Research Article

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The product of a quartic and a sextic number cannot be octic

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Abstract: In this article, we prove that the product of two algebraic numbers of degrees 4 and 6 over $\mathbb{Q}$ cannot be of degree 8. This completes the classification of so-called product-feasible triplets $(a, b, c) \in \mathbb{N}^3$ with $a \le b \le c$ and $b \le 7$. The triplet (a, b, c) is called product-feasible if there are algebraic numbers α, β, and γ of degrees a, b, and c over $\mathbb{Q}$, respectively, such that $\alpha\beta = \gamma$. In the proof, we use a proposition that describes all monic quartic irreducible polynomials in $\mathbb{Q}[x]$ with four roots of equal moduli and is of independent interest. We also prove a more general statement, which asserts that for any integers $n \ge 2$ and $k \ge 1$, the triplet $(a, b, c) = (n, (n-1)k, nk)$ is product-feasible if and only if n is a prime number. The choice $(n, k) = (4, 2)$ recovers the case $(a, b, c) = (4, 6, 8)$ as well.

Keywords: algebraic numbers, product-feasible triplets, field extensions

MSC 2020: 11R04, 11R21, 11R32, 12F10

1 Introduction

In the study by Drungilas et al. [1] a curious version of an *abc*-problem for sums and products of algebraic numbers has been introduced. (It has nothing to do with the *abc*-conjecture of Masser and Oesterlé.) According to the definitions in [1], a triplet $(a, b, c) \in \mathbb{N}^3$ is called

- *sum-feasible* if there exist algebraic numbers α and β of degrees a and b (over $\mathbb{Q}$), respectively, such that the degree of $\alpha + \beta$ is c,
- *product-feasible* if there exist algebraic numbers α and β of degrees a and b (over $\mathbb{Q}$), respectively, such that the degree of $\alpha \cdot \beta$ is c,
- *compositum-feasible* if there exist number fields K and L of degrees a and b (over $\mathbb{Q}$), respectively, such that the degree of their compositum KL is c.

The first named author has been interested in these questions before the publication of the article [1]; independently, the sum question is one of the questions at MathOverflow (http://mathoverflow.net/questions/30151/posed in 2010).

It is clear that the first two definitions are symmetric with respect to a, b, and c in the sense that the triplet (a, b, c) is sum-feasible (resp. product-feasible) if and only if so are six possible permutations of the triplet (a, b, c). So, without restriction of generality, we may assume that $a \le b \le c$. The case of compositum-feasible triplets is less symmetric. Then, only the degrees a and b (but not c) are involved in a symmetric way. However, since the field KL contains both K and L, we can also assume that $a \le b \le c$. Of course, if $(a, b, c) \in \mathbb{N}^3$ with

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$a \le b \le c$ is a sum-feasible, a product-feasible or a compositum-feasible triplet, then (in any of the three cases) we must have $c \le ab$.

In the study by Drungilas et al. [1], it was shown that

Proposition 1. *Each compositum-feasible triplet is also sum-feasible.*

Later, in the study by Drungilas and Dubickas [2], it was proved that

Proposition 2. *Each sum-feasible triplet is also product-feasible.*

Consequently, if a triplet is not product-feasible, then it is neither sum-feasible nor compositum-feasible.

The simple example (2, 3, 3) shows that a triplet can be product-feasible but not sum-feasible, so these two questions are not equivalent. Indeed, the product γ of the quadratic number $\alpha = e^{\frac{2\pi i}{3}}$ and the cubic number $\beta = \sqrt[3]{2}$ is cubic, because $\gamma = \alpha\beta = e^{\frac{2\pi i}{3}}\sqrt[3]{2}$ is conjugate to β. On the other hand, since the integers 2 and 3 are coprime, the sum γ of any quadratic number α and any cubic number β must be of degree 6 (and so $\gamma = \alpha + \beta$ cannot be cubic); see, e.g., [3].

In the study by Drungilas et al. [1], all sum-feasible and also all compositum-feasible triplets $(a, b, c) \in \mathbb{N}^3$ satisfying $a \le b \le c$, with $b \le 6$, have been described except for one special case (6, 6, 8). the study by Drungilas et al. [4], the missing case (6, 6, 8) from that classification has been treated by showing that the triplet (6, 6, 8) is not sum-feasible, and the classification has been extended to $b \le 7$. Then, in the study by Drungilas and Maciulevičius [5], all possible compositum-feasible triplets (a, b, c) satisfying $a \le b \le c$, with $b \le 9$, have been determined. In the special case when $a = b = p$ is a prime number, a complete characterization of all compositum-feasible triplets (p, p, ps) has been given in [6].

