Research Article

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Scattering threshold for the focusing energy-critical generalized Hartree equation

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Abstract: This work investigates the asymptotic behavior of energy solutions to the focusing nonlinear Schrödinger equation of Choquard type

$$i\partial_t u + \Delta u + |u|^{p-2}(I_\alpha * |u|^p)u = 0, \quad p = 1 + \frac{2+\alpha}{N-2}, \quad N \geq 3.$$

Indeed, in the energy-critical spherically symmetric regime, one proves a global existence and scattering versus finite time blow-up dichotomy. Precisely, if the data have an energy less than the ground state one, two cases are possible. If the kinetic energy of the radial data is less than the ground state one, then the solution is global and scatters. Otherwise, if the data have a finite variance or is spherically symmetric and have a finite mass, then the solution is nonglobal. The main difficulty is to deal with the nonlocal source term. The argument is the concentration-compactness-rigidity method introduced by Kenig and Merle (*Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), no. 3, 645–675). This note naturally complements the work by Saanouni (*Scattering theory for a class of defocusing energy-critical Choquard equations*, J. Evol. Equ. **21** (2021), 1551–1571), where the scattering of the defocusing energy-critical generalized Hartree equation was obtained.

Keywords: energy-critical Hartree equation, nonlinear equations, scattering, blow-up

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1 Introduction

This article studies the Cauchy problem for an energy-critical focusing generalized Hartree equation

$$\begin{cases} i\partial_t u + \Delta u + |u|^{p-2}(I_\alpha * |u|^p)u = 0; \\ u_{|t=0} = u_0. \end{cases} \tag{NLS}$$

In this note, the space dimension is $N \geq 3$, the wave function is $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$. Moreover, $0 < \alpha < N$ and the Riesz-potential stands for

$$I_\alpha(x) := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^\alpha\,|x|^{N-\alpha}}, \quad 0 \neq x \in \mathbb{R}^N.$$

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Introduced by Ph. Choquard in 1976, the Choquard equation models a one-component plasma [1]. It appears also in Fröhlich and Pekar's model of the polaron, which describes the interaction of an electron with its own hole [2–4].

If u is a solution to (NLS), one gets a family of solutions

$$\lambda^{\frac{2+\alpha}{2(p-1)}}u(\lambda^2\cdot,\lambda\,\cdot),\quad \lambda>0.$$

The equation (NLS) is called $\dot H^{s_c}$ critical, where

$$s_c := \frac{N}{2}-\frac{2+\alpha}{2(p-1)}$$

is the unique index keeping the following Sobolev norm invariant

$$\left\|\lambda^{\frac{2+\alpha}{2(p-1)}}u(\lambda^2 t,\lambda\,\cdot)\right\|_{\dot H^{s_c}} = \|u(\lambda^2 t)\|_{\dot H^{s_c}}.$$

This note treats the energy-critical case corresponding to $s_c = 1$, equivalently

$$p = p^* := 1+\frac{\alpha+2}{N-2}.$$

The particular case $p^* = 2$ in (NLS) corresponds to the repulsive energy-critical nonlinear equations of Hartree type. The global existence and scattering versus finite time blow-up of solutions were established for radial setting in the study by Miao et al. [5]. This result was extended to the nonradial case [6]. For $p^* > 2$, it seems that the study by Arora and Roudenko [7] is the only work dealing with the energy-critical generalized Hartree equation (NLS). Indeed, the global well-posedness in H^{s_c}, $s_c \geq 0$ was established for small data. A natural complementing of the study by Arora and Roudenko [7] is to establish the scattering of global solutions with a precise data size in $\dot H^1$. This is the goal of this note. The argument is the concentration-compactness-rigidity method used first by [8] in the energy-critical focusing NLS. Note that (NLS) is a part of the nonlinear evolution equations with nonlocal source term [9,10].

This article naturally complements the work [11], where the scattering of the defocusing energy-critical generalized Hartree equation was obtained by the third author in the spirit of the study by Miao et al. [5]. This helps to understand the asymptotic behavior of the energy-critical Choquard problem in different regimes. Note also that the intercritical regime was investigated by Saanouni [12]. It was also revisited with an alternative proof in the study by Arora [13], using a new method suggested by Dodson and Murphy [14], which relies on a scattering criterion [15], combined with the radial Sobolev and Morawetz-type estimates.

This article is organized as follows: Section 2 contains some technical estimates and the main result. Section 3 is devoted to prove the linear profile decomposition of bounded sequences in $\dot H^1$. In Section 4, one collects some variational estimates about energy solutions to (NLS). The goal of Section 5 is to obtain the nonexistence of global solutions. In Section 6, one proves the global existence and scattering of solutions. Finally, in the Appendix, we prove a scattering criteria and give a second proof of the profile decomposition.

Here and hereafter, for simplicity, one denotes the spaces

$$L^r := L^r(\mathbb{R}^N),\quad \dot H^1 := \dot H^1(\mathbb{R}^N);$$
$$\dot H^1_{rd} := \{f\in \dot H^1,\quad f(x)=f(|x|),\quad \forall x\in\mathbb{R}^N\},$$

and the norms

$$\|\cdot\|_r := \|\cdot\|_{L^r},\quad \|\cdot\| := \|\cdot\|_2.$$

Finally, $T^* := T^*(u_0) > 0$ denotes the lifespan for a possible solution to the Schrödinger problem (NLS).

2 Background and main result

This section contains the contribution of this note and some useful estimates.

2.1 Notations

Here and hereafter, define, for $u \in \dot{H}^1$,

$$\mathcal{N} := \mathcal{N}(u) := \int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x;$$
$$\mathcal{F} := \mathcal{F}(u) := (I_\alpha * |u|^{p^*})|u|^{p^*-2}u;$$
$$E(u) := \|\nabla u\|^2 - \frac{1}{p^*}\mathcal{N}.$$

Definition 2.1. A ground state of (NLS) resolves the elliptic problem

$$\Delta Q + (I_\alpha * |Q|^{p^*})|Q|^{p^*-2}Q = 0, \quad 0 \neq Q \in \dot{H}^1 \tag{1}$$

and minimizes the problem

$$\inf_{0 \neq u \in \dot{H}^1} \left\{ E(u) \quad \text{s.t.}\ \|\nabla u\|^2 = \left(\int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x \right)^{\frac{1}{p^*}} \right\}.$$

According to [16], the following explicit function is a ground state of (NLS),

$$Q := S^{\frac{\alpha(2-N)}{4(2+\alpha)}} C_*^{\frac{2-N}{4(2+\alpha)}} \left(1 + \frac{|\cdot|^2}{N(N-2)} \right)^{-\frac{N}{N-2}}, \tag{2}$$

where S is the best Sobolev injection constant in $\|\nabla u\|^2 \geq C\|u\|^2_{\frac{2N}{N-2}}$ and

$$C_* := \left(\frac{1}{2\sqrt{\pi}} \right)^{\alpha} \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)} \left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \right)^{\frac{\alpha}{N}}.$$

Note that global solutions to (NLS) such as this ground state may not scatter. Finally, let us denote, for an interval $I \subset \mathbb{R}$, the Strichartz spaces

$$S(I) := L^{2p^*}(I, L^{\frac{2Np^*}{Np^*-2}}), \quad S^1(I) := L^{2p^*}(I, L^{\frac{2Np^*(p^*-1)}{2+\alpha p^*}}).$$

2.2 Preliminary

Solutions to the Schrödinger problem (NLS) will be considered with an equivalent way as a fixed point of the integral equation:

$$e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}[(I_\alpha * |u|^{p^*})|u|^{p^*-2}u]\mathrm{d}s,$$

where

$$e^{it\Delta}u := \mathcal{F}^{-1}(e^{-it|\cdot|^2}) * u.$$

In the study by Du and Yang [16], a sharp Gagliardo-Nirenberg type inequality related to (NLS) was established.

Proposition 2.1. *Let $N \geq 3$ and $0 < \alpha < N$. Then,*

(1) $\int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x \leq C_{N,\alpha}\|\nabla u\|^{2p^*}$, *for any* $u \in \dot{H}^1$;

(2) *the best constant in the previous inequality is*

$$C_{N,\alpha} = \frac{C_*}{S^{\frac{\alpha+N}{N-2}}}.$$

The problem (NLS) has a local solution in the energy space, which is global for small data [7].

Proposition 2.2. *Let $N \geq 3$, $0 < \alpha < N \leq 4 + \alpha$, and $u_0 \in \dot{H}^1$. Then, there exists $T^* > 0$ such that* (NLS) *admits a unique maximal solution*

$$u \in C_{T^*}(\dot{H}^1).$$

Moreover,

(1) $u \in L^{2p^*}_{\text{loc}}((0, T^*), \dot{W}^{1,\frac{2Np^*}{Np^*-2}})$;

(2) *the energy is conserved* $E(u(t)) = E(u_0)$;

(3) *if* $T^* < \infty$, *then,* $\|u\|_{S^1(0,T^*)} = \infty$.

Furthermore, there exists $\delta(\|\nabla u_0\|) > 0$ *such that if* $\|e^{i\cdot\Delta}u_0\|_{S^1(\mathbb{R})} \leq \delta$, *then,* $T^* = \infty$ *and*

(1) $\|u\|_{S^1(\mathbb{R})} \leq 2\|e^{i\cdot\Delta}u_0\|_{S^1(\mathbb{R})}$;

(2) $\|\nabla u\|_{S(\mathbb{R})} \lesssim \|\nabla u_0\|_{S(\mathbb{R})}$.

For the reader convenience, the next scattering criterion will be proved in the Appendix.

Proposition 2.3. *Let $N \geq 3$, $0 < \alpha < N \leq 4 + \alpha$, and $u \in C(\mathbb{R}, \dot{H}^1)$ be a global solution to* (NLS) *satisfying*

$$\|u\|_{S^1(\mathbb{R})} < \infty.$$

Then, u scatters in $\dot{H}^1$.

The following long-time perturbation theory [12] will be useful.

Proposition 2.4. *Let $N \geq 3$, $0 < \alpha < N \leq 4 + \alpha$. Let $T > 0$, $u \in C_T(\dot{H}^1)$ be a solution to* (NLS) *and $\tilde{u} \in L^\infty_T(\dot{H}^1)$ satisfying for some $\varepsilon, A > 0$,*

$$\|\tilde{u}\|_{L^\infty_T(\dot{H}^1)\cap S^1(0,T)} \leq A;$$
$$i\partial_t\tilde{u} + \Delta\tilde{u} + \mathcal{F}(\tilde{u}) = e;$$
$$\|\nabla e\|_{S'(0,T)} \leq \varepsilon, \quad \|e^{i\cdot\Delta}[u_0 - \tilde{u}_0]\|_{S^1(0,T)} \leq \varepsilon.$$

Thus, there exists $\varepsilon_0 \coloneqq \varepsilon_0(A)$ satisfying for any $0 < \varepsilon < \varepsilon_0$,

$$\|u\|_{S^1(0,T)} \leq C(A).$$

The next linear profile decomposition for bounded radial sequences in $\dot{H}^1$ is a key tool for the scattering proof [5,17].

