



Research Article

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Positive periodic solutions for discrete time-delay hematopoiesis model with impulses

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Abstract: The present article is devoted to the positive periodic solution for impulsive discrete hematopoiesis model. This model is described by a first-order nonlinear difference equation with multiple delays and impulses. The existence result is obtained by applying the Krasnosel'skii fixed point theorem. A sufficient condition that guarantees that there is at least one positive periodic solution is established. Moreover, illustrative example and numerical simulation analysis are provided to show the existence of positive periodic solutions and the effect of impulses on positive periodic solutions.

Keywords: discrete hematopoiesis model, impulses, positive periodic solutions, Krasnosel'skii fixed point theorem

MSC 2020: 39A23, 39A60, 47H10, 92C37

1 Introduction

Consider the following discrete nonlinear hematopoiesis model subject to periodic impulses:

$$\begin{cases} \Delta x(k) = -a(k)x(k) + \sum_{i=1}^m b_i(k)f(x(k - \tau_i(k))), & k \in \mathbb{Z}^+ \text{ and } k \neq k_j, \\ \Delta x(k_j) = \gamma_j x(k_j), & j \in \mathbb{N} \end{cases} \quad (1.1)$$

with $m \in \mathbb{N}$. In this model, Δ is a forward-difference operator defined by $\Delta x(k) = x(k+1) - x(k)$; $f(u) = u/(1+u^n)$ ($u \geq 0, n > 1$) is the production function, which has a unimodal shape and reaches the peak value at point $u_* = \sqrt[n]{1/(n-1)}$. Here,

(A₁) $a : \mathbb{Z} \rightarrow (0, 1)$, $b_i : \mathbb{Z} \rightarrow (0, \infty)$, and $\tau_i : \mathbb{Z} \rightarrow \mathbb{Z}^+$ ($1 \leq i \leq m$) are ω -periodic discrete functions satisfying $a(k) = a(k + \omega)$, $b_i(k) = b_i(k + \omega)$, and $\tau_i(k) = \tau_i(k + \omega)$ for $k \in \mathbb{Z}$ and the common period $\omega \in \mathbb{N}$;

(A₂) $\{\gamma_j\}_{j \in \mathbb{N}}$ with $\gamma_j > -1$ is a ω -periodic sequence of real numbers, and $\{k_j\}_{j \in \mathbb{N}}$ is a ω -periodic sequence of natural numbers, i.e., there exists a natural number $q \in \mathbb{N}$ such that $k_{j+q} = k_j + \omega$ and $\gamma_{j+q} = \gamma_j$. Also, $0 < k_1 < k_2 < \dots < k_j < k_{j+1} < \dots$ for $j \in \mathbb{N}$.

(A₃) $\prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r)) < 1$.

Let $\mathbb{Z}[-\bar{\tau}, 0] = \{-\bar{\tau}, -\bar{\tau} + 1, \dots, 0\}$. We consider the impulsive system (1.1) with the initial value condition

$$x(k) = \phi(k) > 0 \quad \text{for } k \in \mathbb{Z}[-\bar{\tau}, 0], \quad (1.2)$$

in which $\bar{\tau} = \max_{1 \leq i \leq m} \{\max_{0 \leq k < \omega} \tau_i(k)\} \in \mathbb{Z}^+$. Denote the solution of equation (1.1) satisfying the initial value condition (1.2) by $x(\cdot; \phi)$, since $0 < a(k) < 1$ for $k \in \mathbb{Z}$ and $\gamma_j > -1$ for $j \in \mathbb{N}$, we see that $x(\cdot; \phi)$ is positive.

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Remark 1.1. The periodic model (1.1) can be reformulated as an impulsive difference equation of form

$$\begin{cases} x(k+1) = G(k, x(k), x(k - \tau_1(k)), \dots, x(k - \tau_m(k))), & k \in \mathbb{Z}^+ \text{ and } k \neq k_j, \\ x(k_j+1) = (1 + \gamma_j)x(k_j), & j \in \mathbb{N}, \end{cases}$$

where $G(k, u_0, \dots, u_m) = (1 - a(k))u_0 + \sum_{i=1}^m b_i(k)f(u_i)$ is ω -periodic with respect to the first variable k and is continuous with respect to variable $u_i (1 \leq i \leq m)$.

The nonlinear continuous model with single delay

$$x'(t) = -a x(t) + b f(x(t - \tau)) \quad (1.3)$$

was first introduced by Mackey and Glass [1] to understand the dynamics of production of blood cells. Here, x is the density of mature blood cells in the blood circulation; a is the lost rate of blood cells in the circulation; the term $b f(x(t - \tau))$ represents the influx of blood cells into the bloodstream from multipotential hematopoietic stem cells; b is the maximal production rate; τ is the time delay required for immature cells produced in bone marrow to be released into the bloodstream as mature cells. The production function f in this model reflects a mixed feedback control in the hematopoietic system. Specifically, as the blood cell level gradually increases from zero, it increases monotonically at the beginning, saturates at an appropriate blood cell level, and then decreases monotonically. The positive equilibrium of equation (1.3) has been studied in several works (see, e.g., [2–4]).

Clinical experiments have shown that periodic seasonal changes greatly affect the population density and internal composition of organisms [5,6]. However, the autonomous model with constant coefficients and constant time delays cannot reveal such external periodic effects to the blood cell dynamics. Taking the internal periodicity of organisms into account, modifications of equation (1.3) have been developed by assuming that the formation, maturation, and death of blood cells are periodic. The existence and stability of positive periodic solutions of such periodic hematopoiesis models are main research themes. We refer to [7–12].

It has been recognized that when considering a population with nonoverlapping generations or with small population number, the discrete models are more suitable than continuous ones [13–15]. In addition, discrete models can also provide more efficient methods for numerical computations and simulations. Therefore, it is very meaningful to study the discrete hematopoiesis models. For example, we refer to [16–19] for the discrete hematopoietic dynamics.

In the normal hematopoiesis process of organisms, the number of various blood cells is in a dynamic stable state. However, impulsive phenomena that the states are subject to external perturbations at certain moments of time are observed in many evolutionary processes [20]. In the hematopoietic system, such instantaneous disturbance results in sudden changes in blood cell number [21,22]. There are some research works that investigated the dynamics of blood cells in the disturbed environment [23–25]. These studies are well conducted from continuous perspective. By contrast, discrete dynamical models considering the impulse effects are much less explored.

The purpose of this article is to elucidate a sufficient condition that guarantees the existence of positive ω -periodic solutions of discrete impulsive system (1.1). The rest of this article is organized as follows. In Section 2, the solution transformation and some preparations about completely continuous operator in a cone are considered. Section 3 presents the existence of positive ω -periodic solutions. In Section 4, a numerical example and some simulations are given. Finally, a short conclusion is provided in Section 5.