Recently, in the study by Maciulevičius [7], the second named author described all possible product-feasible triplets (a, b, c) satisfying $a \le b \le c$, with $b \le 7$, except for the following five cases:

$$(4, 6, 8),\ (4, 7, 7),\ (4, 7, 14),\ (5, 6, 10),\ (5, 6, 15).$$

By a different approach, Virbalas [8] showed that the triplets (4, 7, 7), (4, 7, 14), (5, 6, 10), and (5, 6, 15) are not product-feasible. More generally, he proved the following:

Proposition 3. (Part of [8, Theorem 3]) *Let α and β be algebraic numbers over $\mathbb{Q}$. Suppose* $\deg\alpha = p$, $\deg\beta = m$, *where $p > 2$ is a prime number, $p \nmid m$ and $p - 1 \nmid m$. Then,* $\deg(\alpha\beta) = mp$.

This leaves us with the only undecided case (4, 6, 8). In this note, we complete the picture of [7] by showing that

Theorem 4. *The product of a quartic and a sextic number over $\mathbb{Q}$ cannot be octic.*

This implies that

Table 1: Triplets (a, b, c), $a \le b \le c$, with $b \le 7$, that are product-feasible

$b\backslash a$	1	2	3	4	5	6	7
1	1						
2	2	2, 4					
3	3	*3*, 6	3, 6, 9				
4	4	4, 8	*6*, 12	4, 6, 8, 12, 16			
5	5	10	15	*5*, *10*, 20	5, 10, 20, 25		
6	6	6, 12	6, *9*, 12, 18	6, 12, 24	30	6, *8*, 9, 12, 15, 18, 24, 30, 36	
7	7	14	21	28	35	*7*, *14*, *21*, 42	7, 14, 21, 28, 42, 49

Corollary 5. *The triplet* $(4, 6, 8)$ *is not product-feasible, and therefore, it is not sum-feasible.*

Combining the aforementioned results with [7, Theorem 1], we obtain the following table that describes all possible product-feasible triplets (a, b, c), where $a \le b \le c$ and $b \le 7$ (Table 1).

In Table 1, there are nine triplets (with c written in italics) that are not sum-feasible by the results in [4] and [1], namely,

$$(2, 3, 3), (3, 4, 6), (4, 5, 5), (4, 5, 10), (3, 6, 9), (6, 6, 8), (6, 7, 7), (6, 7, 14), (6, 7, 21).$$

We also prove the following more general theorem:

Theorem 6. *For any integers* $n \ge 2$ *and* $k \ge 1$, *the triplet* $(a, b, c) = (n, (n-1)k, nk)$ *is product-feasible if and only if* n *is a prime number*.

In the proof of Theorem 4, assuming that there are α and β of degree 4 and 6 over $\mathbb{Q}$, respectively, whose product $\alpha\beta$ is of degree 8, we will first show that the conjugates of α must all be of the same modulus. In 1969, Robinson [9] described algebraic integers whose conjugates including α itself are all of the same moduli; see also [10] for the description of algebraic numbers with conjugates of two distinct moduli. Here, we need a more specific result for quartic algebraic numbers α whose all four conjugates have equal moduli. The proposition below is a new ingredient that turns out to be a useful tool in the proof of Theorem 4.

Proposition 7. *Assume that* $p(x)$ *is a monic quartic polynomial in* $\mathbb{Q}[x]$, *which is irreducible over* $\mathbb{Q}$ *and whose four roots have equal moduli. Then,* $p(x)$ *must be of one of the following forms*:

(*i*) $x^4 - r$, *where* $r \in \mathbb{Q}_{>0}$;

(*ii*) $x^4 + sx^2 + r$, *where* $s, r \in \mathbb{Q}$ *and* $s^2 < 4r$;

(*iii*) $(x^2 + ux + r)(x^2 + u'x + r)$, *where* $r \in \mathbb{Q}$ *and* $u \ne u'$ *are the conjugates of a real quadratic algebraic number satisfying* $\max(u^2, u'^2) < 4r$.

This approach apparently cannot be used in the proof of a more general Theorem 6, although some parts of the proof of Theorem 4 will be used in the more general setting of Theorem 6.

Of course, Theorem 6 immediately implies Theorem 4 by selecting $(n, k) = (4, 2)$. However, we present both proofs of the case $(a, b, c) = (4, 6, 8)$, since a separate proof of Theorem 4 contains some ideas that may be useful in treating similar problems for algebraic numbers of small degrees, but not only. As observed by one of the referees, in the first proof, the case (4, 6, 8) is obtained via descent, using the fact that the triplet (2, 4, 6) is not product-feasible. The main auxiliary result in the proof of Theorem 6 is Lemma 11.