Proposition 2.5. *Let $N \geq 3$, $0 < \alpha < N$ such that p^* is an integer. Take (u_n) be a bounded sequence in $\dot{H}^1_{rd}$ such that $\|e^{i\cdot\Delta}u_n\|_{S^1(\mathbb{R})} \geq \delta > 0$. Then, for any $M \in \mathbb{N}$, there exist a subsequence denoted also (u_n) and*

(1) *for any $1 \leq j \leq M$, a profile $\psi^j \in \dot{H}^1$;*

(2) *for any $1 \leq j \leq M$, a sequence $(t^j_n, \lambda^j_n) \in \mathbb{R}^2$ satisfying for $1 \leq i \neq j \leq M$ and $n \to \infty$,*

$$\frac{\lambda^j_n}{\lambda^i_n} + \frac{\lambda^i_n}{\lambda^j_n} + \frac{|t^j_n - t^i_n|}{(\lambda^j_n)^2} \to \infty;$$

(3) *a sequence of remainders $W^M_n \in \dot{H}^1_{rd}$, such that, for $\psi^j_l \coloneqq e^{i\cdot\Delta}\psi^j$,*

$$u_n = \sum_{j=1}^M \frac{1}{(\lambda^j_n)^{\frac{N-2}{2}}} e^{-\frac{it^j_n}{(\lambda^j_n)^2}\Delta}\psi^j\left(\frac{\cdot}{\lambda^j_n}\right) + W^M_n \coloneqq \sum_{j=1}^M \frac{1}{(\lambda^j_n)^{\frac{N-2}{2}}}\psi^j_l\left(-\frac{t^j_n}{(\lambda^j_n)^2}, \frac{\cdot}{\lambda^j_n}\right) + W^M_n.$$

Moreover, $\|\nabla\psi_1\| > 0$ *and*

$$\lim_{M\to\infty}\left[\lim_{n\to\infty}\|e^{i\cdot\Delta}W_n^M\|_{S^1(\mathbb{R})}\right] = 0.$$

For fixed M, one has the next Pythagorean expansions

$$\|\nabla u_n\|^2 = \sum_{j=1}^{M}\|\nabla\psi^j\|^2 + \|\nabla W_n^M\|^2 + o_n(1);$$

$$E(u_n) = \sum_{j=1}^{M}E\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) + E(W_n^M) + o_n(1).$$

Finally, one recalls the notion of nonlinear profile.

Definition 2.2. Let a sequence of real numbers (t_n) such that $\lim_n t_n \coloneqq \bar{t} \in [-\infty, +\infty]$ and $v_0 \in \dot{H}^1$. One says that $u(t, x)$ is a nonlinear profile associated to (v_0, t_n) if there exists an interval $I \ni \bar{t}$, ($I = [a, \infty)$ or $I = (\infty, a]$, if $\bar{t} = \infty$) and $u(t, x)$ a solution to (NLS) on I such that

$$\lim_n \|u(t_n) - e^{it_n\Delta}v_0\|_{\dot{H}^1} = 0.$$

The main result proved in this article is given in the following subsection.

2.3 Main result

The novelty in this work is the next dichotomy of global existence and scattering versus nonexistence of global energy solutions to the Schrödinger problem (NLS).

Theorem 2.1. *Let* $N \geq 3$, $0 < \alpha < N \leq 4 + \alpha$, Q *be the ground state solution to* (1) *given in* (2) *and* $u_0 \in \dot{H}^1$ *satisfying*

$$E[u_0] < E[Q]. \tag{3}$$

Take $u \in C_{T^*}(\dot{H}^1)$ *be the unique maximal solution to* (NLS). *Then,*

(1) $T^* < \infty$ *if* $xu_0 \in L^2$ *and*

$$\|\nabla u_0\| > \|\nabla Q\|. \tag{4}$$

(2) $T^* = \infty$ *and* u *scatters if* u_0 *is radial and*

$$\|\nabla u_0\| < \|\nabla Q\|. \tag{5}$$

In view of the result stated in the aforementioned theorem, some comments are in order.

Remarks 2.1.
- The condition $\alpha \geq N - 4$ avoids a singular source term if $p^* < 2$;
- the range of space dimensions N containing 3, is $\{3, 4, 5, 6\}$;
- the scattering means the existence of $\psi \in \dot{H}^1$ such that

$$\lim_{t\to\infty}\|u(t) - e^{it\Delta}\psi\|_{\dot{H}^1} = 0;$$

- in the proof of the finite time blow-up, we establish also that there is no global solution to (NLS) in the energy space if $u_0 \in H^1_{rd}$ and $N^2 - (5 + \alpha)N - \alpha \geq 0$. This extra assumption is related to the method and is needed in order to apply Young estimate;
- the major part of the proof is devoted to prove the energy scattering;

- the scattering in the defocusing regime was proved by Saanouni [11];
- the local well-posedness of (NLS) for $1 < p < 2$ was investigated for a class of data in weighted Sobolev spaces in [18];
- the proof of the scattering follows the concentration-compactness-rigidity method due to the pioneering work [8], which has a deep influence in the NLS context.
- the new method of Dodson and Murphy [14], used by Arora [13] in the intercritical regime fails in the energy critical case.

Section 2.4 presents some technical estimates needed in the sequel.

2.4 Tools

The next consequence of Hardy-Littlewood-Sobolev inequality [19] is adapted to estimate a possible solution to the Hartree equation (NLS).

Proposition 2.6. *Let* $0 < \alpha < N \geq 1$ *and* $1 < s, r < \infty$. *Then,*

(1) *if* $1 + \frac{\alpha}{N} = \frac{1}{r} + \frac{1}{s}$,

$$\int_{\mathbb{R}^N} (I_\alpha * g)(x) f(x) \mathrm{d}x \leq C_{N,\alpha,s} \|f\|_r \|g\|_s, \quad \forall (f, g) \in L^r \times L^s. \tag{6}$$

(2) *If* $1 + \frac{\alpha}{N} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$,

$$\|(I_\alpha * f) g\|_{r'} \leq C_{N,\alpha,s} \|f\|_s \|g\|_q, \quad \forall (f, g) \in L^s \times L^q.$$

Definition 2.3. A couple of real numbers (q, r) is said to be $\dot{H}^s$ admissible (admissible if $s = 0$) if

$$N\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{2}{q} + s,$$

where

$$\begin{cases} \dfrac{2N}{N-2s} < r \leq \left(\dfrac{2N}{N-2}\right)^-, & \text{if} \quad N \geq 3; \\ \dfrac{2}{1-s} < r \leq \left(\left(\dfrac{2}{1-s}\right)^+\right)', & \text{if} \quad N = 2; \\ \dfrac{2}{1-2s} < r \leq \infty, & \text{if} \quad N = 1. \end{cases}$$

Here, $(a^+)' = \frac{a_+ a}{a_+ - a}$. Finally, one says that (q, r) is said to be $\dot{H}^{-s}$ admissible if

$$N\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{2}{q} - s,$$

where

$$\begin{cases} \left(\dfrac{2N}{N-2s}\right)^+ < r \leq \left(\dfrac{2N}{N-2}\right)^-, & \text{if} \quad N \geq 3; \\ \left(\dfrac{2}{1-s}\right)^+ < r \leq \left(\left(\dfrac{2}{1-s}\right)^+\right)', & \text{if} \quad N = 2; \\ \left(\dfrac{2}{1-2s}\right)^+ < r \leq \infty, & \text{if} \quad N = 1. \end{cases}$$

For simplicity, one denotes Γ^s the set of s admissible pairs.

A standard tool to control solutions of (NLS) is the Strichartz estimate [20,21].

Proposition 2.7. *Let $N \geq 1$ and $s \in \mathbb{R}$. Then, there exists $C > 0$ such that*
(1) $\sup_{(q,r)\in\Gamma^s} \|e^{i\cdot\Delta}u\|_{L^q(L^r)} \leq C\||\nabla|^s u\|$;
(2) $\sup_{(q,r)\in\Gamma^s} \|\int_0^{\cdot} e^{i(\cdot-\tau)\Delta}f(\tau)\mathrm{d}\tau\|_{L^q(L^r)} \leq C \inf_{(a,b)\in\Gamma^{-s}} \|f\|_{L^{a'}(L^{b'})}$.

To investigate the nonexistence of global solutions to the Schrödinger problem (NLS), one will use the next variance identity [22].

Proposition 2.8. *Let $N \geq 3$, $0 < \alpha < N \leq 4 + \alpha$, and $u \in C_T(\dot{H}^1)$ be a solution to (NLS) satisfying $xu_0 \in L^2$. Then, the variance defined on $[0, T)$ by*

$$V : t \mapsto \int_{\mathbb{R}^N} |xu(t,x)|^2 \mathrm{d}x$$

satisfies $V \in C^2(-T, T)$ and

$$V'' = 8\left(\|\nabla u\|^2 - \int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x\right).$$

3 Profile decomposition

In this section, one proves Proposition 2.5. Taking account of [17], it is sufficient to prove the last equation, which reads

$$\mathcal{N}(u_n) = \sum_{j=1}^{M} \mathcal{N}\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) + \mathcal{N}(W_n^M) + o_n(1). \tag{7}$$

For this, one needs to establish that

$$0 = \lim_n (\mathcal{N}(u_n) - \mathcal{N}(u_n - (\tilde{\psi}_l^1)_n) - \mathcal{N}((\tilde{\psi}_l^1)_n)) := \lim_n I_n, \tag{8}$$

where, one denotes the sequence

$$(\tilde{\psi}_l^1)_n := \frac{1}{(\lambda_n^1)^{\frac{N-2}{2}}} \psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}, \frac{\cdot}{\lambda_n^1}\right).$$

By re-indexing and adjusting profiles, one can assume that one of the two scenarios happens. The first one is $\frac{t_n^1}{(\lambda_n^1)^2} \to \infty$. The second one is $\frac{t_n^1}{(\lambda_n^1)^2} \to -\tilde{t}$.

Take the first case and recall the useful inequality [23], for any $m \geq 2$,

$$||x|^m - |x-y|^m - |y|^m| \leq m2^{m-1}(|x-y|^{m-1}|y| + |x-y||y|^{m-1}). \tag{9}$$

Let us write

$$\begin{aligned} I_n = &\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{p^*})(|u_n|^{p^*} - |u_n - (\tilde{\psi}_l^1)_n|^{p^*} - |(\tilde{\psi}_l^1)_n|^{p^*})\mathrm{d}x + \int_{\mathbb{R}^N} |u_n - (\tilde{\psi}_l^1)_n|^{p^*}[I_\alpha * (|u_n|^{p^*} \\ &- |u_n - (\tilde{\psi}_l^1)_n|^{p^*})]\mathrm{d}x + \int_{\mathbb{R}^N} |(\tilde{\psi}_l^1)_n|^{p^*}[I_\alpha * (|u_n|^{p^*} - |(\tilde{\psi}_l^1)_n|^{p^*})]\mathrm{d}x \\ := &(I_n)_1 + (I_n)_2 + (I_n)_3. \end{aligned} \tag{10}$$

Now, by (9), via Sobolev embeddings, Hölder estimate and (6), one writes for $\frac{2N}{N-2} \coloneqq 2^*$,

$$\begin{aligned}(I_n)_1 &\lesssim \|u_n\|_{2^*} \| |u_n|^{p^*} - |u_n - (\tilde{\psi}_l^1)_n|^{p^*} - |(\tilde{\psi}_l^1)_n|^{p^*} \|_{\frac{2N}{\alpha+N}} \\ &\lesssim \| |u_n|^{p^*} - |u_n - (\tilde{\psi}_l^1)_n|^{p^*} - |(\tilde{\psi}_l^1)_n|^{p^*} \|_{\frac{2N}{\alpha+N}} \\ &\lesssim \| |u_n - (\tilde{\psi}_l^1)_n|^{p^*-1} |(\tilde{\psi}_l^1)_n| + |u_n - (\tilde{\psi}_l^1)_n| |(\tilde{\psi}_l^1)_n|^{p^*-1} \|_{\frac{2N}{\alpha+N}} \\ &\lesssim \|u_n - (\tilde{\psi}_l^1)_n\|_{2^*}^{p^*-1} \|(\tilde{\psi}_l^1)_n\|_{2^*} + \|u_n - (\tilde{\psi}_l^1)_n\|_{2^*} \|(\tilde{\psi}_l^1)_n\|_{2^*}^{p^*-1}.\end{aligned} \tag{11}$$

Using the dispersive estimate [24] of the free Schrödinger operator, $\|e^{it\Delta}\cdot\|_r \le \frac{C}{t^{N\left(\frac{1}{2}-\frac{1}{r}\right)}}\|\cdot\|_{r'}$, for all $r \ge 2$, via (11) and (10), one gets $\lim_n I_n = 0$.