2 Solution transformation and auxiliary preparations

We discuss the nonlinear difference equation with delays

$$\Delta y(k) = -\tilde{a}(k)y(k) + \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \sum_{i=1}^m \tilde{b}_i(k) f \left(y(k - \tau_i(k)) \prod_{0 \leq k_j < k - \tau_i(k)} (1 + \gamma_j) \right), \quad (2.1)$$

where

$$\tilde{a}(k) = \begin{cases} a(k), & k \neq k_j, \\ 0, & k = k_j, j = 1, 2, \dots, \end{cases} \quad \text{and} \quad \tilde{b}_i(k) = \begin{cases} b_i(k), & k \neq k_j, \\ 0, & k = k_j, j = 1, 2, \dots \end{cases}$$

The initial value condition is $y(k) = \phi(k) > 0$ for $k \in \mathbb{Z}[-\bar{\tau}, 0]$. We give the following lemma, which is essential to obtain our main result.

Lemma 2.1. *Suppose that (A_1) and (A_2) hold. Then,*

- (i) *if $y(k)$ is a solution of equation (2.1), then $x(k) = y(k) \prod_{0 \leq k_j < k} (1 + \gamma_j)$ is a solution of equation (1.1);*
- (ii) *if $x(k)$ is a solution of equation (1.1), then $y(k) = x(k) \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j}$ is a solution of equation (2.1).*

Proof of part (i). From equation (2.1), we see that $y(k_j + 1) = y(k_j)$ ($j = 1, 2, \dots$). Then, we have

$$x(k_j + 1) = y(k_j + 1) \prod_{0 \leq k_\ell < k_j + 1} (1 + \gamma_\ell) = (1 + \gamma_j) y(k_j) \prod_{0 \leq k_\ell < k_j} (1 + \gamma_\ell) = (1 + \gamma_j) x(k_j).$$

For any $k \neq k_j$ ($j = 1, 2, \dots$), it follows that

$$\begin{aligned} & x(k+1) - (1 - a(k))x(k) - \sum_{i=1}^m b_i(k)f(x(k - \tau_i(k))) \\ &= y(k+1) \prod_{0 \leq k_j < k+1} (1 + \gamma_j) - (1 - \tilde{a}(k))y(k) \prod_{0 \leq k_j < k} (1 + \gamma_j) - \sum_{i=1}^m \tilde{b}_i(k)f\left(y(k - \tau_i(k)) \prod_{0 \leq k_j < k - \tau_i(k)} (1 + \gamma_j)\right) \\ &= \prod_{0 \leq k_j < k} (1 + \gamma_j) \times \left[y(k+1) - (1 - \tilde{a}(k))y(k) - \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \sum_{i=1}^m \tilde{b}_i(k)f\left(y(k - \tau_i(k)) \prod_{0 \leq k_j < k - \tau_i(k)} (1 + \gamma_j)\right) \right] = 0. \end{aligned}$$

Therefore, $x(k) = y(k) \prod_{0 \leq k_j < k} (1 + \gamma_j)$ is a solution of equation (1.1). \square

Proof of part (ii). It is easy to see that

$$\begin{aligned} y(k_j + 1) &= x(k_j + 1) \prod_{0 \leq k_\ell < k_j + 1} \frac{1}{1 + \gamma_\ell} = (1 + \gamma_j) x(k_j) \prod_{0 \leq k_\ell < k_j + 1} \frac{1}{1 + \gamma_\ell} \\ &= x(k_j) \prod_{0 \leq k_\ell < k_j} \frac{1}{1 + \gamma_\ell} = y(k_j) \end{aligned}$$

for $j = 1, 2, \dots$, namely, equation (2.1) is satisfied by $k = k_j$. For any $k \neq k_j$ ($j = 1, 2, \dots$), from equation (1.1), we have

$$\begin{aligned} \Delta y(k) &= y(k+1) - y(k) \\ &= x(k+1) \prod_{0 \leq k_j < k+1} \frac{1}{1 + \gamma_j} - x(k) \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \\ &= (x(k+1) - x(k)) \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \\ &= \left[-a(k)x(k) + \sum_{i=1}^m b_i(k)f(x(k - \tau_i(k))) \right] \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \\ &= -\tilde{a}(k)y(k) + \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \sum_{i=1}^m \tilde{b}_i(k)f(y(k - \tau_i(k)) \prod_{0 \leq k_j < k - \tau_i(k)} (1 + \gamma_j)). \end{aligned}$$

Thus, we can conclude that $y(k) = x(k) \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j}$ is a solution of equation (2.1). \square

Define

$$F(k, s) = \frac{\prod_{s+1 \leq r < k+\omega} (1 - \tilde{a}(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \quad \text{for } k \leq s \leq k + \omega - 1, k \in \mathbb{Z}. \quad (2.2)$$

The assumptions (A₁) and (A₃) ensure the positivity of F , and basic computation shows that

$$\begin{aligned} F(k + \omega, s + \omega) &= \frac{\prod_{s+\omega+1 \leq r < k+2\omega} (1 - \tilde{a}(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \\ &= \frac{\prod_{s+1 \leq r < k+\omega} (1 - \tilde{a}(r + \omega))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \\ &= \frac{\prod_{s+1 \leq r < k+\omega} (1 - \tilde{a}(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} = F(k, s). \end{aligned} \quad (2.3)$$

Hence, F is ω -periodic with respect to both variables k and s for $k \leq s \leq k + \omega - 1$ and $k \in \mathbb{Z}$.

Let X be the space of all ω -periodic discrete functions defined by $X = \{x : x(k) = x(k + \omega) \in \mathbb{R} \text{ for } k \in \mathbb{Z}\}$ with the norm $\|x\| = \max_{0 \leq k < \omega} |x(k)|$. Then, it follows that X is a finite-dimensional Banach space. For an element x of X satisfying $x(k) > 0$ for $k \in \mathbb{Z}$, we make $\hat{x}(k) = x(k)$ for $k \in \mathbb{Z}[-\bar{\tau}, \infty) \stackrel{\text{def}}{=} [-\bar{\tau}, \infty) \cap \mathbb{Z}$ and $\hat{\phi}(k) = x(k)$ for $k \in \mathbb{Z}[-\bar{\tau}, 0]$.

Lemma 2.2. *A positive ω -periodic discrete function \hat{x} is the solution $x(\cdot; \hat{\phi})$ of equation (1.1) if and only if the original function $x \in X$ satisfies that $x(k) > 0$ and*

$$x(k) = \sum_{s=k}^{k+\omega-1} \left(F(k, s) \prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right), \quad (2.4)$$

where F is defined by equation (2.2).