In the next section, we will prove Proposition 7 and also give several auxiliary lemmas, which will be useful in the proofs of Theorems 4 and 6. Then, in Section 3, we will prove Theorem 4. The proof of Theorem 6 is a bit more involved. First, in Section 4, we will give some preparation under the assumption that the triplet $(n, (n-1)k, nk)$ is product feasible. Second, in Section 5, using some previous results, we will show that the triplet $(a, b, c) = (n, (n-1)k, nk)$, where n is a prime number and $k \ge 1$, is product-feasible. Finally, we will complete the proof of Theorem 6 by getting a contradiction for composite n.

2 Proof of Proposition 7

Write

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4). \tag{1}$$

Assume first that p has a real root. By the assumptions of the proposition, not all four roots of p can be real. Thus, p must have a pair of complex conjugate roots and another real root. So, without loss of generality, we may assume that the four roots of p in equation (1) are $\{\alpha_1, \alpha_2\} = \{\alpha, -\alpha\}$ and $\{\alpha_3, \alpha_4\} = \{\alpha e^{i\phi}, \alpha e^{-i\phi}\}$ for some

$\alpha > 0$ and $\phi \in (0, \pi)$. It follows that $a_0 = \alpha_1\alpha_2\alpha_3\alpha_4 = -\alpha^4$ is a negative rational number, say, $-r$, where $r \in \mathbb{Q}_{>0}$. Furthermore, since p has four roots, at least one real, on the circle $|z| = \alpha$, by Ferguson's result [11] (or even by an earlier result of Boyd [12]), we must have $p(x) = g(x^4)$ with some $g \in \mathbb{Z}[x]$. The polynomials p and g have the same constant coefficient, so g is a monic linear polynomial $x - r$, which implies that p is as in (i).

Next, we consider the case when p has no real roots. Then, there are $\varrho > 0$ and $0 < \phi < \xi < \pi$ such that the roots of p are $\{\alpha_1, \alpha_2\} = \{\varrho e^{i\phi}, \varrho e^{-i\phi}\}$ and $\{\alpha_3, \alpha_4\} = \{\varrho e^{i\xi}, \varrho e^{-i\xi}\}$. This time, by equation (1), we obtain $a_0 = \alpha_1\alpha_2\alpha_3\alpha_4 = \varrho^4 \in \mathbb{Q}_{>0}$. Furthermore, by Vieta's formulas,

$$a_3 = -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = -2\varrho(\cos(\phi) + \cos(\xi)) \tag{2}$$

and

$$\begin{aligned} a_1 &= -a_0\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4}\right) = -a_0\frac{2(\cos(\phi) + \cos(\xi))}{\varrho} \\ &= -\varrho^4\frac{2(\cos(\phi) + \cos(\xi))}{\varrho} = -2\varrho^3(\cos(\phi) + \cos(\xi)) = \varrho^2 a_3. \end{aligned}$$

Likewise, by Vieta's formula, we deduce

$$a_2 = \alpha_1\alpha_2 + \alpha_3\alpha_4 + (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) = 2\varrho^2 + 4\varrho^2\cos(\phi)\cos(\xi). \tag{3}$$

If $a_3 = 0$, then $a_1 = \varrho^2 a_3 = 0$ and $\cos(\phi) = -\cos(\xi)$ by equation (2). Thus, equation (3) yields $a_2 = 2\varrho^2(1 - 2\cos^2(\xi)) = -2\varrho^2\cos(2\xi) \in \mathbb{Q}$. Therefore, setting $r := a_0 = \varrho^4 \in \mathbb{Q}_{>0}$ and $s := a_2 = -2\sqrt{r}\cos(2\xi) \in \mathbb{Q}$, we obtain the polynomial $p(x) = x^4 + sx^2 + r \in \mathbb{Q}[x]$, which is the case (ii), because $s^2 = 4r\cos^2(2\xi) \le 4r$. Here, the inequality must be strict, i.e., $s^2 < 4r$, since otherwise $p(x)$ has a real root.