Now, take the second case and denote

$$s_n \coloneqq \frac{t_n^1}{(\lambda_n^1)^2} \to -\tilde{t}; \tag{12}$$

$$v_n \coloneqq \mathcal{L}_1 u_n \coloneqq (\lambda_n^1)^{\frac{N-2}{2}} u_n(\lambda_n^1 \cdot) \rightharpoonup e^{i\tilde{t}\Delta}\psi^1 \quad \text{in } \dot{H}^1; \tag{13}$$

$$\psi_l^1(-s_n) \to e^{i\tilde{t}\Delta}\psi^1 \quad \text{in } L^{2^*}. \tag{14}$$

Let us write

$$\begin{aligned} I_n &= \mathcal{N}(u_n) - \mathcal{N}(u_n - (\tilde{\psi}_l^1)_n) - \mathcal{N}((\tilde{\psi}_l^1)_n) \\ &= \mathcal{N}(v_n) - \mathcal{N}(v_n - \psi_l^1(-s_n)) - \mathcal{N}(\psi_l^1(-s_n)) \\ &\coloneqq (I_n)_1 + (I_n)_2 + (I_n)_3. \end{aligned} \tag{15}$$

Here, one takes

$$\begin{aligned} (I_n)_1 &\coloneqq \mathcal{N}(v_n) - \mathcal{N}(v_n - e^{i\tilde{t}\Delta}\psi^1) - \mathcal{N}(e^{i\tilde{t}\Delta}\psi^1); \\ (I_n)_2 &\coloneqq \mathcal{N}(v_n - e^{i\tilde{t}\Delta}\psi^1) - \mathcal{N}(v_n - \psi_l^1(-s_n)); \\ (I_n)_3 &\coloneqq \mathcal{N}(e^{i\tilde{t}\Delta}\psi^1) - \mathcal{N}(\psi_l^1(-s_n)). \end{aligned}$$

Moreover, the aforementioned first term reads

$$\begin{aligned} (I_n)_1 = &\int_{\mathbb{R}^N} (I_\alpha * |v_n|^{p^*})(|v_n|^{p^*} - |v_n - e^{i\tilde{t}\Delta}\psi^1|^{p^*} - |e^{i\tilde{t}\Delta}\psi^1|^{p^*})\mathrm{d}x + \int_{\mathbb{R}^N} |v_n - e^{i\tilde{t}\Delta}\psi^1|^{p^*}[I_\alpha * (|v_n|^{p^*} \\ &- |v_n - e^{i\tilde{t}\Delta}\psi^1|^{p^*})]\mathrm{d}x + \int_{\mathbb{R}^N} |e^{i\tilde{t}\Delta}\psi^1|^{p^*}[I_\alpha * (|v_n|^{p^*} - |e^{i\tilde{t}\Delta}\psi^1|^{p^*})]\mathrm{d}x \\ \coloneqq &(I_n)_{11} + (I_n)_{12} + (I_n)_{13}. \end{aligned} \tag{16}$$

By (6) via Hölder estimate and (9), one obtains

$$\begin{aligned} (I_n)_{11} &= \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{p^*})(|v_n|^{p^*} - |v_n - e^{i\tilde{t}\Delta}\psi^1|^{p^*} - |e^{i\tilde{t}\Delta}\psi^1|^{p^*})\mathrm{d}x \\ &\lesssim \|v_n\|_{2^*} \| |v_n - e^{i\tilde{t}\Delta}\psi^1|^{p^*-1} |e^{i\tilde{t}\Delta}\psi^1| + |v_n - e^{i\tilde{t}\Delta}\psi^1| |e^{i\tilde{t}\Delta}\psi^1|^{p^*-1} \|_{\frac{2N}{\alpha+N}} \\ &\coloneqq (I_n)_{111} + (I_n)_{112}. \end{aligned} \tag{17}$$

By density argument, one takes $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\|\chi - e^{i\tilde{t}\Delta}\psi^1\|_{\dot{H}^1} \ll 1$. Thus, by (13), via Hölder estimate, it follows that

$$(I_n)_{111} \lesssim \|v_n - e^{i\tilde{t}\Delta}\psi^1\|_{2^*}^{p^*-1} \|e^{i\tilde{t}\Delta}\psi^1 - \chi\|_{2^*} + \|v_n - e^{i\tilde{t}\Delta}\psi^1\|_{2^*}^{p^*-2} \|\chi(v_n - e^{i\tilde{t}\Delta}\psi^1)\|_{\frac{2^*}{2}} \to 0.$$

Moreover, by Hölder estimate, one has

$$\begin{aligned}(I_n)_{112} &= \||v_n - e^{i\tilde{t}\Delta}\psi^1||e^{i\tilde{t}\Delta}\psi^1 - \chi + \chi|^{p^*-1}\|_{\frac{2N}{\alpha+N}}\\ &\lesssim \|v_n - e^{i\tilde{t}\Delta}\psi^1\|_{2^*}\|e^{i\tilde{t}\Delta}\psi^1 - \chi\|_{2^*}^{p^*-1} + \|\chi(v_n - e^{i\tilde{t}\Delta}\psi^1)\|_{\frac{2^*}{2}}\|\chi\|_{2^*}^{p^*-2} \to 0.\end{aligned} \tag{18}$$

Now, one considers the second term. Arguing as previously and taking account of (14), one obtains

$$\begin{aligned}(I_n)_{12} &= \int_{\mathbb{R}^N} |v_n - e^{i\tilde{t}\Delta}\psi^1|^{p^*}[I_\alpha*(|v_n|^{p^*} - |v_n - e^{i\tilde{t}\Delta}\psi^1|^{p^*})]\mathrm{d}x\\ &\lesssim \int_{\mathbb{R}^N} |v_n - e^{i\tilde{t}\Delta}\psi^1|^{p^*}[I_\alpha*(|e^{i\tilde{t}\Delta}\psi^1 - \psi_l^1(-s_n)|(|v_n|^{p^*-1} + |e^{i\tilde{t}\Delta}\psi^1|^{p^*-1} + |\psi_l^1(-s_n)|^{p^*-1}))]\mathrm{d}x\\ &\lesssim (\|v_n\|_{2^*}^{p^*-1} + \|e^{i\tilde{t}\Delta}\psi^1\|_{2^*}^{p^*-1} + \|\psi_l^1(-s_n)\|_{2^*}^{p^*-1})\|e^{i\tilde{t}\Delta}\psi^1 - \psi_l^1(-s_n)\|_{2^*} \to 0.\end{aligned} \tag{19}$$

The third term is controlled similarly. This finishes the proof.

4 Variational analysis

In this section, one prepares the proof of the main result by collecting some useful estimates. Taking account of [16], the ground state Q given in (2) is a minimizer for

$$S_{N,\alpha} := \inf_{\{0\neq u\in H^1\}}\left\{\frac{\|\nabla u\|^2}{\left(\int_{\mathbb{R}^N}|u|^{p^*}(I_\alpha*|u|^{p^*})\mathrm{d}x\right)^{\frac{1}{p^*}}}\right\}.$$

This ground state is a global solution to (NLS), which does not scatter. Thus, global solutions may not scatter. Moreover, the equation (1) gives

$$\|\nabla Q\|^2 = \int_{\mathbb{R}^N}(I_\alpha*|Q|^{p^*})|Q|^{p^*}\mathrm{d}x.$$

Thus,

$$\|\nabla Q\|^2 = S_{N,\alpha}^{\frac{\alpha+N}{\alpha+2}}, \quad E(Q) = \frac{\alpha+2}{\alpha+N}S_{N,\alpha}^{\frac{\alpha+N}{\alpha+2}}.$$

Lemma 4.1. *For $\delta \in (0,1)$, there exists $\bar{\delta} \in (0,1)$ such that if $u \in \dot{H}^1$ satisfies*

$$\|\nabla u\| < \|\nabla Q\|; \quad E(u) < (1-\delta)E(Q) < E(Q),$$

then,

$$\|\nabla u\|^2 < (1-\bar{\delta})\|\nabla Q\|^2; \tag{20}$$

$$\|\nabla u\|^2 - \mathcal{N} \geq \bar{\delta}\|\nabla u\|^2; \tag{21}$$

$$E(u) > 0. \tag{22}$$

Proof. Take the real function $f(x) := x - \frac{1}{p^*S_{N,\alpha}^{p^*}}x^{p^*}$. Then,

$$\begin{aligned}f(\|\nabla u\|^2) &= \|\nabla u\|^2 - \frac{1}{p^*S_{N,\alpha}^{p^*}}\|\nabla u\|^{2p^*}\\ &\leq \|\nabla u\|^2 - \frac{1}{p^*}\mathcal{N}\\ &\leq E(u)\\ &\leq (1-\delta)E(Q).\end{aligned}$$

The equation $f'(x) = 0$ is equivalent to

$$x = x^* = S_{N,\alpha}^{\frac{p^*}{p^*-1}} = \|\nabla Q\|^2.$$

Moreover, $f(x^*) = E(Q)$. Now, since f is positive and strictly increasing on $[0, x^*]$, one has (20) and (22). Now, let the real function $g(x) \coloneqq x - (\frac{x}{S_{N,\alpha}})^{p^*}$. Then,

$$\|\nabla u\|^2 - \mathcal{N} \geq \|\nabla u\|^2 - \left(\frac{\|\nabla u\|^2}{S_{N,\alpha}}\right)^{p^*} = g(\|\nabla u\|^2).$$

Moreover, $g(x) = 0$ if and only if $x = 0$ or $x = x^*$. Thus, $g(x) \gtrsim x$ on $[0, (1 - \bar{\delta})x^*]$. So, since $0 \leq \|\nabla u\|^2 < (1 - \bar{\delta})x^*$, one obtains (21). □

Corollary 4.1. *If $u \in \dot{H}^1$ satisfies $\|\nabla u\| < \|\nabla Q\|$, then $E(u) \geq 0$.*

Proof. The case $E(u) \geq E(Q) = \frac{\alpha+2}{\alpha+N} S_{N,\alpha}^{\frac{\alpha+N}{\alpha+2}} > 0$ is clear. Otherwise, the Lemma 4.1 gives the result. □

With a continuity argument via the conservation of the energy and Lemma 4.1, one has the following energy trapping.

Proposition 4.1. *For $\delta \in (0, 1)$, there exists $\bar{\delta} \in (0, 1)$ such that if $u_0 \in \dot{H}^1$ satisfies*

$$\|\nabla u_0\| < \|\nabla Q\|; \quad E(u_0) < (1 - \delta)E(Q) < E(Q),$$

then, the maximal solution to (NLS) *satisfies for any $t \in [0, T^*)$,*

$$\|\nabla u(t)\|^2 < (1 - \bar{\delta})\|\nabla Q\|^2; \tag{23}$$

$$\|\nabla u(t)\|^2 - \mathcal{N}(u(t)) \geq \bar{\delta}\|\nabla u(t)\|^2; \tag{24}$$

$$E(u) > 0. \tag{25}$$

Moreover,

$$E(u(t)) \simeq \|\nabla u(t)\|^2 \simeq \|\nabla u_0\|^2.$$

Proof. For the last point, since $E(u(t)) \leq \|\nabla u(t)\|^2$, by (23), one has

$$\begin{aligned} E(u(t)) &\geq \left(1 - \frac{1}{p^*}\right)\|\nabla u(t)\|^2 + \frac{1}{p^*}(\|\nabla u(t)\|^2 - \mathcal{N}(u(t))) \\ &\geq \left(1 - \frac{1}{p^*}\right)\|\nabla u(t)\|^2. \end{aligned}$$

Taking account of Lemma 4.1, it is sufficient to prove that

$$\|\nabla u(t)\| < \|\nabla Q\|, \quad \forall t \in (0, T^*).$$

Assume that there exists $t_0 \in (0, T^*)$ such that

$$\|\nabla u(t)\| < \|\nabla Q\|, \quad \forall t \in (0, t_0) \quad \text{and} \quad \|\nabla u(t_0)\| = \|\nabla Q\|.$$

Thus, by Lemma 4.1, one obtains

$$\|\nabla u(t)\|^2 < (1 - \bar{\delta})\|\nabla Q\|^2, \quad \forall t \in (0, t_0).$$

With a continuity argument, it follows that

$$\|\nabla u(t_0)\| < \|\nabla Q\|.$$

This contradiction finishes the proof. □

5 Nonexistence of global solutions

In this section, one proves the first part of the main result dealing with the nonexistence of global solutions to (NLS). The next result follows like Proposition 4.1.

