A_{29} = 0,$$
$$A_{26} = \frac{ac^2\alpha\beta y^*}{(a + cx^*)^4} + \frac{a\alpha\beta A_{23}}{3(a + cx^*)^2} - \frac{2a^2c\alpha^2\beta^2 y^*}{3(a + cx^*)^5}, \quad A_{27} = \frac{a^2\alpha^2\beta^2}{2(a + cx^*)^4} - \frac{ac\alpha\beta}{(a + cx^*)^3}.$$

The characteristic equation of System (17) at $(u, v) = (0, 0)$ is as follows:

$$\lambda^2 - p(\alpha^*)\lambda + q(\alpha^*) = 0,$$

where

$$p(\alpha^*) = 2 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^*y^*}{(a + cx^*)^2},$$
$$q(\alpha^*) = 1 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^*y^*}{(a + cx^*)^2} + \frac{a\alpha^*\beta^2 x^*y^*}{(a + cx^*)^3}.$$

Since parameters $(r, K, A, a, c, \beta, \alpha, d) \in \Omega_{NS}$, the roots of the characteristic equation are

$$\lambda_{1,2} = -\frac{p(\alpha^*)}{2} \pm \frac{i}{2}\sqrt{4q(\alpha^*) - p^2(\alpha^*)},$$

we have

$$|\lambda_{1,2}| = \sqrt{q(\alpha^*)}$$

and

$$L = \frac{d|\lambda_{1,2}|}{d\alpha^*}\bigg|_{\alpha^*=0} \neq 0.$$

In addition, it is required that $\alpha^* = 0, \lambda_{1,2}^j \neq 1$ $(j = 1, 2, 3, 4)$, which is equivalent to $p(0) \neq -2,\ 0,\ 1,\ 2$. Because $(r, K, A, a, c, \beta, \alpha, d) \in \Omega_{NS}$, $p(0) \neq -2, 2$. We only require $p(0) \neq 0, 1$, so that

$$2 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^*y^*}{(a + cx^*)^2} \neq 0 \quad \text{and} \quad 2 + rx^*(K + A) - 2rx^{*2} + \frac{c\beta x^*y^*}{(a + cx^*)^2} \neq 1. \tag{18}$$

Therefore, eigenvalues λ_1 and λ_2 of the equilibrium point (0,0) of System (17) do not lay in the intersection of the unit circle with the coordinate axes when $\alpha^* = 0$ and the condition (18) holds.

Let $\eta = -\frac{p(0)}{2}$, $\omega = \frac{\sqrt{4q(0) - p^2(0)}}{2}$, we use the following transformation:

$$\begin{bmatrix} u \\ v \end{bmatrix} = T' \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} A_{12} & 0 \\ \eta - A_{11} & -\omega \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix},$$

and System (17) becomes

$$\begin{bmatrix} W_{t+1} \\ Z_{t+1} \end{bmatrix} = \begin{bmatrix} \eta & -\omega \\ \omega & \eta \end{bmatrix} \begin{bmatrix} W_t \\ Z_t \end{bmatrix} + \begin{bmatrix} \overline{f}(W, Z) \\ \overline{g}(W, Z) \end{bmatrix}, \tag{19}$$

where

$$\begin{aligned}
\overline{f}(W, Z) &= \frac{A_{13}}{A_{12}} u^2 + \frac{A_{14}}{A_{12}} uv + \frac{A_{15}}{A_{12}} v^2 + \frac{A_{16}}{A_{12}} u^3 + \frac{A_{17}}{A_{12}} u^2 v + \frac{A_{18}}{A_{12}} uv^2 + \frac{A_{19}}{A_{12}} v^3 + O((|W| + |Z|)^4), \\
\overline{g}(W, Z) &= \left[\frac{A_{13}(\eta - A_{11})}{\omega A_{12}} - \frac{A_{23}}{\omega}\right] u^2 + \left[\frac{A_{14}(\eta - A_{11})}{\omega A_{12}} - \frac{A_{24}}{\omega}\right] uv + \left[\frac{A_{15}(\eta - A_{11})}{\omega A_{12}} - \frac{A_{25}}{\omega}\right] v^2 \\
&\quad + \left[\frac{A_{16}(\eta - A_{11})}{\omega A_{12}} - \frac{A_{26}}{\omega}\right] u^3 + \left[\frac{A_{17}(\eta - A_{11})}{\omega A_{12}} - \frac{A_{27}}{\omega}\right] u^2 v + \left[\frac{A_{18}(\eta - A_{11})}{\omega A_{12}} - \frac{A_{28}}{\omega}\right] uv^2 \\
&\quad + \left[\frac{A_{19}(\eta - A_{11})}{\omega A_{12}} - \frac{A_{29}}{\omega}\right] v^3 + O((|W| + |Z|)^4), \\
u &= A_{12} W, \quad v = (\eta - A_{11}) W - \omega Z.
\end{aligned}$$