It remains to consider the case $a_3 \neq 0$. Then, $a_1 = \varrho^2 a_3 \neq 0$ and $\varrho^2 = a_1/a_3 \in \mathbb{Q}$. Set $r := \varrho^2 \in \mathbb{Q}_{>0}$. Since $\alpha_1\alpha_2 = \alpha_3\alpha_4 = r$, setting

$$u := -(\alpha_1 + \alpha_2) = -2\sqrt{r}\cos(\phi) \tag{4}$$

and

$$u' := -(\alpha_3 + \alpha_4) = -2\sqrt{r}\cos(\xi), \tag{5}$$

by equation (1), we obtain

$$p(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4) = (x^2 + ux + r)(x^2 + u'x + r).$$

Here, $a_3 = u + u' \neq 0$, $a_2 = uu' + 2r$, $a_1 = (u + u')r$, and $a_0 = r^2$. Since $r \in \mathbb{Q}_{>0}$, these four numbers are all rational if and only if $u + u'$ and uu' are both rational. Furthermore, u and u' are both real by equations (4) and (5). They are both irrational, since otherwise $p(x)$ would be reducible. Since u and u' are the roots of the quadratic polynomial

$$x^2 - (u + u')x + uu' \in \mathbb{Q}[x],$$

which is irreducible over $\mathbb{Q}$ due to $u, u' \notin \mathbb{Q}$, the numbers u and u' must be real quadratic conjugates. Moreover, since the roots of $x^2 + ux + r$ and $x^2 + u'x + r$ are all nonreal, we must have $u^2 - 4r < 0$ and $u'^2 - 4r < 0$. Consequently, p is a polynomial of the form described in case (iii). This completes the proof of the proposition.

In the proofs of Theorems 4 and 6, we will also use the following two lemmas:

Lemma 8. [1, Proposition 21] *Suppose that α and β are algebraic numbers of degrees a and b over $\mathbb{Q}$, respectively. Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_a$ be the distinct conjugates of α, and let $\beta_1 = \beta, \beta_2, \ldots, \beta_b$ be the distinct conjugates of β. If β is of degree b over $\mathbb{Q}(\alpha)$, then all the numbers $\alpha_i\beta_j, 1 \le i \le a, 1 \le j \le b$, are conjugate over $\mathbb{Q}$ (although not necessarily distinct).*

Of course, the numbers $\alpha_i\beta_j$, where $1 \le i \le a$ and $1 \le j \le b$, cover all possible conjugates of $\alpha\beta$. Lemma 8 describes the situation when they are all conjugate over $\mathbb{Q}$.

Lemma 9. (Part of [13, Lemma 1]) *Let β_1, β_2, and β_3 be distinct algebraic numbers conjugate over $\mathbb{Q}$. If $\beta_1^2 = \beta_2\beta_3$, then $\beta_1^m = \beta_2^m$ for some positive integer m.*

Let p be a prime number and $n \in \mathbb{N}$. Denote by $\mathrm{ord}_p(n)$ the exponent to which p appears in the prime factorization of n (for $p \nmid n$, we set $\mathrm{ord}_p(n) = 0$). We say that a triplet (a, b, c) satisfies the *exponent triangle inequality with respect to a prime number p* if

$$\mathrm{ord}_p(a) + \mathrm{ord}_p(b) \ge \mathrm{ord}_p(c), \quad \mathrm{ord}_p(a) + \mathrm{ord}_p(c) \ge \mathrm{ord}_p(b), \quad \text{and} \quad \mathrm{ord}_p(b) + \mathrm{ord}_p(c) \ge \mathrm{ord}_p(a).$$

The next lemma will be used in the proof of Theorem 6.

Lemma 10. [1, Proposition 28] *Suppose that the triplet $(a, b, c) \in \mathbb{N}^3$ satisfies the exponent triangle inequality for all prime numbers p. Then, for any product-feasible triplet $(a', b', c') \in \mathbb{N}^3$, the triplet (aa', bb', cc') is also product-feasible.*

3 Proof of Theorem 4

Suppose there exist algebraic numbers α and β satisfying

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4, \quad [\mathbb{Q}(\beta) : \mathbb{Q}] = 6, \quad \text{and} \quad [\mathbb{Q}(\alpha\beta) : \mathbb{Q}] = 8.$$

Since $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\alpha\beta)$ are subfields of $\mathbb{Q}(\alpha, \beta)$, we find that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$ is divisible by both 6 and 8 and therefore divisible by $\mathrm{lcm}(6, 8) = 24$. On the other hand,

$$[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}]\cdot[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] \le [\mathbb{Q}(\alpha) : \mathbb{Q}]\cdot[\mathbb{Q}(\beta) : \mathbb{Q}] = 4 \cdot 6 = 24.$$

This implies $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = 24$, and we have the picture as in Figure 1.