Lemma 5.1. *For* $\delta \in (0,1)$, *there exists* $\bar{\delta} \in (0,1)$ *such that if the maximal solution to* (NLS) *denoted by* $u \in C_{T^*}(\dot{H}^1)$ *satisfies* (4) *and*

$$E(u) < (1-\delta)E(Q) < E(Q).$$

Then, holds on $[0, T^*)$,

$$\|\nabla u\|^2 > (1+\bar{\delta})\|\nabla Q\|^2.$$

Let us discuss two cases.

(1) $xu_0 \in L^2$. Compute

$$\begin{aligned}\|\nabla u\|^2 - \int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})dx &= p^*E(u) - (p^*-1)\|\nabla u\|^2\\ &\leq p^*E(Q) - (p^*-1)(1+\bar{\delta})\|\nabla Q\|^2\\ &\leq \left(\frac{\alpha+N}{N-2}\frac{2+\alpha}{N+\alpha} - \frac{2+\alpha}{N-2}(1+\bar{\delta})\right)S_{N,\alpha}^{\frac{N+\alpha}{2+\alpha}}\\ &\leq -\bar{\delta}(p^*-1)S_{N,\alpha}^{\frac{p^*}{p^*-1}}.\end{aligned}$$

Thus, by the variance identity in Proposition 2.8,

$$\begin{aligned}\partial_t^2\||\cdot|u(t)\|^2 &= 8\left(\|\nabla u\|^2 - \int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})dx\right)\\ &\leq -8\bar{\delta}(p^*-1)S_{N,\alpha}^{\frac{p^*}{p^*-1}}.\end{aligned}$$

Thus, integrating twice, one obtains

$$\||\cdot|u(t)\|^2 \leq -4\bar{\delta}(p^*-1)S_{N,\alpha}^{\frac{p^*}{p^*-1}}t^2 + 4t\int_{\mathbb{R}^N}(x.\,\nabla u_0)u_0 dx + \|xu_0\|^2.$$

This quantity becomes negative for large time. Thus, $T^* < \infty$.

(2) Second case $u_0 \in H^1_{rd}$.

In the rest of this section, take a smooth radial function $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\psi'' \leq 1$ and

$$\psi : x \mapsto \begin{cases}\frac{|x|^2}{2}, & \text{if} \quad |x| \leq 1;\\ 0, & \text{if} \quad |x| \geq 2.\end{cases}$$

Then, the truncated function $\psi_R := R^2\psi(\frac{\cdot}{R})$ satisfies

$$\psi_R'' \leq 1, \quad \psi_R'(r) \leq r \quad \text{and} \quad \Delta\psi_R \leq N.$$

Denote the localized virial

$$V_\psi[u(t)] := \int_{\mathbb{R}^N} \psi\,|u(t)|^2 dx \quad \text{and} \quad V_R := V_{\psi_R}.$$

The first derivative reads using the convention of summed repeated index,

$$\partial_t V_\psi[u(t)] \coloneqq M_\psi[u(t)] = 2\Im \int_{\mathbb{R}^N} \bar{u}(t)\partial_k\psi\partial_k u(t)\mathrm{d}x.$$

Compute using the equation (NLS),

$$\begin{aligned}\partial_t\Im(\partial_k u\bar{u}) &= \Im(\partial_k\partial_t u\bar{u}) + \Im(\partial_k u\overline{\partial_t u})\\ &= \Re(i\partial_t u\partial_k\bar{u}) - \Re(i\partial_k\partial_t u\bar{u})\\ &= \Re(\partial_k\bar{u}(-\Delta u - \mathcal{F})) - \Re(\bar{u}\partial_k(-\Delta u - \mathcal{F}))\\ &= \Re(\bar{u}\partial_k\Delta u - \partial_k\bar{u}\Delta u) + \Re(\bar{u}\partial_k\mathcal{F} - \partial_k\bar{u}\mathcal{F}).\end{aligned}$$

Recall the identity

$$\frac{1}{2}\partial_k\Delta(|u|^2) - 2\partial_l\Re(\partial_k u\partial_l\bar{u}) = \Re(\bar{u}\partial_k\Delta u - \partial_k\bar{u}\Delta u).$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^N}\partial_k\psi\Re(\bar{u}\partial_k\Delta u - \partial_k\bar{u}\Delta u)\mathrm{d}x &= \int_{\mathbb{R}^N}\partial_k\psi\left(\frac{1}{2}\partial_k\Delta(|u|^2) - 2\partial_l\Re(\partial_k u\partial_l\bar{u})\right)\mathrm{d}x\\ &= -\frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\psi\,|u|^2\mathrm{d}x + 2\int_{\mathbb{R}^N}\partial_l\partial_k\psi\Re(\partial_k u\partial_l\bar{u})\mathrm{d}x.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N}\partial_k\psi\Re(\bar{u}\partial_k\mathcal{F} - \partial_k\bar{u}\mathcal{F})\mathrm{d}x &= \int_{\mathbb{R}^N}\partial_k\psi\Re(\partial_k[\bar{u}\mathcal{F}] - 2\partial_k\bar{u}\mathcal{F})\mathrm{d}x\\ &= -\int_{\mathbb{R}^N}(\Delta\psi\bar{u}\mathcal{F} + 2\Re(\partial_k\psi\partial_k\bar{u}\mathcal{F}))\mathrm{d}x\\ &= -\int_{\mathbb{R}^N}\Delta\psi\,|u|^{p^*}(I_\alpha*|u|^{p^*})\mathrm{d}x - 2\int_{\mathbb{R}^N}\partial_k\psi\Re(\partial_k\bar{u}\mathcal{F})\mathrm{d}x\\ &= -\int_{\mathbb{R}^N}\Delta\psi\,|u|^{p^*}(I_\alpha*|u|^{p^*})\mathrm{d}x - \frac{2}{p^*}\int_{\mathbb{R}^N}\partial_k\psi\partial_k(|u|^{p^*})(I_\alpha*|u|^{p^*})\mathrm{d}x.\end{aligned}$$

By using the identity $\nabla I_\alpha = -(N-\alpha)\frac{\cdot}{|\cdot|^2}I_\alpha$, one obtains

$$\begin{aligned}\int_{\mathbb{R}^N}\partial_k\psi\partial_k(|u|^{p^*})(I_\alpha*|u|^{p^*})\mathrm{d}x &= -\int_{\mathbb{R}^N}\Delta\psi\,|u|^{p^*}(I_\alpha*|u|^{p^*})\mathrm{d}x - \int_{\mathbb{R}^N}\partial_k\psi(\partial_k I_\alpha*|u|^{p^*})|u|^{p^*}\mathrm{d}x\\ &= -\int_{\mathbb{R}^N}\Delta\psi\,|u|^{p^*}(I_\alpha*|u|^{p^*})\mathrm{d}x + (N-\alpha)\int_{\mathbb{R}^N}\partial_k\psi\left(\frac{x_k}{|\cdot|^2}I_\alpha*|u|^{p^*}\right)|u|^{p^*}\mathrm{d}x.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^N}\partial_k\psi\Re(\bar{u}\partial_k\mathcal{F} - \partial_k\bar{u}\mathcal{F})\mathrm{d}x &= -\int_{\mathbb{R}^N}\Delta\psi\,|u|^{p^*}(I_\alpha*|u|^{p^*})\mathrm{d}x - \frac{2}{p^*}\int_{\mathbb{R}^N}\partial_k\psi\partial_k(|u|^{p^*})(I_\alpha*|u|^{p^*})\mathrm{d}x\\ &= \left(\frac{2}{p^*} - 1\right)\int_{\mathbb{R}^N}\Delta\psi\,|u|^{p^*}(I_\alpha*|u|^{p^*})\mathrm{d}x - \frac{2}{p^*}(N-\alpha)\int_{\mathbb{R}^N}\partial_k\psi\left(\frac{x_k}{|\cdot|^2}I_\alpha*|u|^{p^*}\right)|u|^{p^*}\mathrm{d}x.\end{aligned}$$

Regrouping previous computation, one obtains

$$\begin{aligned}\partial_t M_\psi[u(t)] = &-\int_{\mathbb{R}^N}\Delta^2\psi\,|u|^2\mathrm{d}x + 4\int_{\mathbb{R}^N}\partial_l\partial_k\psi\Re(\partial_k u\partial_l\bar{u})\mathrm{d}x\\ &+ 2\left(\frac{2}{p^*} - 1\right)\int_{\mathbb{R}^N}\Delta\psi\,|u|^{p^*}(I_\alpha*|u|^{p^*})\mathrm{d}x - \frac{4}{p^*}(N-\alpha)\int_{\mathbb{R}^N}\partial_k\psi\left(\frac{x_k}{|\cdot|^2}I_\alpha*|u|^{p^*}\right)|u|^{p^*}\mathrm{d}x\end{aligned}$$

$$= -\int_{\mathbb{R}^N} \Delta^2\psi\, |u|^2 \mathrm{d}x + 4\int_{\mathbb{R}^N} \partial_l\partial_k\psi \Re(\partial_k u \partial_l \bar{u})\mathrm{d}x + 2\left(\frac{2}{p^*} - 1\right)\int_{\mathbb{R}^N} \Delta\psi\, |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x$$
$$- \frac{2}{p^*}(N-\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\partial_k\psi(x) - \partial_k\psi(y))(x_k - y_k)\frac{I_\alpha(x-y)}{|x-y|^2}\, |u(x)|^{p^*}|u(y)|^{p^*}\mathrm{d}x\mathrm{d}y.$$

Denote, for simplicity, $M_R := M_{\psi_R}$. By using the properties of ψ, one has

$$\partial_t M_R[u(t)] \le \frac{C}{R^2} + 4\|\nabla u\|^2 + 2\left(\frac{2}{p^*} - 1\right)\int_{\mathbb{R}^N} \Delta\psi_R (I_\alpha * |u|^{p^*})|u|^{p^*}\mathrm{d}x$$
$$- \frac{2}{p^*}(N-\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\partial_k\psi_R(x) - \partial_k\psi_R(y))(x_k - y_k)\frac{I_\alpha(x-y)}{|x-y|^2}\, |u(x)|^{p^*}|u(y)|^{p^*}\mathrm{d}x\mathrm{d}y.$$

Following lines in [11], one has

$$(M) := \int_{\mathbb{R}^N\times\mathbb{R}^N} (\nabla\psi_R(x) - \nabla\psi_R(y))\frac{x-y}{|x-y|^2} I_\alpha(x-y)|u(y)|^{p^*}|u(x)|^{p^*}\mathrm{d}y\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x + O\left(\int_{\{|x|>R\}} (I_\alpha * |u|^{p^*})|u|^{p^*}\mathrm{d}x\right).$$