System (17) undergoes the Neimark-Sacker bifurcation if the following quantity is not zero:

$$\mathfrak{L} = -\mathrm{Re}\left[\frac{(1 - 2\lambda_1)\lambda_2^2}{1 - \lambda_1} L_{11} L_{12}\right] - \frac{1}{2} |L_{11}|^2 - |L_{21}|^2 + \mathrm{Re}(\lambda_2 L_{22}), \tag{20}$$

where

$$\begin{aligned}
L_{11} &= \frac{1}{4}[(\overline{f}_{WW} + \overline{f}_{ZZ}) + i(\overline{g}_{WW} + \overline{g}_{ZZ})], \\
L_{12} &= \frac{1}{8}[(\overline{f}_{WW} - \overline{f}_{ZZ} + 2\overline{g}_{WZ}) + i(\overline{g}_{WW} - \overline{g}_{ZZ} - 2\overline{f}_{WZ})], \\
L_{21} &= \frac{1}{8}[(\overline{f}_{WW} - \overline{f}_{ZZ} - 2\overline{g}_{WZ}) + i(\overline{g}_{WW} - \overline{g}_{ZZ} + 2\overline{f}_{WZ})], \\
L_{22} &= \frac{1}{16}[(\overline{f}_{WWW} + \overline{f}_{ZZZ} + \overline{g}_{WWZ} + \overline{g}_{ZZZ}) + i(\overline{g}_{WWW} + \overline{g}_{WZZ} - \overline{f}_{WWZ} - \overline{f}_{ZZZ})].
\end{aligned}$$

If $\mathfrak{L} \neq 0$, the Neimark-Sacker bifurcation will occur in System (1), and the following theorem holds:

Theorem 11. *System* (1) *undergoes a Neimark-Sacker bifurcation at the positive equilibrium point* $E^*(x^*, y^*)$ *if conditions in* (18) *are satisfied and* $\mathfrak{L} \neq 0$ *in* (20). *Moreover, if* $\mathfrak{L} < 0$ (*resp.*, $\mathfrak{L} > 0$), *an attracting* (*resp., repelling*) *invariant closed curve bifurcates from the steady state for* $\alpha > \alpha^*$ (*resp.*, $\alpha < \alpha^*$).

Remark 1. When the system has the Neimark-Sacker bifurcation, it will dispose of the expenditure invariant curve from the fixed point, which indicates that prey and predator can coexist, and the dynamic behavior can be periodic or quasi-periodic.

3 Numerical simulations

This section will show the bifurcation diagram, phase diagram, and maximum Lyapunov exponent diagram with the Allee effect model to verify the correctness of theoretical analysis.

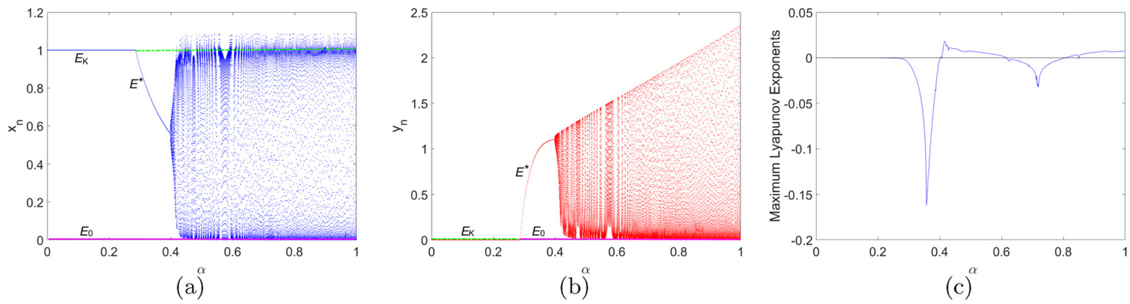

Figure 3: Neimark-Sacker bifurcation diagram and maximum Lyapunov exponent diagram for System (1).