Let $\alpha_1 := \alpha, \alpha_2, \alpha_3, \alpha_4$ be the four distinct conjugates of α over $\mathbb{Q}$, and let $\beta_1 := \beta, \beta_2, \ldots, \beta_6$ be the 6 distinct conjugates of β over $\mathbb{Q}$. For $i = 1, 2, 3, 4$, we denote

$$\Gamma_i := \{\alpha_i\beta_1, \alpha_i\beta_2, \ldots, \alpha_i\beta_6\}.$$

The diagram and Lemma 8 with $(a, b) = (4, 6)$ implies that the numbers in the union $\Gamma_1 \cup \cdots \cup \Gamma_4$ are all conjugate over $\mathbb{Q}$. Since this set contains the number $\alpha\beta$, it must have exactly 8 distinct elements. Therefore,

$$8 = |\Gamma_1 \cup \cdots \cup \Gamma_4| \ge |\Gamma_1 \cup \Gamma_2| = |\Gamma_1| + |\Gamma_2| - |\Gamma_1 \cap \Gamma_2| = 6 + 6 - |\Gamma_1 \cap \Gamma_2|,$$

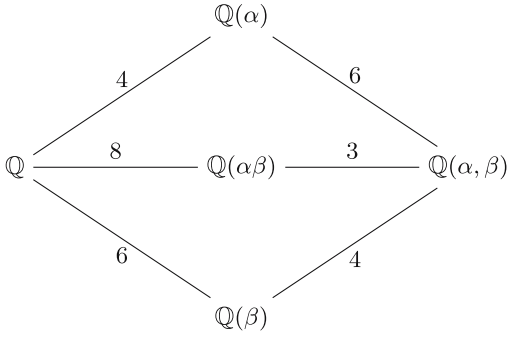

Figure 1: Diagram for the product-feasible triplet (4, 6, 8).

which implies $|\Gamma_1 \cap \Gamma_2| \geq 4$. Consequently, there exist distinct indices i_1, i_2, i_3, and i_4 and distinct indices j_1, j_2, j_3, and j_4 in $\{1, 2, \ldots, 6\}$ such that

$$\alpha_1\beta_{i_1} = \alpha_2\beta_{j_1}, \tag{6}$$

$$\alpha_1\beta_{i_2} = \alpha_2\beta_{j_2}, \tag{7}$$

$$\alpha_1\beta_{i_3} = \alpha_2\beta_{j_3},$$

$$\alpha_1\beta_{i_4} = \alpha_2\beta_{j_4}.$$

Clearly, $\{i_1, i_2, i_3, i_4\} \cap \{j_1, j_2, j_3, j_4\} \neq \emptyset$. Also, $i_1 \neq j_1$, since $\alpha_1 \neq \alpha_2$. Without loss of generality, we may assume that $i_1 = j_2$. Dividing equation (6) by equation (7) we obtain $\beta_{i_1}^2 = \beta_{j_1}\beta_{i_2}$. If $j_1 \neq i_2$, then Lemma 9 implies $\beta_{j_1}^{m_1} = \beta_{i_1}^{m_1}$ for some $m_1 \in \mathbb{N}$. Hence, by (6),

$$\alpha_1^{m_1} = \alpha_2^{m_1}. \tag{8}$$

If $j_1 = i_2$, then multiplying (6) and (7) we obtain $\alpha_1^2 = \alpha_2^2$, and so equation (8) holds for $m_1 = 2$.

Repeating the analogous argument for the pairs (Γ_1, Γ_3) and (Γ_1, Γ_4), we deduce

$$\alpha_1^{m_2} = \alpha_3^{m_2} \quad \text{and} \quad \alpha_1^{m_3} = \alpha_4^{m_3} \tag{9}$$

for some $m_2, m_3 \in \mathbb{N}$. Now, by equations (8) and (9), we conclude that all four conjugates of $\alpha_1 = \alpha$ have equal moduli.

By Lemma 8 with $(a, b) = (4, 6)$, the full list of conjugates of $\alpha\beta$ of degree 8 is, for instance,

$$\underbrace{\alpha\beta_1,\ \alpha\beta_2, \ldots, \alpha\beta_6}_{\Gamma_1},\ \alpha_i\beta_t,\ \alpha_j\beta_l \tag{10}$$

for some $i, j \in \{2, 3, 4\}$ and some $t, l \in \{1, 2, 3, 4, 5, 6\}$. In equation (10) we can choose any distinct products $\alpha_i\beta_t$ and $\alpha_j\beta_l$, which do not belong to Γ_1. In particular, for indices i and j, we must have

$$\Gamma_i \neq \Gamma_1 \quad \text{and} \quad \Gamma_j \neq \Gamma_1. \tag{11}$$