Thus, using the equality $\Delta\psi_R(r) - N = 0$ if $r \le R$,

$$(N) := -\frac{2(N-\alpha)}{p^*}(M) + 2\left(\frac{2}{p^*} - 1\right)\int_{\mathbb{R}^N} \Delta\psi_R (I_\alpha * |u|^{p^*})|u|^{p^*}\mathrm{d}x$$
$$= -2\left(\frac{N-\alpha}{p^*} + N\left(1 - \frac{2}{p^*}\right)\right)\int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x + O\left(\int_{\{|x|>R\}} (I_\alpha * |u|^{p^*})|u|^{p^*}\mathrm{d}x\right)$$
$$+ 2\left(\frac{2}{p^*} - 1\right)\int_{\{|x|>R\}} (\Delta\psi_R - N)(I_\alpha * |u|^{p^*})|u|^{p^*}\mathrm{d}x$$
$$= -4\int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x + O\left(\int_{\{|x|>R\}} (I_\alpha * |u|^{p^*})|u|^{p^*}\mathrm{d}x\right).$$

By using Hardy-Littlewood-Sobolev inequality, one has

$$\int_{\{|x|>R\}} (I_\alpha * |u|^{p^*})|u|^{p^*}\mathrm{d}x \lesssim \|u\|^{p^*}_{L^{\frac{2N}{N-2}}(|x|>R)}\|u\|^{p^*}_{L^{\frac{2N}{N-2}}} \lesssim \|u\|^{p^*}_{L^{\frac{2N}{N-2}}}\|u\|^{p^*-1-\frac{\alpha}{N}}_{L^\infty(|x|>R)}\|u\|^{\frac{\alpha+N}{N}}.$$

Thanks to Hardy-Littlewood-Sobolev and Strauss inequalities via the mass conservation and Sobolev injections, one obtains

$$\int_{\{|x|>R\}} (I_\alpha * |u|^{p^*})|u|^{p^*}\mathrm{d}x \lesssim \|\nabla u\|^{p^*}\|u\|^{p^*-1-\frac{\alpha}{N}}_{L^\infty(|x|>R)}$$
$$\lesssim \|\nabla u\|^{p^*}\left(\frac{1}{R^{N-1}}\|\nabla u\|_{L^2(|x|>R)}\|u\|\right)^{\frac{1}{2}(p^*-1-\frac{\alpha}{N})}$$
$$\lesssim \frac{1}{R^{\frac{(N-1)(\alpha+N)}{N(N-2)}}}\|\nabla u\|^{\frac{(\alpha+N)(1+N)}{N(N-2)}}.$$

So, by Young inequality, since $N^2 - (5+\alpha)N - \alpha \ge 0$, there exists $C(R) \to 0$ when $R \to \infty$, such that

$$\partial_t M_R[u(t)] \le 4\|\nabla u\|^2 - 4\int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x + \frac{C}{R^2} + \frac{C}{R^{\frac{(N-1)(\alpha+N)}{N(N-2)}}}\|\nabla u\|^{\frac{(1+N)(\alpha+N)}{N(N-2)}}$$
$$\le 4\|\nabla u\|^2 - 4\int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\mathrm{d}x + C(R)(1 + \|\nabla u\|^2).$$

Compute using the previous lemma

$$
\begin{aligned}
(1+\varepsilon)\|\nabla u\|^2 - \int_{\mathbb{R}^N} |u|^{p^*}(I_\alpha * |u|^{p^*})\,dx &= p^*E(u) - (p^* - 1 - \varepsilon)\|\nabla u\|^2 \\
&\leq p^*E(Q) - (p^* - 1 - \varepsilon)(1+\bar{\delta})\|\nabla Q\|^2 \\
&\leq \left(\frac{\alpha+N}{N-2}\frac{2+\alpha}{N+\alpha} - \left[\frac{2+\alpha}{N-2} - \varepsilon\right](1+\bar{\delta})\right) S_{N,\alpha}^{\frac{N+\alpha}{2+\alpha}} \\
&\leq -(\bar{\delta}(p^*-1) - \varepsilon(1+\bar{\delta}))S_{N,\alpha}^{\frac{p^*}{p^*-1}}.
\end{aligned}
$$

Finally, taking $R \gg 1$, there exists $\varepsilon > 0$ such that

$$\partial_t M_R[u(t)] < -\varepsilon \quad \text{for all } 0 < t < T^*.$$

By integrating twice the previous estimate, one obtains

$$0 < V_R < V_R(0) + tV_R'(0) - \frac{\varepsilon}{2}t^2 \quad \text{for all } 0 < t < T^*.$$

This implies that $T^* < \infty$ and the proof is achieved.

6 Global existence and scattering of radial solutions

Let us recall for any time slab $I \subset \mathbb{R}$, the Strichartz spaces

$$S(I) := L^{2p^*}(I, L^{\frac{2Np^*}{Np^*-2}}), \quad S^1(I) := L^{2p^*}(I, L^{\frac{2Np^*(p^*-1)}{2+\alpha p^*}}).$$

Remarks 6.1.

(1) $(2p^*, \frac{2Np^*}{Np^*-2}) \in \Gamma^0$ and $(2p^*, \frac{2Np^*(p^*-1)}{2+\alpha p^*}) \in \Gamma^1$;

(2) $S := S(\mathbb{R})$ and $S^1 := S^1(\mathbb{R})$.

One says that the statement $(SC)(u_0)$ holds if:

For $u_0 \in \dot{H}^1$ with $\|\nabla u_0\| < \|\nabla Q\|$ and $E(u_0) < E(Q)$, the corresponding solution to (NLS) is global and satisfies

$$u \in S^1(\mathbb{R}).$$

By using Sobolev injection and Strichartz estimate, one has

$$\|e^{i\cdot\Delta}u_0\|_{S^1} \lesssim \|\nabla e^{i\cdot\Delta}u_0\|_S \lesssim \|\nabla u_0\|.$$

Thus, if $\|\nabla u_0\| \ll 1$, by the small data theory in Proposition 2.2, $(SC)(u_0)$ holds. Now, for each $\delta > 0$, define the set

$$S_\delta := \{u_0 \in \dot{H}^1_{rd}, \quad E[u_0] < \delta \quad \text{and} \quad \|\nabla u_0\| < \|\nabla Q\|\}.$$

Define also

$$E_c := \sup\{\delta > 0 \quad \text{s.t.} \quad u_0 \in S_\delta \Rightarrow (SC)(u_0) \quad \text{holds}\}.$$

The goal is to prove that $E_c = E[Q]$. By contradiction, assume that

$$E_c < E[Q]. \tag{26}$$

Then, there is a sequence u_n of solutions to (NLS) such that the data $u_{n,0} \in \dot{H}^1$ satisfies $\|\nabla u_{n,0}\| < \|\nabla Q\|$ and $E[u_{n,0}] \to E_c$ as $n \to \infty$ and $(SC)(u_{n,0})$ does not hold for any n.

The goal in this section is to show the existence of a critical solution u_c to (NLS) with data $u_{c,0}$ such that

$$\|\nabla u_{c,0}\| < \|\nabla Q\|, \quad E[u_c] = E_c \quad \text{and} \quad (SC)(u_{c,0}) \quad \text{does not hold}.$$

In the next result, the constant δ is given by Proposition 2.2.

Lemma 6.1. *Let a sequence $u_n \in \dot H^1_{rd}$ satisfying*

$$\|\nabla u_n\| < \|\nabla Q\| \quad \textit{and} \quad E(u_n) \to E_c.$$

Assume that $\|e^{i\cdot\Delta}u_n\|_{S^1(\mathbb{R})} \geq \delta > 0$ and that the profile decomposition of u_n satisfies one of the two following hypothesis

(1) $\liminf_n E(\psi_l^1(-\frac{t_n^1}{(\lambda_n^1)^2})) < E_c$;

(2) $\liminf_n E(\psi_l^1(-\frac{t_n^1}{(\lambda_n^1)^2})) = E_c$. *Moreover, if $s_n^1 := -\frac{t_n^1}{(\lambda_n^1)^2} \to s^* \in [-\infty, +\infty]$ and U_1 is the nonlinear profile associated to (ψ_1, s_n^1), then U_1 is global and $\|U_1\|_{S^1(\mathbb{R})} < \infty$.*

Then, $(SC)(u_n)$ holds, for a subsequence and large n.

Proof. Thanks to the profile decomposition, one has

$$\begin{aligned} u_n &= \sum_{j=1}^{M} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} e^{-\frac{it_n^j}{(\lambda_n^j)^2}\Delta} \psi^j\left(\frac{\cdot}{\lambda_n^j}\right) + W_n^M \\ &= \sum_{j=1}^{M} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} \psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right) + W_n^M. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\nabla Q\|^2 &> \|\nabla u_n\|^2 \\ &= \sum_{j=1}^{M} \|\nabla \psi^j\|^2 + \|\nabla W_n^M\|^2 + o_n(1) \\ &= \sum_{j=1}^{M} \left\|\nabla \psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right\|^2 + \|\nabla W_n^M\|^2 + o_n(1); \end{aligned}$$

and

$$E_c \leftarrow E(u_n) = \sum_{j=1}^{M} E\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) + E(W_n^M) + o_n(1).$$

By Corollary 4.1, one obtains, for $1 \leq j \leq M$,

$$\min\left\{E\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right), E(W_n^M)\right\} \geq 0.$$

One discusses two cases.

(1) Take for a subsequence $\lim_n E(\psi_l^1(-\frac{t_n^1}{(\lambda_n^1)^2})) < E_c$. Write

$$\left\|\nabla \psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}\right)\right\| = \|\nabla \psi^1\| < \|\nabla Q\|;$$

$$E(\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}\right)) \leq E_c + o_n(1) < E(Q).$$

Lemma 4.1 implies that

$$\begin{aligned} E\left(\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}\right)\right) &= \left\|\nabla\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}\right)\right\|^2 - \frac{1}{p^*}\int_{\mathbb{R}^N}\left(I_\alpha * \left|\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}\right)\right|^{p^*}\right)\left|\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}, x\right)\right|^{p^*} \mathrm{d}x \\ &= \frac{1}{p^*}\left(\left\|\nabla\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}\right)\right\|^2 - \int_{\mathbb{R}^N}\left(I_\alpha * \left|\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}\right)\right|^{p^*}\right)\left|\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}, x\right)\right|^{p^*} \mathrm{d}x\right) \\ &\quad + \left(1 - \frac{1}{p^*}\right)\left\|\nabla\psi_l^1\left(-\frac{t_n^1}{(\lambda_n^1)^2}\right)\right\|^2 \\ &\gtrsim \|\nabla\psi^1\|^2. \end{aligned}$$

Thus,

$$E_c \leftarrow E(u_n) \geq C\|\nabla\psi^1\|^2 + \sum_{j=2}^{M} E\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) + E(W_n^M) + o_n(1).$$

So,

$$\liminf_n E\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) < E_c, \quad \forall 2 \leq j \leq M.$$

Let U_j be the nonlinear profile associated to $(\psi^j, -\frac{t_n^j}{(\lambda_n^j)^2})$. Then, for large n,

$$\begin{aligned} \|\nabla U_j\| &= \left\|\nabla\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right\| + o_n(1) < \|\nabla Q\|; \\ E(U_j) &= E\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) + o_n(1) < E(Q). \end{aligned} \tag{27}$$

Then, with Proposition 4.1, one has

$$E(U_j) \simeq \|\nabla U_j(t)\|^2 \simeq \|\nabla U_j(0)\|^2.$$

By the aforementioned estimates, one has

$$\sum_{j=1}^{M} \|\nabla\psi^j\|^2 \leq \|\nabla u_n\|^2 + o_n(1) \leq \|\nabla Q\|^2.$$

So, there exists $j_0 > 0$ such that for $j \geq j_0$, $\|\nabla\psi^j\|$ is small enough. With Strichartz estimate, it follows that $\|e^{i\cdot\Delta}\psi^j\|_{S^1(\mathbb{R})} < \delta$ given by the small data theory. Then, the integral formula via the small data theory gives

$$\|U_j\|_{S^1(\mathbb{R})} \lesssim \|\nabla\psi^j\|, \quad \forall j \geq j_0.$$