3.1 Weak Allee effect

Assuming that the parameter is $r = 1.7$, $K = 1$, $A = -0.1$, $\beta = 0.7$, $a = 1$, $c = 1$, and $d = 0.1$, α is the bifurcation parameter, and the initial value of the system is $(x_0, y_0) = (0.8, 0.5)$. Lemma 2 shows that when $\alpha < 0.2857$, the coexistence equilibrium point does not exist; when $\alpha > 0.2857$, there is a unique coexistence equilibrium point. It can be seen from Theorem 8 and Figures 3(a) and (b) when $\alpha = 0.2857$, System (1) has a transcritical bifurcation at the boundary equilibrium point E_K. And according to Theorem 10, when $\alpha = 0.3981$, System (1) will have a supercritical Neimark-Sacker bifurcation at the coexistence equilibrium point. The bifurcation diagram and the maximum Lyapunov exponent diagram are shown in Figure 3. Combined with Figures 3 and 4, when $\alpha < 0.3981$, the fixed point is stable. When $\alpha = 0.39$, the phase and evolution diagrams of predator and prey with time are given in Figures 4(b) and 5(a). It can be seen from Theorem 6 that the coexistence equilibrium points of System (1) are globally asymptotically stable. However, when $\alpha > 0.3981$, the coexistence equilibrium loses its stability, and a stable invariant loop appears. At this time, the periodic solution of System (1) appears (Figures 4(c) and (d) and 5(b)). When α increases, it can be seen from Figures 3(c), 4(e) and (f), and 5(c) that System (1) will have quasi-periodic solutions and chaos.

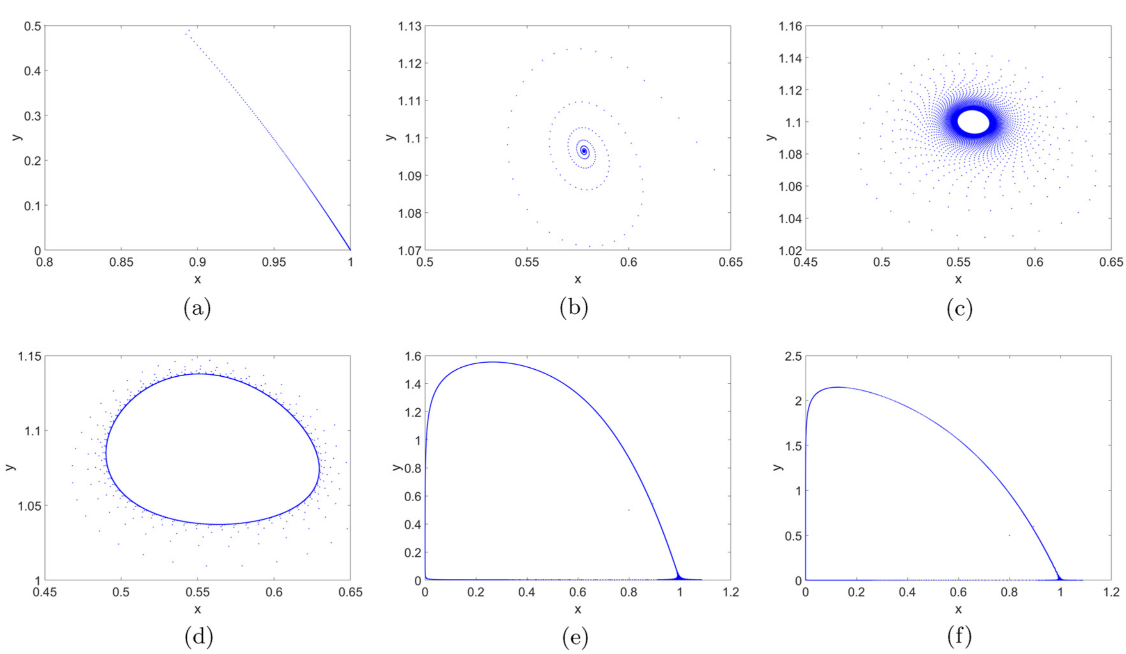

Figure 4: Phase diagram corresponding to Figure 3 when α takes different values: (a) $\alpha = 0.25$, (b) $\alpha = 0.39$, (c) $\alpha = 0.3981$, (d) $\alpha = 0.4$, (e) $\alpha = 0.6$, and (f) $\alpha = 0.9$.