Proof. If \hat{x} is the solution $x(\cdot; \hat{\phi})$ of equation (1.1), then it is clear that $x(k) > 0$ for $k \in \mathbb{Z}$. Equation (2.1) can be rewritten into the following form:

$$y(k+1) - (1 - \tilde{a}(k))y(k) = \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \sum_{i=1}^m \tilde{b}_i(k) f \left(y(k - \tau_i(k)) \prod_{0 \leq k_j < k - \tau_i(k)} (1 + \gamma_j) \right).$$

By Lemma 2.1, multiplying both sides of the above equality by $\prod_{0 \leq r < k+1} 1/(1 - \tilde{a}(r))$ leads to

$$\begin{aligned} &x(k+1) \prod_{0 \leq k_j < k+1} \frac{1}{1 + \gamma_j} \prod_{0 \leq r < k+1} \frac{1}{1 - \tilde{a}(r)} - x(k) \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \prod_{0 \leq r < k} \frac{1}{1 - \tilde{a}(r)} \\ &= \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \sum_{i=1}^m \tilde{b}_i(k) f(x(k - \tau_i(k))) \prod_{0 \leq r < k+1} \frac{1}{1 - \tilde{a}(r)}. \end{aligned}$$

Note that $x(k + \omega) = x(k)$. Sum up both sides from k to $k + \omega - 1$, which produces

$$\begin{aligned} &x(k) \left(\prod_{0 \leq k_j < k+\omega} \frac{1}{1 + \gamma_j} \prod_{0 \leq r < k+\omega} \frac{1}{1 - \tilde{a}(r)} - \prod_{0 \leq k_j < k} \frac{1}{1 + \gamma_j} \prod_{0 \leq r < k} \frac{1}{1 - \tilde{a}(r)} \right) \\ &= \sum_{s=k}^{k+\omega-1} \left(\prod_{0 \leq k_j < s} \frac{1}{1 + \gamma_j} \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \prod_{0 \leq r < s+1} \frac{1}{1 - \tilde{a}(r)} \right). \end{aligned}$$

Then, it turns out that

$$x(k) \left(1 - \prod_{k \leq k_j < k+\omega} (1 + \gamma_j) \prod_{k \leq r < k+\omega} (1 - \tilde{a}(r)) \right) = \sum_{s=k}^{k+\omega-1} \left(\prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \prod_{s+1 \leq r < k+\omega} (1 - \tilde{a}(r)) \right),$$

and hence, the periodicity of \tilde{a} and γ_j results in

$$\begin{aligned} x(k) &= \frac{1}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \sum_{s=k}^{k+\omega-1} \left(\prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \prod_{s+1 \leq r < k+\omega} (1 - \tilde{a}(r)) \right) \\ &= \sum_{s=k}^{k+\omega-1} \left(F(k, s) \prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \end{aligned}$$

for $k \in \mathbb{Z}$. Thus, x has the expression (2.4).

Conversely, from equations (2.3) and (2.4), one has

$$\begin{aligned} x(k + \omega) &= \sum_{s=k+\omega}^{k+2\omega-1} \left(F(k + \omega, s) \prod_{s \leq k_j < k+2\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \\ &= \sum_{s=k}^{k+\omega-1} \left(F(k + \omega, s + \omega) \prod_{s+\omega \leq k_j < k+2\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s + \omega) f(x(s + \omega - \tau_i(s + \omega))) \right) \\ &= \sum_{s=k}^{k+\omega-1} \left(F(k, s) \prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) = x(k). \end{aligned}$$

Hence, an element $x \in X$ verifies equation (2.4). We now make \hat{x} and $\hat{\phi}$ from such an x . To prove that \hat{x} is a positive ω -periodic solution $x(\cdot; \hat{\phi})$ of equation (1.1), we only need to show that x satisfies equation (1.1). To this end, we should first prove that $x(k+1) - (1 - a(k))x(k) = \sum_{i=1}^m b_i(k) f(x(k - \tau_i(k)))$ for $k \neq k_j$. The first term $x(k+1)$ of the left side equals

$$\begin{aligned} & \sum_{s=k+1}^{k+\omega} \left(F(k+1, s) \prod_{s \leq k_j < k+\omega+1} (1 + \gamma_j) \sum_{i=1}^m b_i(s) f(x(s - \tau_i(s))) \right) \\ &= \sum_{s=k+1}^{k+\omega-1} \left(F(k+1, s) \prod_{s \leq k_j < k+\omega+1} (1 + \gamma_j) \sum_{i=1}^m b_i(s) f(x(s - \tau_i(s))) \right) \\ & \quad + F(k+1, k+\omega) \prod_{k+\omega \leq k_j < k+\omega+1} (1 + \gamma_j) \sum_{i=1}^m b_i(k+\omega) f(x(k+\omega - \tau_i(k+\omega))) \\ &= \sum_{s=k+1}^{k+\omega-1} \left(\frac{\prod_{s \leq k_j < k+\omega+1} (1 + \gamma_j) \prod_{s+1 \leq r < k+\omega+1} (1 - a(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - a(r))} \sum_{i=1}^m b_i(s) f(x(s - \tau_i(s))) \right) \\ & \quad + \frac{\prod_{k+\omega+1 \leq r < k+\omega+1} (1 - a(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - a(r))} \sum_{i=1}^m b_i(k) f(x(k - \tau_i(k))) \\ &= \sum_{s=k+1}^{k+\omega-1} \left(\frac{\prod_{s \leq k_j < k+\omega+1} (1 + \gamma_j) \prod_{s+1 \leq r < k+\omega+1} (1 - a(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - a(r))} \sum_{i=1}^m b_i(s) f(x(s - \tau_i(s))) \right) \\ & \quad + \frac{1}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - a(r))} \sum_{i=1}^m b_i(k) f(x(k - \tau_i(k))). \end{aligned}$$

Moreover, the second term $(1 - a(k))x(k)$ of the left side equals

$$\begin{aligned}
& \sum_{s=k}^{k+\omega-1} \left((1-a(k))F(k, s) \prod_{s \leq k_j < k+\omega} (1+\gamma_j) \sum_{i=1}^m b_i(s) f(x(s-\tau_i(s))) \right) \\
&= \sum_{s=k+1}^{k+\omega-1} \left((1-a(k))F(k, s) \prod_{s \leq k_j < k+\omega} (1+\gamma_j) \sum_{i=1}^m b_i(s) f(x(s-\tau_i(s))) \right) \\
&\quad + (1-a(k))F(k, k) \prod_{k \leq k_j < k+\omega} (1+\gamma_j) \sum_{i=1}^m b_i(k) f(x(k-\tau_i(k))) \\
&= \sum_{s=k+1}^{k+\omega-1} \left((1-a(k+\omega)) \frac{\prod_{s \leq k_j < k+\omega} (1+\gamma_j) \prod_{s+1 \leq r < k+\omega} (1-a(r))}{1 - \prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{0 \leq r < \omega} (1-a(r))} \sum_{i=1}^m b_i(s) f(x(s-\tau_i(s))) \right) \\
&\quad + (1-a(k)) \frac{\prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{k+1 \leq r < k+\omega} (1-a(r))}{1 - \prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{0 \leq r < \omega} (1-a(r))} \sum_{i=1}^m b_i(k) f(x(k-\tau_i(k))) \\
&= \sum_{s=k+1}^{k+\omega-1} \left(\frac{\prod_{s \leq k_j < k+\omega+1} (1+\gamma_j) \prod_{s+1 \leq r < k+\omega+1} (1-a(r))}{1 - \prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{0 \leq r < \omega} (1-a(r))} \sum_{i=1}^m b_i(s) f(x(s-\tau_i(s))) \right) \\
&\quad + \frac{\prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{0 \leq r < \omega} (1-a(r))}{1 - \prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{0 \leq r < \omega} (1-a(r))} \sum_{i=1}^m b_i(k) f(x(k-\tau_i(k))).
\end{aligned}$$