Now, take for $\varepsilon_0 > 0$, the near solution

$$H_{n,\varepsilon_0} \coloneqq \sum_{j=1}^{M(\varepsilon_0)} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot - t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right), \quad \|e^{i\cdot\Delta}W_n^{M(\varepsilon_0)}\|_{S^1(\mathbb{R})} < \varepsilon_0,$$

and the rest

$$R_{n,\varepsilon_0} \coloneqq \sum_{j=1}^{M(\varepsilon_0)} \mathcal{F}\left[\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot - t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right] - \mathcal{F}(H_{n,\varepsilon_0}).$$

Thus,

$$u_n = H_{n,\varepsilon_0} + R_{n,\varepsilon_0}, \quad \|e^{i\cdot\Delta}R_{n,\varepsilon_0}\|_{S^1(\mathbb{R})} < 2\varepsilon_0.$$

Using (27), one writes

$$\|\nabla H_{n,\varepsilon_0}\|^2 \leq \|\nabla Q\|^2.$$

Moreover,

$$\begin{aligned}\|H_{n,\varepsilon_0}\|_{S(\mathbb{R})}^{2p^*} &\leq \int_{\mathbb{R}^N}\left(\sum_{j=1}^{M(\varepsilon_0)} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} \left\| U_j\left(\frac{t-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right\|_{\frac{2Np^*}{Np^*-2}}\right)^{2p^*} \mathrm{d}t \\ &\leq \sum_{j=1}^{M(\varepsilon_0)} \int_{\mathbb{R}^N} \left\| \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{t-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right\|_{\frac{2Np^*}{Np^*-2}}^{2p^*} \mathrm{d}t \\ &\quad + \sum_{I\subset[1,2p^*],|I|<2p^*} \int_{\mathbb{R}^N} \prod_{j_k\in I} \left\| \frac{1}{(\lambda_n^{j_k})^{\frac{N-2}{2}}} U_{j_k}\left(\frac{t-t_n^{j_k}}{(\lambda_n^{j_k})^2}, \frac{\cdot}{\lambda_n^{j_k}}\right)\right\|_{\frac{2Np^*}{Np^*-2}} \mathrm{d}t \\ &:= (A_n) + (B_n).\end{aligned}$$

Now, there exists C_0 independent of $M(\varepsilon_0)$ such that

$$\begin{aligned}(A_n) &\leq \left(\sum_{j=1}^{j_0} + \sum_{1+j_0}^{M(\varepsilon_0)}\right)\|U_j\|_{S^1(\mathbb{R})}^{2p^*} \\ &\leq \sum_{j=1}^{j_0}\|U_j\|_{S^1(\mathbb{R})}^{2p^*} + \sum_{1+j_0}^{M(\varepsilon_0)}\|\nabla\psi^j\|^{2p^*} \\ &\leq C_0.\end{aligned}$$

Arguing as previously, the orthogonality condition gives

$$\lim_n (B_n) = 0.$$

Thus,

$$\|H_{n,\varepsilon_0}\|_{L^{2p^*}(\mathbb{R},L^{\frac{2Np^*}{Np^*-2}})} \leq C_0.$$

Moreover, for large n,

$$\|e^{i\cdot\Delta}(u_n - H_{n,\varepsilon_0}(0))\|_{S^1(\mathbb{R})} \leq \|e^{i\cdot\Delta}\tilde{W}_n^{M(\varepsilon_0)}\|_{S^1(\mathbb{R})} \to 0.$$

Now,

$$\begin{aligned}\|\nabla R_{n,\varepsilon_0}\|_{S'(\mathbb{R})} &= \left\| \nabla\left[\mathcal{F}\left(\sum_{j=1}^{M(\varepsilon_0)} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right) - \sum_{j=1}^{M(\varepsilon_0)} \mathcal{F}\left(\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right)\right]\right\|_{S'(\mathbb{R})} \\ &= \Bigg\| \nabla\Bigg[\left(I_\alpha * \left|\sum_{j=1}^{M(\varepsilon_0)} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right|^{p^*}\right) \left|\sum_{j=1}^{M(\varepsilon_0)} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right|^{p^*-2} \\ &\quad\times \sum_{j=1}^{M(\varepsilon_0)} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right) \\ &\quad - \sum_{j=1}^{M(\varepsilon_0)} \frac{1}{(\lambda_n^j)^{\frac{(N-2)(2p^*-1)}{2}}} \left[I_\alpha * \left|U_j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right|^{p^*}\right] \left|U_j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right|^{p^*-2} U_j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\Bigg]\Bigg\|_{S'(\mathbb{R})}.\end{aligned}$$

Take a term of the previous quantity expansion

$$\begin{aligned}
(I) = \Bigg|\Bigg[I_a * \Bigg| \sum_{\substack{I\subset[1,M(\varepsilon_0)],\\ |I|=p^*-2}} \prod_{j\in I} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot - t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\Bigg| \mathfrak{R}\left(\frac{1}{(\lambda_n^k)^{N-1}} \nabla U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right) \bar{U}_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\right)\Bigg] \\
\times \Bigg| \sum_{\substack{I\subset[1,M(\varepsilon_0)],\\ |I|=p^*-2}} \prod_{j\in I} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot - t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\Bigg| \frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right) \\
- \Bigg[I_a * \Bigg| \frac{1}{(\lambda_n^k)^{\frac{a_k(N-2)}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\Bigg|^{p^*-2} \mathfrak{R}\left(\frac{1}{(\lambda_n^k)^{N-1}} \nabla U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right) \bar{U}_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\right)\Bigg] \\
\times \Bigg| \frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\Bigg|^{p^*-2} \frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right) \Bigg| \\
= \Bigg|\Bigg[I_a * \Bigg(\Bigg| \sum_{\substack{I\subset[1,M(\varepsilon_0)],\\ |I|=p^*-2}} \prod_{j\in I} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot - t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\Bigg| - \Bigg| \frac{1}{(\lambda_n^k)^{\frac{a_k(N-2)}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\Bigg|^{p^*-2}\Bigg) \\
\times \left\{\mathfrak{R}\left(\frac{1}{(\lambda_n^k)^{N-1}} \nabla U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right) \bar{U}_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\right)\right\}\Bigg] \Bigg| \frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\Bigg|^{p^*-2} \\
\times \frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right) \Bigg| + (II),
\end{aligned}$$

where (II) is a mixed term. Now, with the triangular and Hardy-Littlewood-Paley inequalities, one obtains

$$\begin{aligned}
\|(I) - (II)\|_{S'(\mathbb{R})} &\le \sum_{\substack{I\subset[1,M(\varepsilon_0)],\\ |I|=p^*-2}} \Bigg\|\Bigg[I_a * \Bigg(\Bigg| \prod_{j\in I, j\neq k} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot - t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\Bigg|\Bigg)\Bigg(\mathfrak{R}\left(\frac{1}{(\lambda_n^k)^{N-1}} \nabla U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right) \times \bar{U}_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\right)\Bigg)\Bigg] \\
&\quad \times \Bigg| \frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\Bigg|^{p^*-1} \Bigg\|_{S'(\mathbb{R})} \\
&\lesssim \left\| \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} U_j\left(\frac{\cdot - t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right) \frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}} U_k\left(\frac{\cdot - t_n^k}{(\lambda_n^k)^2}, \frac{\cdot}{\lambda_n^k}\right)\right\|_{L^{p^*}(\mathbb{R}, L^{\frac{r_1}{2}})}, \quad j \neq k \\
&\lesssim \left\| \left(\frac{\lambda_n^k}{\lambda_n^j}\right)^{\frac{2(N-1)(N-2)}{a+N} + \frac{N}{2} - 1} U_j\left((\cdot - t_n^j)\frac{(\lambda_n^k)^2}{(\lambda_n^j)^2}, \frac{\lambda_n^k}{\lambda_n^j}\right) U_k(\cdot - t_n^k)\right\|_{L^{p^*}\left(\mathbb{R}, L^{\frac{r_1}{2}}\right)}, \quad j \neq k.
\end{aligned}$$

Here, $r_1 := \frac{2Np^*(p^*-1)}{2+ap^*}$ and one used the estimate

$$\|\nabla \mathcal{F} \cdot\|_{S'} \lesssim \|\cdot\|_{S^1}^{2(p-1)} \|\nabla \cdot\|_S.$$

Thus, with the orthogonality property of the profile decomposition, it follows that

$$\|\nabla R_{n,\varepsilon_0}\|_{S'(\mathbb{R})} \to 0.$$

Then, applying the perturbation theory in Proposition 2.4, the property $SC(u_n)$ holds.

(2) By Proposition 4.1, one obtains, for $2 \le j \le M$,

$$\psi^j = 0, \quad \|\nabla W_n^M\| \to 0.$$

Thus, $\|\nabla W_n^1\| \to 0$ and

$$u_n = \frac{1}{(\lambda_n^1)^{\frac{N-2}{2}}} \psi_l^1\left(s_n^1, \frac{\cdot}{\lambda_n^1}\right) + W_n^1.$$

Taking the scaling

$$z_n := (\lambda_n^1)^{\frac{N-2}{2}} u_n(\lambda_n^1 \cdot), \quad \bar{W}_n := (\lambda_n^1)^{\frac{N-2}{2}} W_n(\lambda_n^1 \cdot).$$

Then,

$$\|\nabla z_n\| = \|\nabla u_n\| < \|\nabla Q\|$$

and

$$z_n = \psi_l^1(s_n^1, \cdot) + \bar{W}_n^1, \quad \|\nabla \bar{W}_n^1\| \to 0.$$

Moreover,

$$\|\nabla U_1(s_n^1) - \nabla \psi_l^1(s_n^1)\| \to 0.$$

Then,

$$\begin{aligned} &z_n = \psi_l^1(s_n^1, \cdot) + \bar{W}_n^1; \\ &E(U_1(s_n^1)) = E(\psi_l^1(s_n^1)) + o_n(1) \to E_c; \\ &\|\nabla U_1(s_n^1)\| = \|\nabla \psi_l^1(s_n^1)\| + o_n(1) = \|\nabla \psi_1\| + o_n(1) < \|\nabla Q\|. \end{aligned}$$

This case follows by applying the long time perturbation theory in Proposition 2.4 with $e = 0$, $\tilde{u} = U_1$ and $u_0 = z_n$. □

The proof of the next result, about the existence of a critical solution with a pre-compact flow, is omitted because it follows like in he study by Miao et al. [5].

Proposition 6.1. *Assume that $E_c < E[Q]$. Then, there exists a global solution u_c to* (NLS) *with data $u_{c,0}$ such that*

$$\|\nabla u_{c,0}\| < \|\nabla Q\|, \quad E[u_c] = E_c \quad \text{and} \quad (SC)(u_{c,0}) \quad \text{does not hold.}$$

Moreover, there exists a real function $\lambda : (T_*, T^*) \to \mathbb{R}_+$ such that the following set is pre-compact in $\dot{H}^1$,

$$K := \left\{ \frac{1}{\lambda(t)^{\frac{N-2}{2}}} u_c\left(t, \frac{x}{\lambda(t)}\right), \quad t \in (T_*, T^*) \right\}.$$

6.1 Rigidity theorem

The aim of this subsection is to establish the following Liouville-type result.

Theorem 6.1. *Let $N \ge 3$ and $0 < \alpha < N \le 4 + \alpha$. Let $u_0 \in \dot{H}_{rd}^1$ satisfying* (3) *and* (5) *and $u \in C((T_*, T^*), \dot{H}_{rd}^1)$ be the maximal solution to* (NLS). *If there exists a real function $\lambda : (T_*, T^*) \to \mathbb{R}_+$ such that the following set is pre-compact in $\dot{H}^1$,*

$$K := \left\{ \frac{1}{\lambda(t)^{\frac{N-2}{2}}} u_c\left(t, \frac{x}{\lambda(t)}\right), \quad t \in (T_*, T^*) \right\}.$$

Then $u_0 = 0$.