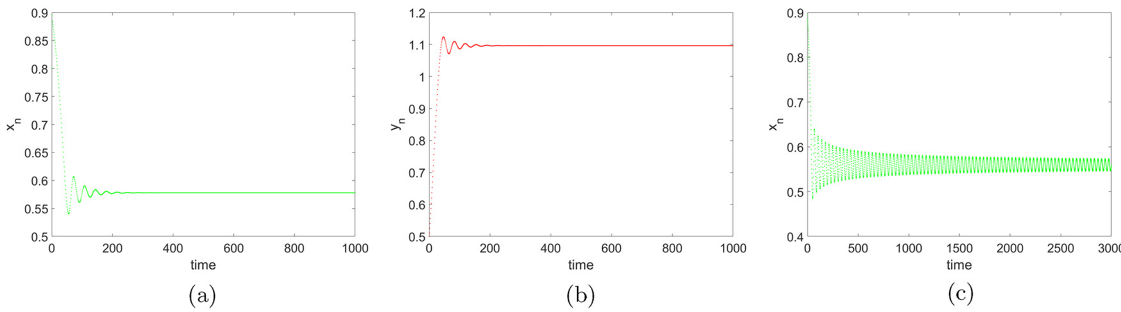

Figure 5: Evolution of predator and prey population with time: (a) $\alpha = 0.39$, (b) $\alpha = 0.39$, and (c) $\alpha = 0.6$.

3.2 Strong Allee effect

When $A = 0.1$ and other parameters remain unchanged, it can also be obtained that when $\alpha < 0.2857$, the coexistence equilibrium point does not exist; when $\alpha > 0.2857$, there is a unique coexistence equilibrium point, and when $\alpha = 0.2857$, System (1) has a transcritical bifurcation at the boundary equilibrium point E_K. According to Theorem 10, when $\alpha = 0.3729$, System (1) has the Neimark-Sacker bifurcation at the coexistence equilibrium point. Figure 6(a) and (b) is the bifurcation graph of α on [0, 1], and Figure 6(c) is the maximum Lyapunov exponent graph corresponding to Figure 6(a). Figure 6(d–f) is a locally enlarged view of Figure 6(a–c). It can be seen from Figures 6 and 7 that when $\alpha < 0.3729$, the fixed point is stable; when $\alpha > 0.3729$, the fixed point loses its stability and a stable invariant loop appears. At this time, System (1) generates a periodic solution (Figures 7(d) and 8(b)). When α increases, System (1) produces quasi-periodic solutions and chaotic phenomena. However, when we continue to increase α, the population will become extinct (Figure 8(c)). Combining Theorem 6 and Figure 8(a), when $0.2875 < \alpha < 0.3729$, the coexistence equilibrium point of System (1) exists and is globally asymptotically stable.

It can be found from Figures 3 and 6 that under the weak Allee effect, the growth rate of the population at low density is always positive and the population will not become extinct. Under the strong Allee effect, the population will become extinct at low density. This is consistent with the conclusion obtained from Figures 1 and 2.