Hence, one obtains

$$\begin{aligned}
x(k+1) - (1-a(k))x(k) &= \frac{1}{1 - \prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{0 \leq r < \omega} (1-a(r))} \sum_{i=1}^m b_i(k) f(x(k-\tau_i(k))) \\
&\quad - \frac{\prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{0 \leq r < \omega} (1-a(r))}{1 - \prod_{0 \leq k_j < \omega} (1+\gamma_j) \prod_{0 \leq r < \omega} (1-a(r))} \sum_{i=1}^m b_i(k) f(x(k-\tau_i(k))) \\
&= \sum_{i=1}^m b_i(k) f(x(k-\tau_i(k))).
\end{aligned}$$

Second, we prove that $x(k_j+1) = (1+\gamma_j)x(k_j)$. Since $k_j+\omega$ is impulse point, we see that $\tilde{b}_i(k_j+\omega) = 0$ for $1 \leq i \leq m$. Hence, representation (2.4) derives that $x(k_j+1)$ equals

$$\begin{aligned}
& \sum_{s=k_j+1}^{k_j+\omega} \left(F(k_j+1, s) \prod_{s \leq k_t < k_j+\omega+1} (1+\gamma_t) \sum_{i=1}^m \tilde{b}_i(s) f(x(s-\tau_i(s))) \right) \\
&= \sum_{s=k_j+1}^{k_j+\omega-1} \left(F(k_j+1, s) \prod_{s \leq k_t < k_j+\omega+1} (1+\gamma_t) \sum_{i=1}^m \tilde{b}_i(s) f(x(s-\tau_i(s))) \right) \\
&\quad + F(k_j+1, k_j+\omega) \prod_{k_j+\omega \leq k_t < k_j+\omega+1} (1+\gamma_t) \sum_{i=1}^m \tilde{b}_i(k_j+\omega) f(x(k_j+\omega-\tau_i(k_j+\omega))) \\
&= \sum_{s=k_j+1}^{k_j+\omega-1} \left(\frac{\prod_{s \leq k_t < k_j+\omega+1} (1+\gamma_t) \prod_{s+1 \leq r < k_j+\omega+1} (1-\tilde{a}(r))}{1 - \prod_{0 \leq k_t < \omega} (1+\gamma_t) \prod_{0 \leq r < \omega} (1-\tilde{a}(r))} \sum_{i=1}^m \tilde{b}_i(s) f(x(s-\tau_i(s))) \right).
\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}
x(k_j) &= \sum_{s=k_j}^{k_j+\omega-1} \left(F(k_j, s) \prod_{s \leq k_t < k_j+\omega} (1 + \gamma_t) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \\
&= \sum_{s=k_j+1}^{k_j+\omega-1} \left(F(k_j, s) \prod_{s \leq k_t < k_j+\omega} (1 + \gamma_t) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \\
&\quad + F(k_j, k_j) (1 + \gamma_j) \prod_{k_j \leq k_t < k_j+\omega} (1 + \gamma_t) \sum_{i=1}^m \tilde{b}_i(k_j) f(x(k_j - \tau_i(k_j))) \\
&= \sum_{s=k_j+1}^{k_j+\omega-1} \left(\frac{\prod_{s \leq k_t < k_j+\omega} (1 + \gamma_t) \prod_{s+1 \leq r < k_j+\omega} (1 - \tilde{a}(r))}{1 - \prod_{0 \leq k_t < \omega} (1 + \gamma_t) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right).
\end{aligned}$$

Since γ_j is ω -periodic, we see that $x(k_j + 1) = (1 + \gamma_j)x(k_j)$. Hence, x satisfies system (1.1). The proof is complete. \square

Denote a constant $0 \leq \lambda = \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r)) < 1$. Let $P = \{x \in X : x(k) \geq \lambda \|x\| \text{ for } k \in \mathbb{Z}\}$. Then, P is a cone in X . Define an operator Φ by

$$\Phi x(k) = \sum_{s=k}^{k+\omega-1} \left(F(k, s) \prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \quad (2.5)$$

for $x \in P$. It is obvious that Φ is an operator from cone P in the finite dimensional space X into itself. Due to the expression (2.4) of $x \in X$ and its periodicity, we obtain Φ is ω -periodic, i.e., $\Phi x(k + \omega) = \Phi x(k)$ for $k \in \mathbb{Z}$. Therefore, Lemma 2.2 reveals that \hat{x} , which is made from a fixed point x of Φ , is the positive ω -periodic solution $x(\cdot; \hat{\phi})$ of (1.1).

From the above discussion, it is obvious that $\Phi x \in X$. In addition, we can estimate that

$$\begin{aligned}
F(k, s) &\geq \frac{\prod_{k+1 \leq r < k+\omega} (1 - \tilde{a}(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \\
&\geq \frac{\prod_{k \leq r < k+\omega} (1 - \tilde{a}(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \\
&= \frac{\prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r))} = \mu,
\end{aligned}$$

and

$$\begin{aligned}
F(k, s) &\leq \frac{\prod_{k+\omega \leq r < k+\omega} (1 - \tilde{a}(r))}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \\
&= \frac{1}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{0 \leq r < \omega} (1 - \tilde{a}(r))} \\
&= \frac{1}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r))} = \nu.
\end{aligned}$$

Then, we have

$$\Phi x(k) \geq \mu \sum_{s=k}^{k+\omega-1} \left(\prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \quad (2.6)$$

and

$$\Phi x(k) \leq \nu \sum_{s=k}^{k+\omega-1} \left(\prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right). \quad (2.7)$$

This leads to

$$\|\Phi\| \leq \nu \sum_{s=k}^{k+\omega-1} \left(\prod_{s \leq k_j < k+\omega} (1 + \gamma_j) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \leq \frac{\nu}{\mu} \Phi x(k),$$

and therefore,

$$\Phi x(k) \geq \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r)) \|\Phi\| = \lambda \|\Phi\|.$$

Thus, we see that $\Phi x \in P$. We can verify that Φ is a completely continuous operator by straightforward calculations, namely, Φ is continuous and maps any bounded subset of P into a relatively compact subset of P (see, e.g., [26], Lemma 4). The Krasnosel'skii fixed point theorem [27] utilized for the above cone P is as follows:

Lemma 2.3. *Let $(X, \|\cdot\|)$ be a Banach space, and let $P \subset X$ be a cone in X . Suppose that Ω_1 and Ω_2 are open bounded subsets of X with $\theta \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. Let $\Phi : P \rightarrow P$ be a completely continuous operator on P such that, either*

- (i) $\|\Phi x\| \leq \|x\|$ for $x \in P \cap \partial\Omega_1$ and $\|\Phi x\| \geq \|x\|$ for $x \in P \cap \partial\Omega_2$, or
- (ii) $\|\Phi x\| \geq \|x\|$ for $x \in P \cap \partial\Omega_1$ and $\|\Phi x\| \leq \|x\|$ for $x \in P \cap \partial\Omega_2$.