Adding and multiplying all eight conjugates in equation (10), we obtain

$$\alpha r_1 + \alpha_i\beta_t + \alpha_j\beta_l \coloneqq r_2 \in \mathbb{Q} \quad \text{and} \quad \alpha^6 r_3(\alpha_i\beta_t)(\alpha_j\beta_l) \coloneqq r_4 \in \mathbb{Q}\setminus\{0\},$$

where $r_1 \coloneqq \beta_1 + \beta_2 + \cdots + \beta_6 \in \mathbb{Q}$ and $r_3 \coloneqq \beta_1\beta_2 \cdots \beta_6 \in \mathbb{Q}\setminus\{0\}$. This yields

$$\alpha r_1 + \alpha_i\beta_t + \frac{r_4}{\alpha^6 r_3(\alpha_i\beta_t)} = r_2,$$

and hence,

$$\alpha_i\beta_t + \frac{r_4}{\alpha^6 r_3(\alpha_i\beta_t)} = r_2 - \alpha r_1. \tag{12}$$

Squaring equation (12) and multiplying it by $\alpha_i^2\beta_t^2$, we find that

$$\alpha_i^4\beta_t^4 + \left(\frac{2r_4}{\alpha^6 r_3} - (r_2 - \alpha r_1)^2\right)\alpha_i^2\beta_t^2 + \frac{r_4^2}{\alpha^{12} r_3^2} = 0. \tag{13}$$

Thus, β_t is a root of a degree 4 polynomial over the field $\mathbb{Q}(\alpha, \alpha_i^2)$. As we have shown earlier, all the conjugates of α have equal moduli. Hence, there are three possible cases for the minimal polynomials of α that are listed in Proposition 7.

In case (i), we have $\alpha_i = \alpha\varepsilon$ for some $\varepsilon \in \{-1, \pm i\}$. Then, $\alpha_i^2 = \alpha^2$ or $\alpha_i^2 = -\alpha^2$, which implies $\mathbb{Q}(\alpha, \alpha_i^2) = \mathbb{Q}(\alpha)$. But then the degree of β_t over $\mathbb{Q}(\alpha)$ is at most 4. Since the degree of β over $\mathbb{Q}(\alpha)$ equals 6 (see the diagram), β and β_t are conjugate over $\mathbb{Q}(\alpha)$. So the degree of β_t over $\mathbb{Q}(\alpha)$ equals 6, a contradiction.

In case (ii), since both α^2 and α_i^2 are the roots of the polynomial $x^2 + sx + r$, we must have either $\alpha_i^2 = \alpha^2$ or $\alpha_i^2 = -s - \alpha^2$. In both cases, $\mathbb{Q}(\alpha, \alpha_i^2) = \mathbb{Q}(\alpha)$, and we obtain the same contradiction.

It remains to consider case (*iii*). Then, α is nonreal. Assume, without loss of generality, that α_2 is the complex conjugate of $\alpha_1 = \alpha$. Note that in case (*iii*), both α_1 and $\alpha_2 = \overline{\alpha_1}$ are the roots of the same quadratic factor $x^2 + ux + r$ or $x^2 + u'x + r$, since $u, u', r \in \mathbb{R}$. Hence, $\alpha_2\alpha = r$.

If $\Gamma_1 \neq \Gamma_2$, then we can take in equation (10) $i = 2$, and so $\alpha_i = \alpha_2$. From $\alpha_i = \alpha_2 = r/\alpha$, we obtain $\mathbb{Q}(\alpha, \alpha_i^2) = \mathbb{Q}(\alpha)$, so that equation (13) leads to the same contradiction again.

In the alternative case, when $\Gamma_1 = \Gamma_2$, we obtain

$$\alpha\beta_\ell = \alpha_2\beta_{\tau(\ell)} \quad \text{for } \ell = 1, 2, \ldots, 6, \tag{14}$$

where $\{\tau(1), \tau(2), \ldots, \tau(6)\} = \{1, 2, \ldots, 6\}$. This time, by equation (11), we cannot take $i = 2$, so that $i \in \{3, 4\}$. Multiplying all equalities in equation (14) we obtain $\alpha^6 = \alpha_2^6$. Since $\alpha_2 = \overline{\alpha}$ and $\alpha_2\alpha = r$, this yields $r^6 = \alpha^6\alpha_2^6 = \alpha^{12}$. Consequently, α^6 is a rational number $r_5 \in \{-r^3, r^3\}$. Adding all equalities in equation (14), we derive that $r_1 = \beta_1 + \beta_2 + \ldots + \beta_6 = 0$, since $\alpha \neq \alpha_2$. Thus, by equation (12), we must have

$$\alpha_i\beta_t + \frac{r_4}{r_5 r_3 \alpha_i \beta_t} = r_2.$$

This means that $\delta = \alpha_i\beta_t$ is a rational number or a quadratic number. Evidently, δ cannot be rational, since α_i and β_t are of distinct degrees 4 and 6, respectively. On the other hand, if δ were quadratic, then the product of the quadratic number δ and the quartic number $1/\alpha_i$ would be the sextic number β_t. However, the triplet (2, 4, 6) is not product-feasible by [7, Theorem 1]. This completes the proof of Theorem 4.