Proof. Let $u := u_c$ given in the previous theorem and $v := \frac{1}{\lambda^{\frac{N-2}{2}}}u_c(\cdot,\frac{\cdot}{\lambda})$. One discusses two cases.

(1) First case: $\inf\lambda(t) > \lambda_0 > 0$. Following lines in the study by Miao et al. [5], one has $T^* = \infty$. Moreover, by Hardy inequality

$$\|\frac{u}{|\cdot|}\| \leq \frac{2}{N-2}\|\nabla u\|, \quad \forall N \geq 3$$

and the variational analysis, one has, for any $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\int_{|x|>R(\varepsilon)}\left(\frac{|u|^2}{|x|^2} + |\nabla u|^2\right)dx + \int_{|x|>R(\varepsilon)\cup|y|>R(\varepsilon)} (I_\alpha * |u|^{p^*})|u|^{p^*}dxdy < \varepsilon.$$

Indeed, with Hardy-Littlewood-Sobolev inequality, via the fact that $\nabla u \in L^\infty(\mathbb{R}, L^2)$, one has

$$\begin{aligned}\int_{|x|>R(\varepsilon)\cup|y|>R(\varepsilon)} (I_\alpha * |u|^{p^*})|u|^{p^*}dxdy &\leq C\|u\|^{p^*}_{L^{\frac{2N}{N-2}}(|x|>R(\varepsilon))}\|u\|^{p^*}_{L^{\frac{2N}{N-2}}}\\ &\leq C\|\nabla u\|^{p^*}_{L^2(|x|>R(\varepsilon))}\|\nabla u\|^{p^*}\\ &< \frac{\varepsilon}{2}.\end{aligned}$$

Take the truncated variance defined in Section 5,

$$V_R := \int_{\mathbb{R}^N} \psi_R(x)|u(.,x)|^2 dx.$$

Taking account of computation in section 5 via Hardy inequality, one writes

$$\frac{1}{4}V_R'' = \|\nabla u\|^2 - \int_{\mathbb{R}^N}|u|^{p^*}(I_\alpha * |u|^{p^*})dx + o_R(1).$$

By computing using the variational estimates, we obtain

$$\begin{aligned}F\left(\frac{\|\nabla u\|^2}{\|\nabla Q\|^2}\right) &:= \left(\frac{\|\nabla u\|}{\|\nabla Q\|}\right)^2 - \left(\frac{\|\nabla u\|}{\|\nabla Q\|}\right)^{2p^*}\\ &\leq \left(\frac{\|\nabla u\|}{\|\nabla Q\|}\right)^2 - \left(\frac{1}{\|\nabla Q\|}\right)^{2p^*}\left(S_{N,\alpha}^{p^*}\int_{\mathbb{R}^N}|u|^{p^*}(I_\alpha * |u|^{p^*})dx\right)\\ &\leq \frac{1}{\|\nabla Q\|^2}\left(\|\nabla u\|^2 - \int_{\mathbb{R}^N}|u|^{p^*}(I_\alpha * |u|^{p^*})dx\right).\end{aligned}$$

Now, since $p^* > 1$, there exists $C_\delta > 0$ such that $F(x) > C_\delta x$ for $0 < x < 1 - \delta$. Thus, by Proposition 4.1, one obtains

$$V_R'' > C_\delta\|\nabla u_0\|^2.$$

On the other hand, by Hardy and Hölder inequalities,

$$\begin{aligned}|V_R'| &= 2R\left|\Im\left(\int_{\mathbb{R}^N}\nabla\psi\left(\frac{x}{R}\right)\bar{u}\nabla u\right)dx\right|\\ &\lesssim R\int_{\{|x|<2R\}}|u\nabla u|dx\\ &\lesssim R^2\left\|\frac{u}{|\cdot|}\nabla u\right\|_1\\ &\lesssim R^2\|\nabla u_0\|^2.\end{aligned}$$

Then, integrating in time, one obtains

$$t\|\nabla u_0\|^2 \lesssim |V_R'(t) - V_R'(0)| \lesssim R^2\|\nabla u_0\|^2.$$

This gives $u_0 = 0$.

(2) The case $\inf_t \lambda(t) = 0$ follows similarly as in the study by Kenig and Merle [8]. □

6.2 Proof of the global existence and scattering

Now, we are ready to establish the second part of Theorem 2.1. Theorem 6.1 applied to u_c, via Proposition 6.1, gives $u_{c,0} = 0$. This contradicts the equality $\|u_c\|_{S^1(\mathbb{R})} = \infty$. This absurdity means that (26) is false. Thus, the $\dot{H}^1$ scattering holds by Proposition 2.3.

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Appendix

This section contains two parts. The first one is about a proof of the scattering criteria in Proposition 2.3, and the second one gives an elementary proof of the linear profile decomposition in Proposition 2.5.

A.1 Proof of a scattering criteria

Let us prove Proposition 2.3. By Duhamel formula, one writes

$$u - e^{i\cdot\Delta}u_0 = i\int_0^{\cdot} e^{i(\cdot-s)\Delta}\mathcal{F}ds;$$

$$\phi - u_0 = i\int_0^{\infty} e^{-is\Delta}\mathcal{F}ds;$$

$$u(t) - e^{i\cdot\Delta}\phi = -i\int_{\cdot}^{\infty} e^{i(\cdot-s)\Delta}\mathcal{F}ds.$$

Take the admissible pair $(q,r) := (2p^*, \frac{2Np^*}{Np^*-2}) \in \Gamma$. Thanks to Strichartz estimate and Hardy-Littlewood-Sobolev inequality, it follows that

$$\begin{aligned}
&\|\nabla(u - e^{it\Delta}\phi)\|_{S(\mathbb{R})}\\
&\lesssim \|(I_\alpha * |u|^{p^*})|u|^{p^*-2}\nabla u\|_{L^{q'}((t,\infty),L^{r'})} + \|(I_\alpha * |u|^{p^*-2}\Re(\bar{u}\nabla u))|u|^{p^*-2}u\|_{L^{q'}((t,\infty),L^{r'})}\\
&\lesssim \|u\|^{2(p^*-1)}_{L^{2p^*}((t,\infty),L^{\frac{2Np^*(p^*-1)}{2+\alpha p^*}})}\|\nabla u\|_{L^{2p^*}((t,\infty),L^{\frac{2Np^*}{Np^*-2}})}\\
&\lesssim \|u\|^{2(p^*-1)}_{S^1((t,\infty))}\|\nabla u\|_{S((t,\infty))}.
\end{aligned}$$

Letting $t \gg 1$ so that $\|u\|_{S^1(t,\infty)} \ll 1$, one obtains

$$\nabla u \in S(\mathbb{R}).$$

Then,

$$\|\nabla(u - e^{it\Delta}\phi)\|_{S(\mathbb{R})} \lesssim \|u\|^{2(p^*-1)}_{S^1(t,\infty)}\|\nabla u\|_{S(t,\infty)} \to 0, \quad \text{if } t \to \infty.$$

This achieves the proof.

A.2 Proof of the profile decomposition

This subsection contains a variant proof of Proposition 2.5. This elementary proof is available for an integer p^*. Taking account of [17], the last equality is the only point to prove. It is sufficient to prove that

$$\mathcal{N}(u_n) = \sum_{j=1}^{M}\mathcal{N}\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) + \mathcal{N}(W_n^M) + o_n(1).$$

First step: one proves that

$$\mathcal{N}\left(\sum_{j=1}^{M}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}, \frac{\cdot}{\lambda_n^j}\right)\right) = \sum_{j=1}^{M}\mathcal{N}\left(\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right) + o_n(1).$$

By re-indexing and adjusting profiles, one can assume that there exists $1 \le M_0 \le M$ such that

(1) $\frac{t_n^j}{(\lambda_n^j)^2} = 0$, for any $1 \le j \le M_0$;

(2) $\frac{|t_n^j|}{(\lambda_n^j)^2} \to \infty$, for any $M_0 < j \le M$.

Now, for $1 \le j < k \le M_0$, we computed with some integrations by parts

$$\begin{aligned}
(A_n) &:= \mathcal{N}\left(\sum_{j=1}^{M_0} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right)\right) - \sum_{j=1}^{M_0}\mathcal{N}(\psi^j)\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)\left[\left|\sum_{j=1}^{M_0}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{y}{\lambda_n^j}\right)\right|^{p^*}\left|\sum_{k=1}^{M_0}\frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}}\psi^k\left(\frac{x}{\lambda_n^k}\right)\right|^{p^*} - \sum_{j=1}^{M_0}|\psi^j(y)|^{p^*}|\psi^j(x)|^{p^*}\right]\mathrm{d}x\mathrm{d}y\\
&\le \sum_{\substack{l_1+\dots+l_{M_0}=p^*,\\ h_1+\dots+h_{M_0}=p^*,\\ (l_j,h_j)\ne(p^*,p^*)}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[I_\alpha(x-y)\prod_{j,k=1}^{M_0}\left|\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{y}{\lambda_n^j}\right)\right|^{l_j}\left|\frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}}\psi^k\left(\frac{x}{\lambda_n^k}\right)\right|^{h_k}\right]\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Thus, with Hardy-Littlewood-Sobolev inequality and denoting for short $\int\cdot := \int_{\mathbb{R}^N\times\mathbb{R}^N}\cdot$,

$$\begin{aligned}
(A_n) &\le \sum_{\substack{l_1+\dots+l_{M_0}=p^*,\\ h_1+\dots+h_{M_0}=p^*,\\ (l_j,h_j)\ne(p^*,p^*)}}\int\left[\frac{I_\alpha(x-y)}{(\lambda_n^1)^{N-\alpha}}(\lambda_n^1)^{2N}\prod_{j,k=1}^{M_0}\left|\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{\lambda_n^1}{\lambda_n^j}y\right)\right|^{l_j}\left|\frac{1}{(\lambda_n^k)^{\frac{N-2}{2}}}\psi^k\left(\frac{\lambda_n^1}{\lambda_n^k}x\right)\right|^{h_k}\right]\mathrm{d}x\mathrm{d}y\\
&\le \sum_{\substack{l_1+\dots+l_{M_0}=p^*,\\ h_1+\dots+h_{M_0}=p^*,\\ (l_j,h_j)\ne(p^*,p^*)}}\int\left[I_\alpha(x-y)\prod_{j,k=1}^{M_0}\left|\left(\frac{\lambda_n^1}{\lambda_n^j}\right)^{\frac{N-2}{2}}\psi^j\left(\frac{\lambda_n^1}{\lambda_n^j}y\right)\right|^{l_j}\left|\left(\frac{\lambda_n^1}{\lambda_n^k}\right)^{\frac{N-2}{2}}\psi^k\left(\frac{\lambda_n^1}{\lambda_n^k}x\right)\right|^{h_k}\right]\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where without loss of generality, one assumes that , for $1 < j \le M_0$, yield $\frac{\lambda_n^j}{\lambda_n^1} \to \infty$ and $l_1 \ne 0$. Then,

$$\mathcal{N}\left(\sum_{j=1}^{M_0}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right)\right) = \sum_{j=1}^{M_0}\mathcal{N}(\psi^j) + o_n(1).$$

Now, an expansion gives

$$\begin{aligned}
(B_n) &:= \mathcal{N}\left[\sum_{j=1}^{M_0}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right) + \sum_{j=1+M_0}^{M}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi_l^j\left(\frac{\cdot}{\lambda_n^j}\right)\right] - \mathcal{N}\left[\sum_{j=1}^{M_0}\psi^j\right]\\
&= \int_{\mathbb{R}^N}\left(I_\alpha * \left|\sum_{j=1}^{M_0}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right) + \sum_{j=1+M_0}^{M}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right)\right|^{p^*}\right)\\
&\quad\times\left|\sum_{j=1}^{M_0}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{x}{\lambda_n^j}\right) + \sum_{j=1+M_0}^{M}\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi^j\left(\frac{x}{\lambda_n^j}\right)\right|^{p^*}\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N}\left(I_\alpha * \left|\sum_{j=1}^{M_0}\psi^j\right|^{p^*}\right)\left|\sum_{j=1}^{M_0}\psi^j\right|^{p^*}\mathrm{d}x.
\end{aligned}$$