Then, Φ has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Existence result

In this section, we state the result concerning the existence of positive ω -periodic solution of system (1.1). The main result is assured by the following theorem.

Theorem 3.1. *Let $\underline{b}_i = \min_{k \neq k_j, 0 \leq k < \omega} b_i(k)$ for each $i = 1, 2, \dots, m$ and $\gamma_j^- = \min\{\gamma_j, 0\}$ for $j = 1, 2, \dots$. Suppose that*

$$\prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r)) \sum_{i=1}^m \underline{b}_i \prod_{0 \leq k_j < \omega} (1 + \gamma_j^-) f(\lambda u_*) > \left(1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r)) \right) u_*. \quad (3.1)$$

Then, system (1.1) has at least one positive ω -periodic solution.

Proof. As it follows from the facts exposed in Section 2, the positive ω -periodic solution of system (1.1) is a fixed point of the completely continuous operator Φ defined by equation (2.5). Owing to this solution transformation, we have only to ensure the existence of the fixed point of Φ . Let $\bar{b}_i = \max_{k \neq k_j, 0 \leq k < \omega} b_i(k)$ for each $i = 1, 2, \dots, m$ and $\gamma_j^+ = \max\{\gamma_j, 0\}$ for $j = 1, 2, \dots$. Recall that $f(u) = u/(1 + u^n)$ and $n > 1$. Since $\lim_{u \rightarrow \infty} f(u) = 0$, we can choose a constant ε_0 satisfying $0 < \varepsilon_0 < 1/(\nu \omega \sum_{i=1}^m \bar{b}_i \prod_{0 \leq k_j < \omega} (1 + \gamma_j^+))$ and a number M with $0 < u_* < M$ such that

$$f(u) < \varepsilon_0 M \quad \text{for } u \geq M, \quad (3.2)$$

in which

$$u_* = \sqrt[n]{\frac{1}{n-1}} \quad \text{and} \quad \nu = \frac{1}{1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j) \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r))}.$$

Recall that $\lambda = \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r))$. Let $u^* = M/\lambda > u_*$. Then, we define two subsets as follows:

$$\Omega_1 = \{x \in X : \|x\| < u_*\} \quad \text{and} \quad \Omega_2 = \{x \in X : \|x\| < u^*\}.$$

It is obvious that Ω_i ($1 \leq i \leq 2$) are open bounded subsets with the inclusion relation $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \subset \bar{\Omega}_2$. Needless to say, the zero element belongs to Ω_1 .

Case (i): If an element x belongs to $P \cap \partial\Omega_1 \subset X$, then x is ω -periodic discrete function satisfying $u_* \geq x(k) \geq \lambda \|x\| = \lambda u_*$ for $k \in \mathbb{Z}$. By the monotonically increasing property of function f over the interval $[0, u_*]$, we see that

$$\min_{\lambda u_* \leq u \leq u_*} f(u) = f(\lambda u_*). \quad (3.3)$$

From equations (2.6) and (3.3), we obtain

$$\begin{aligned} (\Phi x)(k) &\geq \mu \sum_{s=k}^{k+\omega-1} \left(\prod_{s \leq k_j < k+\omega} (1 + \gamma_j^-) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \\ &\geq \mu \sum_{s=k}^{k+\omega-1} \left(\prod_{k \leq k_j < k+\omega} (1 + \gamma_j^-) \sum_{i=1}^m \tilde{b}_i(s) f(x(s - \tau_i(s))) \right) \\ &\geq \mu \prod_{0 \leq k_j < \omega} (1 + \gamma_j^-) f(\lambda u_*) \sum_{s=k}^{k+\omega-1} \sum_{i=1}^m \tilde{b}_i(s) \\ &\geq \mu \sum_{i=1}^m \underline{b}_i \prod_{0 \leq k_j < \omega} (1 + \gamma_j^-) f(\lambda u_*) \\ &= \sum_{i=1}^m \underline{b}_i \prod_{0 \leq k_j < \omega} (1 + \gamma_j^-) \times \frac{\prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r))}{\left(1 - \prod_{0 \leq k_j < \omega} (1 + \gamma_j^-) \prod_{r \neq k_j, 0 \leq r < \omega} (1 - a(r)) \right)} f(\lambda u_*) \end{aligned}$$

for $k \in \mathbb{Z}$. Hence, the condition (3.1) leads to the relationship $(\Phi x)(k) > u_*$. Therefore, $\|\Phi x\| > u_* = \|x\|$ for $x \in P \cap \partial\Omega_1$.

Case (ii): If an element x belongs to $P \cap \partial\Omega_2 \subset X$, then x is ω -periodic discrete function satisfying $\lambda u^* \leq x(k) \leq u^*$ for $k \in \mathbb{Z}$. From equation (2.7), we have

$$\begin{aligned} (\Phi x)(k) &\leq \nu \sum_{s=k}^{k+\omega-1} \left(\prod_{s \leq k_j < k+\omega} (1 + \gamma_j^+) \sum_{i=1}^m \bar{b}_i f(x(s - \tau_i(s))) \right) \\ &\leq \nu \sum_{i=1}^m \bar{b}_i \left(\sum_{s=k}^{k+\omega-1} \prod_{k \leq k_j < k+\omega} (1 + \gamma_j^+) f(x(s - \tau_i(s))) \right) \\ &\leq \nu \sum_{i=1}^m \bar{b}_i \prod_{0 \leq k_j < \omega} (1 + \gamma_j^+) \left(\sum_{s=k}^{k+\omega-1} f(x(s - \tau_i(s))) \right) \end{aligned}$$

for $k \in \mathbb{Z}$. Using equation (3.2), we obtain $\sum_{s=k}^{k+\omega-1} f(x(s - \tau_i(s))) < \varepsilon_0 \omega M$ for $i = 1, 2, \dots, m$. Hence, we see that

$$(\Phi x)(k) < \nu \varepsilon_0 \omega \sum_{i=1}^m \bar{b}_i \prod_{0 \leq k_j < \omega} (1 + \gamma_j^+) M < M \quad \text{for } k \in \mathbb{Z}.$$