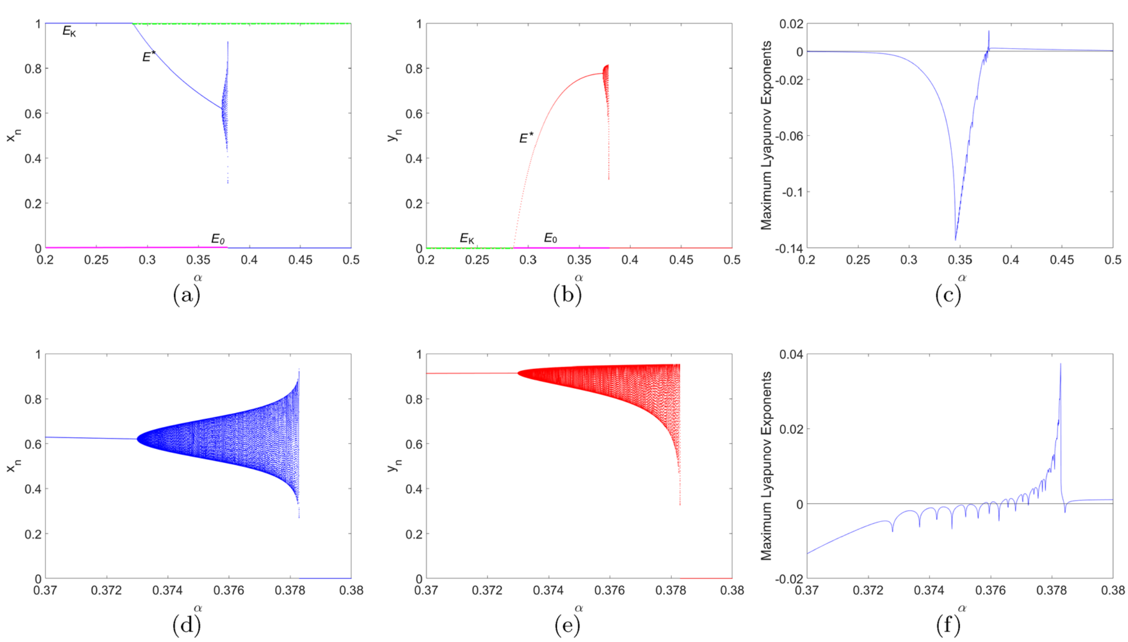

Figure 6: Neimark-Sacker bifurcation diagram and maximum Lyapunov exponent diagram for System (1).

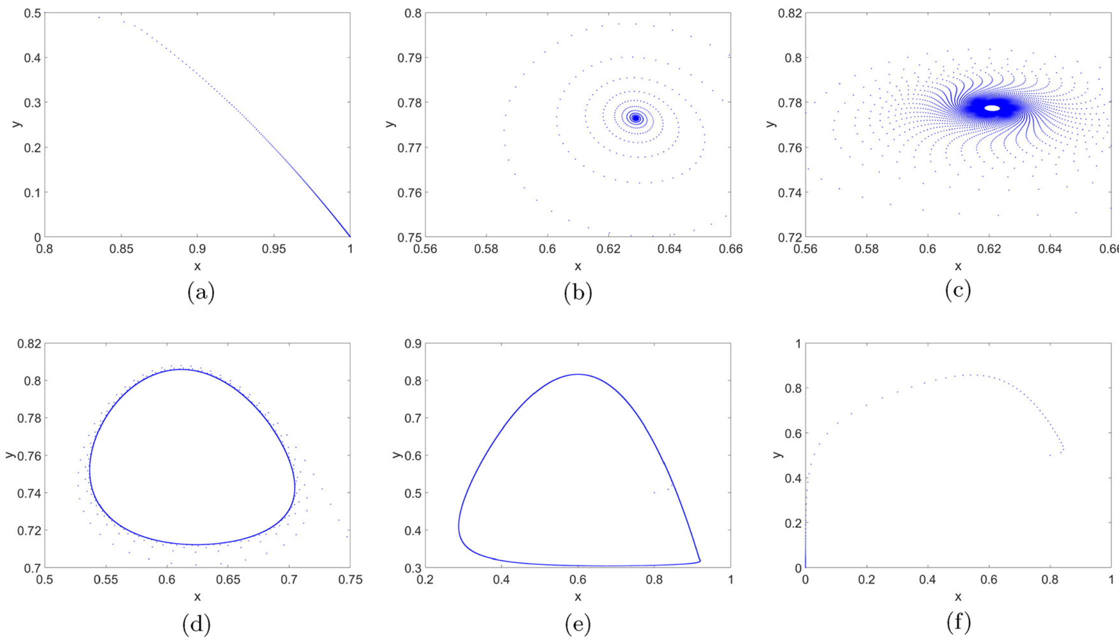

Figure 7: Phase diagram corresponding to Figure 6 when α takes different values: (a) $\alpha = 0.25$, (b) $\alpha = 0.37$, (c) $\alpha = 0.3729$, (d) $\alpha = 0.375$, (e) $\alpha = 0.379$, and (f) $\alpha = 0.4$.