Thus,

$$
\begin{aligned}
(B_n) \leq & \int I_\alpha(x-y)\Bigg| \left(\sum_{j=1}^{M_0} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{x}{\lambda_n^j}\right) + \sum_{j=1+M_0}^{M} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi^j\left(\frac{x}{\lambda_n^j}\right)\right)^{p^*} \\
& \cdot \left(\sum_{j=1}^{M_0} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{y}{\lambda_n^j}\right) + \sum_{j=1+M_0}^{M} \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi^j\left(\frac{y}{\lambda_n^j}\right)\right)^{p^*} - \left(\sum_{j=1}^{M_0}\psi^j(x)\right)^{p^*}\left(\sum_{j=1}^{M_0}\psi^j(y)\right)^{p^*}\Bigg|\, \mathrm{d}x\mathrm{d}y \\
\leq & \sum_{\substack{k_1+\cdots+k_M=p^*,\\ h_1+\cdots+h_M=p^*,\\ k_i+h_i\neq 2p^*}} \int I_\alpha(x-y)\left[\prod_{j=1}^{M_0}\left|\frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\psi^j\left(\frac{x}{\lambda_n^j}\right)\right|^{k_j} \prod_{j'=1+M_0}^{M}\left|\frac{1}{(\lambda_n^{j'})^{\frac{N-2}{2}}} e^{-i\frac{t_n^{j'}}{(\lambda_n^{j'})^2}\Delta}\psi^{j'}\left(\frac{x}{\lambda_n^{j'}}\right)\right|^{k_{j'}}\right. \\
& \times \left.\prod_{i=1}^{M_0}\left|\frac{1}{(\lambda_n^i)^{\frac{N-2}{2}}}\psi^i\left(\frac{y}{\lambda_n^i}\right)\right|^{h_i} \prod_{i'=1+M_0}^{M}\left|\frac{1}{(\lambda_n^{i'})^{\frac{N-2}{2}}} e^{-i\frac{t_n^{i'}}{(\lambda_n^{i'})^2}\Delta}\psi^{i'}\left(\frac{y}{\lambda_n^{i'}}\right)\right|^{h_{i'}}\right]\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Thus, taking account of Corollary 2.6, one obtains

$$
\begin{aligned}
(B_n) \lesssim & \sum_{\substack{k_1+\cdots+k_M=p^*,\\ h_1+\cdots+h_M=p^*,\\ k_i+h_i\neq 2p^*}} \left[\prod_{j=1}^{M_0}\frac{1}{(\lambda_n^j)^{\frac{k_j(N-2)}{2}}}\left\|\psi^j\left(\frac{\cdot}{\lambda_n^j}\right)\right\|_{\frac{2Np^*}{\alpha+N}}^{k_j} \prod_{j'=1+M_0}^{M}\frac{1}{(\lambda_n^{j'})^{\frac{k_{j'}(N-2)}{2}}}\left\|e^{-i\frac{t_n^{j'}}{(\lambda_n^{j'})^2}\Delta}\psi^{j'}\left(\frac{\cdot}{\lambda_n^{j'}}\right)\right\|_{\frac{2Np^*}{\alpha+N}}^{k_{j'}}\right. \\
& \times \left.\prod_{i=1}^{M_0}\frac{1}{(\lambda_n^j)^{\frac{h_i(N-2)}{2}}}\left\|\psi^i\left(\frac{\cdot}{\lambda_n^i}\right)\right\|_{\frac{2Np^*}{\alpha+N}}^{h_i} \prod_{i'=1+M_0}^{M}\frac{1}{(\lambda_n^{i'})^{\frac{h_{i'}(N-2)}{2}}}\|e^{-it_n^{i'}\Delta}\psi^{i'}\|_{\frac{2Np^*}{\alpha+N}}^{h_{i'}}\right] \\
\lesssim & \sum_{\substack{k_1+\cdots+k_M=p^*,\\ h_1+\cdots+h_M=p^*,\\ k_i+h_i\neq 2p}} \left[\prod_{j=1}^{M_0}\|\psi^j\|_{\dot H^1}^{k_j}\prod_{j'=1+M_0}^{M}\left\|e^{-i\frac{t_n^{j'}}{(\lambda_n^{j'})^2}\Delta}\psi^{j'}\right\|_{\frac{2N}{N-2}}^{k_{j'}}\prod_{i=1}^{M_0}\|\psi^i\|_{\dot H^1}^{h_i}\prod_{i'=1+M_0}^{M}\left\|e^{-i\frac{t_n^{i'}}{(\lambda_n^{i'})^2}\Delta}\psi^{i'}\right\|_{\frac{2N}{N-2}}^{h_{i'}}\right].
\end{aligned}
$$

For $M_0 < j \leq M$, via Hardy-Littlewood-Sobolev inequality, the free kernel Schrödinger decay and a density argument [25], one obtains

$$
\left\|\psi_l^j\left(-\frac{t_n^j}{(\lambda_n^j)^2}\right)\right\|_{\frac{2N}{N-2}} \lesssim \frac{(\lambda_n^j)^2}{t_n^j}\|\psi^j\|_{\frac{2N}{N+2}} \to 0.
$$

Thus, $(B_n) \to 0$. This finishes the first step. Now, one writes using Hardy-Littlewood-Sobolev injection

$$
\begin{aligned}
\mathcal{N}[W_n^{M_1}] &\leq \sup_t \mathcal{N}[e^{it\Delta}W_n^{M_1}] \\
&\leq \sup_t \int_{\mathbb{R}^N}(I_\alpha * |e^{it\Delta}W_n^{M_1}|^{p^*})|e^{it\Delta}W_n^{M_1}|^{p^*}\,\mathrm{d}x \\
&\leq \sup_t \|e^{it\Delta}W_n^{M_1}\|_{\frac{2N}{N-2}}^{2p^*}.
\end{aligned}
$$

Then, by the asymptotic smallness,

$$
\lim_{M_1\to\infty}(\lim_{n\to\infty}\mathcal{N}[W_n^{M_1}]) = 0.
$$

Take $M \geq 1$ and $\varepsilon > 0$. By Sobolev injections, Hölder inequality and Proposition 2.1, one obtains

$$
\sup_n \mathcal{N}[u_n] + \sup_n \mathcal{N}[W_n^M] \lesssim 1.
$$

Thus, one can choose $M_1 \geq M$ and $N_1 > 0$ such that for $n \geq N_1$,

$$\begin{aligned}
(C_n) &:= |\mathcal{N}[u_n - W_n^{M_1}] - \mathcal{N}[u_n]| + |\mathcal{N}[W_n^M - W_n^{M_1}] - \mathcal{N}[W_n^M]| \\
&\lesssim \int I_\alpha(x-y)|(u_n(x) - W_n^{M_1}(x))^{p^*}(u_n(y) - W_n^{M_1}(y))^{p^*} - (u_n(x)u_n(y))^{p^*}|\mathrm{d}x\mathrm{d}y \\
&\quad + \int I_\alpha(x-y)|(W_n^M(x) - W_n^{M_1}(x))^{p^*}(W_n^M(y) - W_n^{M_1}(y))^{p^*} - (W_n^M(x)W_n^M(y))^{p^*}|\mathrm{d}x\mathrm{d}y \\
&\lesssim \sum_{\substack{i_j+l_j=i_k+l_k=p^*,\\ i_j,i_k<p^*}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y) \prod_{1\le j,l\le p} |u_n(x)|^{i_j}|u_n(y)|^{i_k}|W_n^{M_1}(x)|^{l_j}|W_n^{M_1}(y)|^{l_k}\mathrm{d}x\mathrm{d}y \\
&\quad + \sum_{\substack{i_j+l_j=i_k+l_k=p,\\ i_j,i_k<p}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y) \prod_{1\le j,l\le p} |W_n^M(x)|^{i_j}|W_n^M(y)|^{i_k}|W_n^{M_1}(x)|^{l_j}|W_n^{M_1}(y)|^{l_k}\mathrm{d}x\mathrm{d}y \\
&\lesssim \sum_{\substack{i_j+l_j=i_k+l_k=p^*,\\ i_j,i_k<p^*}} \|u_n\|_{\frac{2Np^*}{\alpha+N}}^{i_j+i_k}\|W_n^{M_1}\|_{\frac{2Np^*}{\alpha+N}}^{l_j+l_k} + \sum_{\substack{i_j+l_j=i_k+l_k=p^*,\\ i_j,i_k<p^*}} \|W_n^M\|_{\frac{2Np^*}{\alpha+N}}^{i_j+i_k}\|W_n^{M_1}\|_{\frac{2Np^*}{\alpha+N}}^{l_j+l_k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(C_n) &\lesssim \sum_{\substack{i_j+l_j=i_k+l_k=p^*,\\ i_j,i_k<p^*}} \|u_n\|_{\frac{2N}{N-2}}^{i_j+i_k}\|W_n^{M_1}\|_{\frac{2N}{N-2}}^{l_j+l_k} + \sum_{\substack{i_j+l_j=i_k+l_k=p,\\ i_j,i_k<p}} \|W_n^M\|_{\frac{2N}{N-2}}^{i_j+i_k}\|W_n^{M_1}\|_{\frac{2N}{N-2}}^{l_j+l_k} \\
&\lesssim \sum_{\substack{i_j+l_j=i_k+l_k=p^*,\\ i_j,i_k<p^*}} \|u_n\|_{\dot H^1}^{i_j+i_k}\|W_n^{M_1}\|_{\frac{2N}{N-2}}^{l_j+l_k} + \sum_{\substack{i_j+l_j=i_k+l_k=p,\\ i_j,i_k<p}} \|W_n^M\|_{\dot H^1}^{i_j+i_k}\|W_n^{M_1}\|_{\frac{2N}{N-2}}^{l_j+l_k} \\
&\lesssim \sum_{\substack{i_j+l_j=i_k+l_k=p^*,\\ i_j,i_k<p^*}} \|W_n^{M_1}\|_{\frac{2N}{N-2}}^{l_j+l_k} \\
&\le \varepsilon.
\end{aligned}$$

By the first step and the profile expansion, one takes $N_2 \ge N_1$ such that for $n \ge N_2$,

$$\mathcal{N}[u_n - W_n^{M_1}] - \sum_{j=1}^{M_1} \mathcal{N}\left[e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right)\right] < \varepsilon.$$

By the profile expansion,

$$W_n^M - W_n^{M_1} = \sum_{j=1+M}^{M_1} \frac{1}{(\lambda_n^j)^{\frac{N-2}{N}}} e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right).$$

So, with the first step, one takes $N_3 \ge N_2$ such that for $n \ge N_3$,

$$\mathcal{N}[W_n^M - W_n^{M_1}] - \sum_{j=1+M}^{M_1} \mathcal{N}\left[\frac{1}{(\lambda_n^j)^{\frac{N-2}{N}}} e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right)\right] < \varepsilon.$$

Taking account of the last three inequalities with right hand side ε, one obtains

$$\left|\mathcal{N}[u_n] - \sum_{j=1}^{M} \mathcal{N}\left[\frac{1}{(\lambda_n^j)^{\frac{N-2}{N}}} e^{-i\frac{t_n^j}{(\lambda_n^j)^2}\Delta}\psi^j\left(\frac{\cdot}{\lambda_n^j}\right)\right] - \mathcal{N}[W_n^M]\right| \lesssim \varepsilon.$$

This finishes the proof.