Note that $u_* < M = \lambda u^* < u^*$. It turns out that $(\Phi x)(k) < u^*$, which implies that $\|\Phi x\| < u^* = \|x\|$ for $x \in P \cap \partial\Omega_2$. Thus, Lemma 2.3 ensures that operator Φ has a fixed point x in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$. The fixed point x is a positive ω -periodic discrete function satisfying $x(k) \geq \lambda \|x\|$ for $k \in \mathbb{Z}$ and $0 < u_* \leq \|x\| \leq u^*$. Let $\hat{x}(k) = x(k)$ for $k \in \mathbb{Z}[-\bar{\tau}, \infty)$ and $\hat{\phi}(k) = x(k)$ for $k \in \mathbb{Z}[-\bar{\tau}, 0]$. Now, we are in a position to conclude that \hat{x} is a positive ω -periodic solution of equation (1.1) with the initial discrete function $\hat{\phi}$. \square

4 Example and numerical simulations

To illustrate Theorem 3.1, we give an example in this section. The existence of a positive ω -periodic solution is obtained, and the region where a positive ω -periodic solution is located can be clarified.

Consider the impulsive delay difference system

$$\begin{cases} \Delta x(k) = -a(k)x(k) + \frac{b_1(k)x(k-7)}{1+x^2(k-7)} + \frac{b_2(k)x(k-2)}{1+x^2(k-2)}, & k \in \mathbb{Z}^+ \text{ and } k \neq k_j, \\ \Delta x(k_j) = \gamma_j x(k_j), & j = 1, 2, \dots \end{cases} \quad (4.1)$$

with $k_j = 2j - 1$. Let

$$a(k) = \begin{cases} 1/5 & \text{if } k = 0, \\ 1/2 & \text{if } k = 2, \end{cases} \quad (4.2)$$

$$b_1(k) = \begin{cases} 4 & \text{if } k = 0, \\ 2 & \text{if } k = 2, \end{cases} \quad b_2(k) = \begin{cases} 6 & \text{if } k = 0, \\ 7 & \text{if } k = 2. \end{cases} \quad (4.3)$$

Moreover, the common period ω is 4. It is obvious that $m = n = 2$. Fix the impulse values $\gamma_1 = 5/6 > 0$ and $\gamma_2 = -1/11 < 0$. We can verify in detail that there exists at least one positive 4-periodic solution of system (4.1) with the coefficients defined by equations (4.2) and (4.3).

Note that $f(u) = u/(1+u^n) = u/(1+u^2)$ for $u \geq 0$. The unimodal function f reaches its maximum at $u_* = \sqrt{1/(2-1)} = 1$. By a basic calculation, it follows that

$$\prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r)) = \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{2}\right) = \frac{2}{5},$$

$$\prod_{0 \leq k_j < 4} (1 + \gamma_j^-) = 1 \times \frac{10}{11} = \frac{10}{11},$$

and

$$\prod_{0 \leq k_j < 4} (1 + \gamma_j) = \left(1 + \frac{5}{6}\right) \times \left(1 - \frac{1}{11}\right) = \frac{5}{3}.$$

Hence, we obtain

$$\prod_{0 \leq k_j < 4} (1 + \gamma_j) \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r)) = \frac{5}{3} \times \frac{2}{5} = \frac{2}{3} < 1,$$

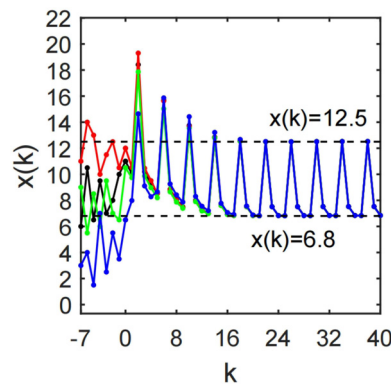


Figure 1: A graph of four arbitrary positive solutions of system (4.1). The numerical simulations show that there is a positive 4-periodic solution of system (4.1) located in $[6.8, 12.5]$.

namely, the assumption (A_3) holds. Moreover, the minimum values of b_1 and b_2 are $\underline{b}_1 = 2$ and $\underline{b}_2 = 6$. Then, we can check that

$$\frac{\prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2 \prod_{0 \leq k_j < 4} (1 + \gamma_j^{-})^{\sum_{i=1}^2 \underline{b}_i}}{(1 + \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2)} = \frac{(2 + 6) \times \frac{4}{25} \times \frac{10}{11}}{\left(1 + \frac{4}{25}\right)} = \frac{320}{319} > 1,$$

and

$$\left(1 - \prod_{0 \leq k_j < 4} (1 + \gamma_j) \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))\right) u_* = 1 - \frac{2}{3} = \frac{1}{3}.$$

Therefore, condition (3.1) is satisfied. Theorem 3.1 shows that system (4.1) has at least one positive 4-periodic solution (see Figure 1).

If we consider system (4.1) with constant impulsive force $\gamma_1 = \gamma_2 = \gamma$, then there are two important cases to be discussed: **(a)** $-1 < \gamma < 0$; **(b)** $\gamma > 0$.

Case (a). For the negative impulse value $-1 < \gamma < 0$, we see that condition (3.1) becomes to

$$\frac{(1 + \gamma)^2 \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2 \sum_{i=1}^2 \underline{b}_i}{\left(1 + \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2\right)} > \left(1 - (1 + \gamma)^2 \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))\right). \tag{4.4}$$

Note that assumption (A_3) holds for any $-1 < \gamma < 0$. From equation (4.4), we have

$$\begin{aligned} (1 + \gamma)^2 &> \frac{1 + \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2}{\prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^3 + \sum_{i=1}^2 \underline{b}_i \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2 + \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))} \\ &= \frac{1 + 0.4^2}{0.4^3 + 8 \times 0.4^2 + 0.4} = \frac{145}{218}, \end{aligned}$$

which implies that $\gamma > -0.184 \dots = \gamma^*$. Hence, system (4.1) has a positive 4-periodic solution for $\gamma^* < \gamma < 0$ (see Figure 2(a)). As shown in Figure 2(b), there also exists a positive 4-periodic solution of system (4.1) when condition (3.1) is not satisfied by $\gamma = -0.3 < \gamma^*$. This suggests that equation (3.1) is a sufficient but not necessary condition for the existence of positive 4-periodic solution.