4 The case $(a, b, c) = (n, (n-1)k, nk)$

Set $a = n$, $b = (n-1)k$, and $c = nk$, where $n, k > 1$ are integers. Throughout this section, we assume that there exist algebraic numbers α and β satisfying

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = a, \quad [\mathbb{Q}(\beta) : \mathbb{Q}] = b \quad \text{and} \quad [\mathbb{Q}(\alpha\beta) : \mathbb{Q}] = c.$$

The beginning of the argument is essentially the same as that in Section 3. Since $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\alpha\beta)$ are subfields of $\mathbb{Q}(\alpha, \beta)$, we find that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$ is divisible by both b and c and therefore divisible by

$$\operatorname{lcm}(b, c) = \frac{bc}{\gcd(b, c)} = \frac{(n-1)k \cdot nk}{k} = ab.$$

On the other hand, $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \leq [\mathbb{Q}(\alpha) : \mathbb{Q}][\mathbb{Q}(\beta) : \mathbb{Q}] = ab$. This implies $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = ab$ and

$$[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha\beta)] = \frac{ab}{c} = n - 1 = a - 1.$$

Thus, we have the following diagram:

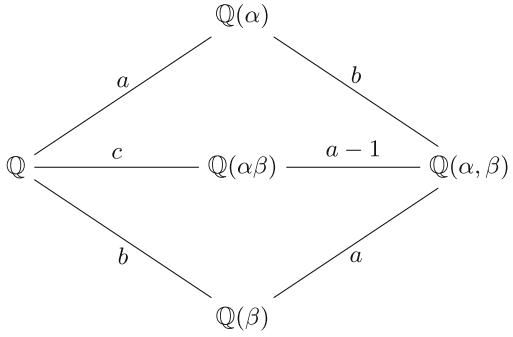

Figure 2: Degree diagram for α, β and $\alpha\beta$.

Let $\alpha_1 := \alpha, \alpha_2, \ldots, \alpha_a$ be the distinct conjugates of α over $\mathbb{Q}$, and let $\beta_1 := \beta, \beta_2, \ldots, \beta_b$ be the distinct conjugates of β over $\mathbb{Q}$. Denote

$$A := \{1, 2, \ldots, a\} \quad \text{and} \quad B := \{1, 2, \ldots, b\}.$$

Figure 2 and Lemma 8 imply that the numbers in the set

$$\Gamma := \{\alpha_i\beta_j : \ i \in A, \ j \in B\}$$

are all conjugate over $\mathbb{Q}$. Since the set Γ contains the number $\alpha\beta$, it must have exactly c distinct elements. This means that $[\mathbb{Q}(\alpha_i\beta_j) : \mathbb{Q}] = c$ for any indices $i \in A$ and $j \in B$.

Therefore, the same degree diagram as that in Figure 2 holds for any pair of indices $i \in A$, $j \in B$ (Figure 3).

We conclude this section with the following lemma and its corollary:

Lemma 11. *Let $n \geq 3$ and $k \geq 2$ be integers. Assume that α and β are algebraic numbers of degrees $a = n$ and $b = (n-1)k$ over $\mathbb{Q}$, respectively, whose product $\alpha\beta$ is of degree $c = nk$. Then, for any pair of conjugates $\alpha_i \neq \alpha_j$ of α over $\mathbb{Q}$, the degree of α_i over $\mathbb{Q}(\alpha_j)$ is equal to $a - 1$.*

Proof. Assume that $\alpha_i \neq \alpha_j$ are two distinct conjugates of α, so $i \in A\backslash\{j\}$. By the diagram in Figure 3, we can assume that $j = 1$ and so $\alpha_j = \alpha$. Since β is of degree b over $\mathbb{Q}(\alpha)$, all the numbers

$$\gamma_t := \alpha\beta_t, \ \ t = 1, 2, \ldots, b,$$

are conjugate to $\gamma := \alpha\beta$ over $\mathbb{Q}$. Let the remaining conjugates of γ over $\mathbb{Q}$ be

$$\gamma_{b+1}, \gamma_{b+2}, \ldots, \gamma_c.$$

Take the polynomials

$$g(x) := (x - \gamma_1)(x - \gamma_2)\cdots(x - \gamma_b) = \alpha^b P_\beta\left(\frac{x}{\alpha}\right),$$
$$h(x) := (x - \gamma_{b+1})(x - \gamma_{b+2})\cdots(x - \gamma_c) = \frac{P_\gamma(x)}{g(x)},$$