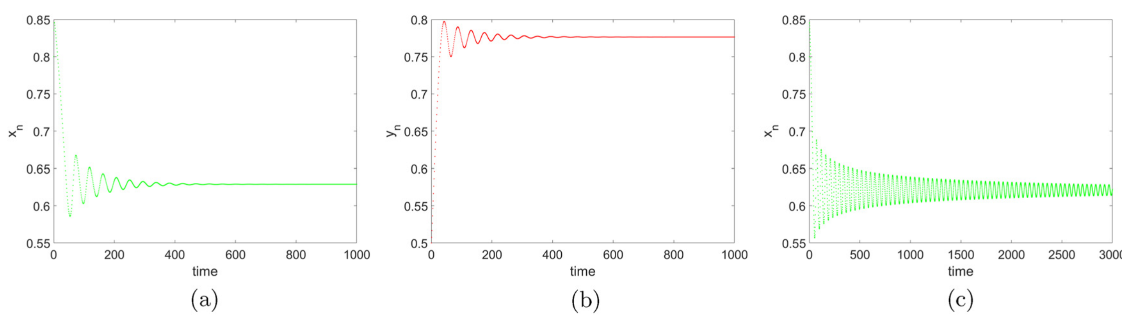

Figure 8: Evolution of predator and prey population with time. (a) $\alpha = 0.37$, (b) $\alpha = 0.375$, and (c) $\alpha = 0.4$.

4 Conclusion

In this article, we study the stability and bifurcation of equilibrium points in a discrete predator-prey model with the Allee effect. The stability analysis shows that the model with strong Allee effect has two positive boundary equilibrium points, and the boundary equilibrium point corresponding to the Allee effect threshold is always unstable. Under the strong Allee effect, when $0 < A < \frac{ad}{\alpha\beta - cd} < K$, the discrete System (1) has a unique coexistence equilibrium, and when the population density is low, the predator and prey populations will be extinct. Under the weak Allee effect, when $0 < \frac{ad}{\alpha\beta - cd} < K$, the discrete System (1) has a unique coexistence equilibrium, and when the population density is low, the growth rate of the prey population always remains positive, and the population will not be extinct. The bifurcation analysis shows that when $\frac{ad}{\alpha\beta - cd} = K$, the boundary equilibrium point E_K will have a transcritical bifurcation, and when the coexistence equilibrium E^* exists and loses stability, System (1) will have a Neimark-Sacker bifurcation. The numerical simulation reveals that when the energy conversion rate of predator α increases gradually, System (1) will produce periodic, quasi-periodic windows, and chaos.

Funding information: This work was supported by Gansu Province University Teacher Innovation Fund (No. 2023A-178) and Graduate Research Support project of Northwest Normal University (No. 2023KYZZ-B051).

Author contributions: JW: conceptualization, methodology, investigation, supervision, writing – review and editing, visualization. CL: conceptualization, methodology, software, formal analysis, investigation, writing – original draft.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] M. Kot, *Elements of Mathematical Ecology*, Cambridge University Press, New York, 2001.

[2] R. M. May, *Simple mathematical models with very complicated dynamics*, Nature **261** (1976), no. 5560, 459–467, DOI: https://doi.org/10.1038/261459a0.

[3] A. Hastings and T. Powell, *Chaos in three-species food chain*, Ecology **72** (1991), no. 3, 896–903, DOI: https://doi.org/10.2307/1940591.

[4] A. Q. Khan and T. Khalique, *Neimark-Sacker bifurcation and hybrid control in a discrete-time Lotka-Volterra model*, Math. Methods Appl. Sci. **43** (2020), no. 9, 5887–5904, DOI: https://doi.org/10.1002/mma.6331.

[5] H. N. Agiza, E. M. Elabbasy, H. EL-Metwally, and A. A. Elsadany, *Chaotic dynamics of a discrete prey-predator model with Holling type II*, Nonlinear Anal. Real World Appl. **10** (2009), no. 1, 116–129, DOI: https://doi.org/10.1016/j.nonrwa.2007.08.029.

[6] E. M. Elabbasy, A. A. Elsadany, and Y. Zhang, *Bifurcation analysis and chaos in a discrete reduced Lorenz system*, Appl. Math. Comput. **228** (2014), 184–194, DOI: https://doi.org/10.1016/j.amc.2013.11.088.

[7] L. Cheng and H. Cao, *Bifurcation analysis of a discrete-time ratio-dependent predator-prey model with Allee effect*, Commun. Nonlinear Sci. Numer. Simul. **38** (2016), 288–302, DOI: https://doi.org/10.1016/j.cnsns.2016.02.038.