Case (b). For the positive impulse value $\gamma > 0$, we see that condition (3.1) becomes

$$\frac{\prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2 \sum_{i=1}^2 \underline{b}_i}{\left(1 + \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2\right)} > \left(1 - (1 + \gamma)^2 \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))\right). \tag{4.5}$$

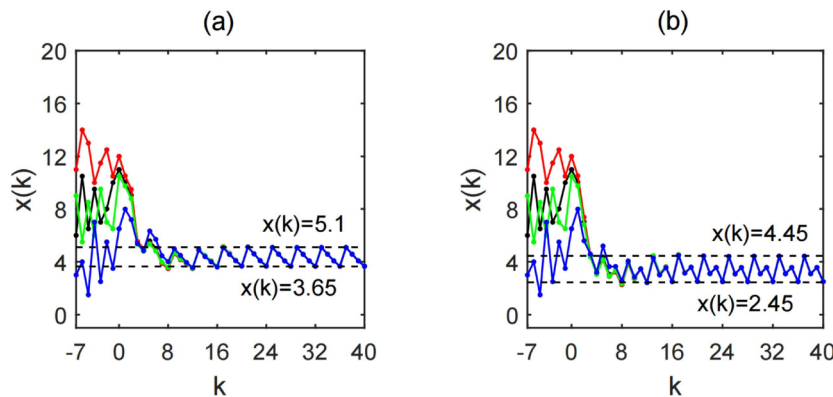


Figure 2: The numerical simulations of four arbitrary positive solutions of system (4.1). (a) There is a positive 4-periodic solution of system (4.1) located in $[3.65, 5.1]$ with $\gamma = 0.1 > \gamma^*$ and (b) there is a positive 4-periodic solution of system (4.1) located in $[2.45, 4.45]$ with $\gamma = 0.3 < \gamma^*$.

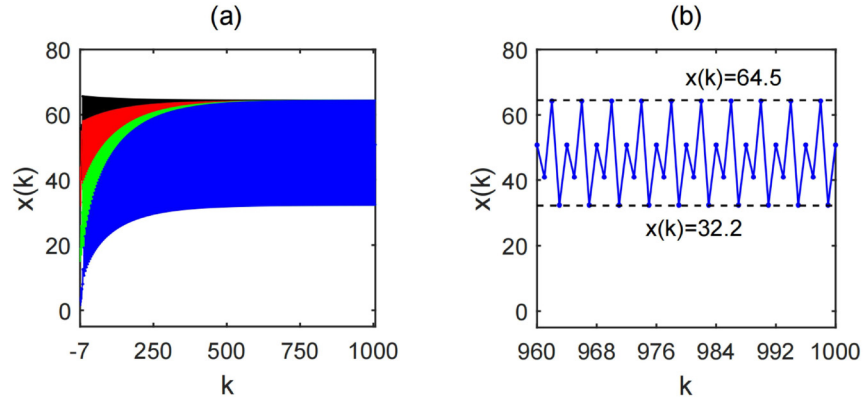


Figure 3: The numerical simulations of four arbitrary positive solutions of system (4.1). There is a positive 4-periodic solution of system (4.1) located in $[32.2, 64.5]$ with $\gamma = 0.57 < \sqrt{5/2} - 1 = \gamma_*$.

In this case, assumption (A_3) holds if

$$\gamma < \sqrt{\frac{1}{\prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))}} - 1 = \sqrt{\frac{1}{0.4}} - 1 = 0.581 \dots$$

Consequently, we obtain from equation (4.5) that

$$\begin{aligned} (1 + \gamma)^2 &> \frac{1 + \left(1 - \sum_{i=1}^2 b_i\right) \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^2}{\prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))^3 + \prod_{r \neq k_j, 0 \leq r < 4} (1 - a(r))} \\ &= \frac{1 + 0.16(1 - 8)}{0.4^3 + 0.4} = -0.258 \dots, \end{aligned}$$

which is always true. Hence, system (4.1) has a positive 4-periodic solution for $0 < \gamma < \sqrt{5/2} - 1$. Then, we can obtain an approximate threshold $\gamma_* = \sqrt{5/2} - 1$, which determines the existence or nonexistence of positive 4-periodic solution. Specifically, we state as follows:

- (i) When $0 < \gamma < \gamma_*$, there is a positive 4-periodic solution of system (4.1).
- (ii) When $\gamma > \gamma_*$, there is no positive 4-periodic solution of system (4.1). In fact, assumption (A_3) does not hold in this case, and the periodic solution of system (4.1) loses its positivity.

Note that for such γ_* , the periodic solution of system (4.1) is undefined if $\gamma = \gamma_*$. Hence, there is no need to consider the case that $\gamma = \gamma_* = \sqrt{5/2} - 1$. These analytical results are also verified by simulations (Figures 3 and 4).

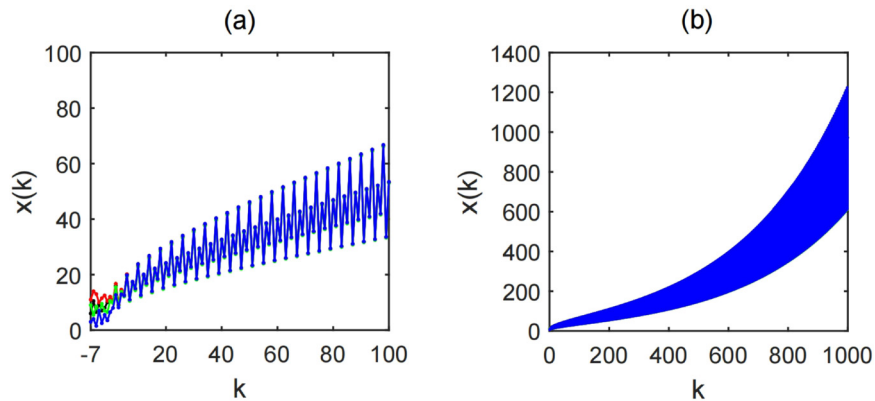


Figure 4: The numerical simulations of four arbitrary positive solutions of system (4.1). The four positive solutions increase continuously, and there is no positive 4-periodic solution of system (4.1) with $\gamma = 0.59 > \sqrt{5/2} - 1 = \gamma_*$.

In Figure 3(a), one observes that the arbitrary four positive solutions of system (4.1) have the same behavior (periodic behavior in fact) for large values of k . This means the existence of a positive 4-periodic solution with $\gamma = 0.57 < \sqrt{5/2} - 1 = \gamma_*$. The details of this positive 4-periodic solution are revealed in Figure 3(b). Figure 4 presents the nonexistence of positive 4-periodic solution of system (4.1). Although the four arbitrary positive solutions increase slowly at the beginning, they grow rapidly after a certain value of k .

5 Conclusion

The discrete hematopoiesis model that incorporates the impulses effect is studied in this article. This model provides a more realistic illustration of the sudden change of blood cell number. By applying the well-known Krasnosel'skii fixed point theorem, we explored the sufficient condition under which there is a positive periodic solution. In the case where the impulse is constant, we obtained a threshold for the existence of positive periodic solution. Through the numerical simulations, the fact that the impulse affects the maximum and minimum of positive periodic solutions is also revealed.

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References

- [1] M. C. Mackey and L. Glass, *Oscillation and chaos in physiological control system*, *Science* **197** (1977), no. 4300, 287–289, DOI: <https://doi.org/10.1126/science.267326>.
- [2] K. Gopalsamy, M. R. S. Kulenović, and G. Ladas, *Oscillations and global attractivity in models of hematopoiesis*, *J. Dynam. Differential Equations* **2** (1990), no. 2, 117–132, DOI: <https://doi.org/10.1007/bf01057415>.
- [3] K. Gopalsamy, S. I. Trofimchuk, and N. R. Bantsur, *A note on global attractivity in models of hematopoiesis*, *Ukrainian Math. J.* **50** (1998), 3–12, DOI: <https://doi.org/10.1007/bf02514684>.
- [4] G. Röst, *On the global attractivity controversy for a delay model of hematopoiesis*, *Appl. Math. Comput.* **190** (2007), no. 1, 846–850, DOI: <https://doi.org/10.1016/j.amc.2007.01.103>.
- [5] S. Lynch, *Analysis of a blood cell population model*, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **15** (2005), no. 7, 2311–2316, DOI: <https://doi.org/10.1142/S0218127405013186>.
- [6] M. Maes, S. Scharpe, W. Cooreman, A. Wauters, H. Neels, R. Verkerd, et al., *Components of biological, including seasonal, variation in hematological measurements and plasma fibrinogen concentrations in normal humans*, *Experientia* **51** (1995), no. 2, 141–149, DOI: <https://doi.org/10.1007/BF01929358>.
- [7] L. Berezansky, E. Braverman, and L. Idels, *Mackey-Glass model of hematopoiesis with non-monotone feedback: stability, oscillation and control*, *Appl. Math. Comput.* **219** (2013), no. 11, 6268–6283, DOI: <https://doi.org/10.1016/j.amc.2012.12.043>.
- [8] T. Faria and J. J. Oliveira, *Existence of positive periodic solutions for scalar delay differential equations with and without impulses*, *J. Dynam. Differential Equations* **31** (2019), 1223–1245, DOI: <https://doi.org/10.1007/s10884-017-9616-0>.
- [9] F.-C. Kong, *Dynamical behaviors of the generalized hematopoiesis model with discontinuous harvesting terms*, *Int. J. Biomath.* **12** (2019), no. 1, 195–231, DOI: <https://doi.org/10.1142/S1793524519500098>.
- [10] S. Padhi, J. R. Graef, and P. D. N. Srinivasu, *Periodic Solutions of First-Order Functional Differential Equations in Population Dynamics*, Springer, New Delhi, 2014.

- [11] X.-M. Wu, J.-W. Li, and H.-Q. Zhou, *A necessary and sufficient condition for the existence of positive periodic solutions of a model of hematopoiesis*, *Comput. Math. Appl.* **54** (2007), no. 6, 840–849, DOI: <https://doi.org/10.1016/j.camwa.2007.03.004>.
- [12] D. Ye, M. Fan, and H.-Y. Wang, *Periodic solutions for scalar functional differential equations*, *Nonlinear Anal.* **62** (2005), no. 7, 1157–1181, DOI: <https://doi.org/10.1016/j.na.2005.03.084>.
- [13] S.-J. Gao and L.-S. Chen, *Dynamic complexities in a single-species discrete population model with stage structure and birth pulses*, *Chaos Solitons Fractals* **23** (2005), no. 2, 519–527, DOI: <https://doi.org/10.1016/j.chaos.2004.05.047>.
- [14] Z. Yao, *Existence and global attractivity of the unique positive periodic solution for discrete hematopoiesis model*, *Topol. Methods Nonlinear Anal.* **45** (2015), no. 2, 423–437, DOI: <https://doi.org/10.12775/TMNA.2015.021>.
- [15] J.-S. Yu and B. Zheng, *Modeling Wolbachia infection in mosquito population via discrete dynamical models*, *J. Difference Equ. Appl.* **25** (2019), no. 11, 1549–1567, DOI: <https://doi.org/10.1080/10236198.2019.1669578>.
- [16] J. G. Dix, S. Padihi, and S. Patic, *Multiple positive periodic solutions for a nonlinear first order functional difference equation*, *J. Difference Equ. Appl.* **16** (2010), no. 9, 1037–1046, DOI: <https://doi.org/10.1080/1023619802705701>.
- [17] D. Jiang, D. O'Regan, and R. P. Agarwal, *Optimal existence theory for single and multiple positive periodic solutions to functional difference equations*, *Appl. Math. Comput.* **161** (2005), no. 2, 441–462, DOI: <https://doi.org/10.1016/j.amc.2003.12.097>.
- [18] Y. Yan and J. Sugie, *Global attractivity of a unique positive periodic solution for a first-order nonlinear difference equation with time delays*, *J. Difference Equ. Appl.* **26** (2020), no. 6, 855–870, DOI: <https://doi.org/10.1080/10236198.2020.1801662>.
- [19] Z.-J. Zeng, *Existence of positive periodic solutions for a class of nonautonomous difference equations*, *Electron. J. Differential Equations* **2006** (2006), no. 3, 1–18.
- [20] X.-D. Li, J.-D. Cao, and D. W. C Ho, *Impulsive control of nonlinear systems with time-varying delay and applications*, *IEEE T. Cybernetics.* **50** (2020), 2661–2673, DOI: <https://doi.org/10.1109/TCYB.2019.2896340>.
- [21] C. J. Gregory, E. A. McCulloch, and J. K. Till, *Erythropoietic progenitors capable of colony formation in culture: State of differentiation*, *J. Cell. Physiol.* **81** (1973), no. 3, 411–420.
- [22] D. D. Porteous and L. G. Lajtha, *On stem-cell recovery after irradiation*, *Brit. J. Haemat.* **12** (1966), 177–188, DOI: <https://doi.org/10.1111/j.1365-2141.1966.tb05623.x>.
- [23] S. H. Saker and J. O. Alzabut, *On the impulsive delay hematopoiesis model with periodic coefficients*, *Rocky Mountain J. Math.* **39** (2009), no. 5, 1657–1688, DOI: <https://doi.org/10.1216/RMJ-2009-39-5-1657>.
- [24] T. Faria and J. J. Oliveira, *Global asymptotic stability for a periodic delay hematopoiesis model with impulses*, *Appl. Math. Model.* **79** (2020), 843–864, DOI: <https://doi.org/10.1016/j.apm.2019.10.063>.
- [25] Z.-N. Xia and D.-J. Wang, *Pseudo-almost periodic solution for impulsive hematopoiesis model with infinite delays and linear harvesting term*, *Int. J. Biomath.* **9** (2016), no. 5, 1650078, DOI: <https://doi.org/10.1142/S1793524516500789>.
- [26] J. Sugie and Y. Yan, *Existence of multiple positive periodic solutions for discrete hematopoiesis models with a unimodal production function*, *Commun. Nonlinear Sci. Numer. Simul.* **89** (2020), no. 34, 105273, DOI: <https://doi.org/10.1016/j.cnsns.2020.105273>.
- [27] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, 1964.