[8] Q. Din, *Complexity and chaos control in a discrete-time prey-predator model*, Commun. Nonlinear Sci. Numer. Simul. **49** (2017), 113–134, DOI: https://doi.org/10.1016/j.cnsns.2017.01.025.

[9] C. Lei, X. Han, and W. Wang, *Bifurcation analysis and chaos control of a discrete-time prey-predator model with fear factor*, Math. Biosci. Eng. **19** (2022), no. 7, 6659–6679, DOI: https://doi.org/10.3934/mbe.2022313.

[10] Z. AlSharawi, N. Pal, and J. Chattopadhyay, *The role of vigilance on a discrete-time predator-prey model*, Discrete Contin. Dyn. Syst. Ser. B **27** (2022), no. 11, 6723–6744, DOI: https://doi.org/10.3934/dcdsb.2022017.

[11] S. H. Streipert, G. S. K. Wolkowicz, and M. Bohner, *Derivation and analysis of a discrete predator-prey model*, Bull. Math. Biol. **84** (2022), no. 7, 67, DOI: https://doi.org/10.1007/s11538-022-01016-4.

[12] W. C. Allee, *Animal Aggregations: A Study in General Sociology*, University of Chicago Press, Chicago, 1931.

[13] J.-B. Ferdy, F. Austerlitz, J. Moret, P.-H. Gouyon, and B. Godelle, *Pollinator-induced density dependence in deceptive species*, Oikos **87** (1999), no. 3, 549–560, DOI: https://doi.org/10.2307/3546819.

[14] M. J. Groom, *Allee effects limit population viability of an annual plant*, Am. Nat. **151** (1998), no. 6, 487–496, DOI: https://doi.org/10.1086/286135.

[15] M. Kuussaari, I. Saccheri, M. Camara, and I. Hanski, *Allee effect and population dynamics in the Glanville fritillary butterfly*, Oikos **82** (1998), no. 2, 384–392, DOI: https://doi.org/10.2307/3546980.

[16] A. W. Stoner and M. Ray-Culp, *Evidence for Allee effects in an over-harvested marine gastropod: density-dependent mating and egg production*, Mar. Ecol. Prog. Ser. **202** (2000), 297–302, DOI: https://doi.org/10.3354/meps202297.

[17] F. Courchamp, B. T. Grenfell, and T. H. Clutton-Brock, *Impact of natural enemies on obligately cooperative breeders*, Oikos **91** (2000), no. 2, 311–322, DOI: https://doi.org/10.1034/j.1600-0706.2000.910212.x.

[18] S. Pal, S. K. Sasmal, and N. Pal, *Chaos control in a discrete-time predator-prey model with weak Allee effect*, Int. J. Biomath. **11** (2018), no. 7, 1850089, DOI: https://doi.org/10.1142/S1793524518500894.

[19] S. Işik, *A study of stability and bifurcation analysis in discrete-time predator-prey system involving the Allee effect*, Int. J. Biomath. **12** (2019), no. 1, 1950011, DOI: https://doi.org/10.1142/S1793524519500116.

[20] P. Chakraborty, S. Sarkar, and U. Ghosh, *Stability and bifurcation analysis of a discrete prey-predator model with sigmoid functional response and Allee effect*, Rend. Circ. Mat. Palermo (2) **70** (2021), no. 1, 253–273, DOI: https://doi.org/10.1007/s12215-020-00495-5.

[21] L. Zhang and L. Zou, *Bifurcations and control in a discrete predator-prey model with strong Allee effect*, Int. J. Bifurcat. Chaos **28** (2018), no. 5, 1850062, DOI: https://doi.org/10.1142/S0218127418500621.

[22] L.-G. Yuan and Q.-G. Yang, *Bifurcation, invariant curve and hybrid control in a discrete-time predator-prey system*, Appl. Math. Model. **39** (2015), no. 8, 2345–2362, DOI: https://doi.org/10.1016/j.apm.2014.10.040.
[23] N. Britton, *Essential Mathematical Biology*, Springer, London, 2003.
[24] S. N. Elaydi, *Discrete Chaos: With Applications in Science and Engineering*, Chapman and Hall/CRC, London, 2007.