where $P_\beta(x)$ and $P_\gamma(x)$ are the minimal (monic) polynomials of β and γ over $\mathbb{Q}$, respectively. Set $K = \mathbb{Q}(\alpha)$. Clearly, $g(x) \in K[x]$, and hence $h(x) \in K[x]$. This means that for t satisfying $b + 1 \leq t \leq c$, we have

$$[\mathbb{Q}(\alpha, \gamma_t) : \mathbb{Q}(\alpha)] = [K(\gamma_t) : K] \leq \deg h = c - b = nk - (n-1)k = k.$$

In particular, this implies that $\gamma_c \neq \alpha\beta_l$ for $l \in B$, since otherwise, by Figure 3, the degree of γ_c over the field $\mathbb{Q}(\alpha)$ would be $b = (n-1)k > k$, which is a contradiction. Hence, $\gamma_c = \alpha_\ell\beta_l$ for some $\ell \in A\backslash\{j\}$ and some $l \in B$.

Evidently, $[K(\alpha_\ell) : K] \leq a - 1$. In addition, $[K(\beta_l) : K] = b$ by Figure 3. Therefore,

$$b = [K(\beta_l) : K] = [K(\gamma_c \cdot \alpha_\ell^{-1}) : K] \leq [K(\gamma_c) : K][K(\alpha_\ell^{-1}) : K] \leq k \cdot [K(\alpha_\ell) : K] \leq k(a-1) = (n-1)k = b.$$

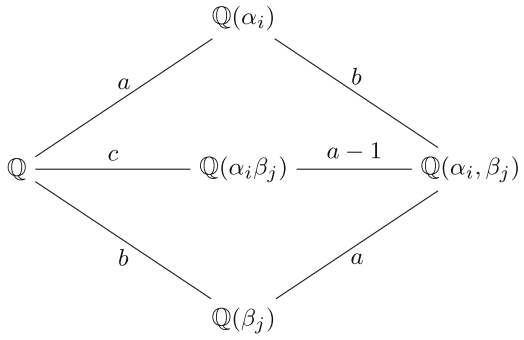

Figure 3: Degree diagram for α_i, β_j and $\alpha_i\beta_j$.

This implies $[K(\gamma_c) : K] = k$ and $[K(\alpha_\ell) : K] = a - 1$. Thus, α_ℓ is of degree $a - 1$ over $K = \mathbb{Q}(\alpha)$, and hence so must be the degree of α_i over K for each $i \in A\backslash\{j\}$. This completes the proof of the lemma. □

Corollary 12. *Under the conditions of* Lemma 11, *we assume that there exists* $m \in \mathbb{N}$ *such that* $\alpha_i^m \in \mathbb{Q}(\alpha_j)$ *for two distinct conjugates* α_i *and* α_j *of* α *over* $\mathbb{Q}$. *If* $a = n \geq 3$, *then there exists a rational number* r_6 *such that* $\alpha_1^m = \alpha_2^m = \cdots = \alpha_n^m = r_6$.

Proof. Write $\alpha_i^m = G(\alpha_j)$, where $G \in \mathbb{Q}[x]$. Then, by Lemma 11, for each $t \notin A\backslash\{j\}$, there is an automorphism of the Galois group $\mathrm{Gal}(E/\mathbb{Q}(\alpha_j))$, where E is the Galois closure of $\mathbb{Q}(\alpha_j, \alpha_i)$ over $\mathbb{Q}(\alpha_j)$, which maps α_i to α_t. Applying it to $\alpha_i^m = G(\alpha_j)$, we obtain $\alpha_t^m = G(\alpha_j)$. Hence, α_t^m are equal for $t \in A\backslash\{j\}$. Choosing $t \notin A\backslash\{i, j\}$ we can apply the same argument to the equality $\alpha_i^m = \alpha_t^m \in \mathbb{Q}(\alpha_t)$ by mapping α_i to α_j. This yields $\alpha_j^m = \alpha_t^m$, and therefore, $\alpha_1^m = \alpha_2^m = \cdots = \alpha_n^m$. Adding all these n equal elements, we obtain $\sum_{t=1}^{n} \alpha_t^m$, which is a rational number. Thus, α_t^m, $t = 1, 2, \ldots, n$, are all equal to this rational number divided by n. □

5 Proof of Theorem 6

The case $k = 1$ of Theorem 6 follows from the previous result of the second named author:

Lemma 13. [7, Theorem 2] *The triplet* $(n, n - 1, n)$, $n \geq 2$, *is product-feasible if and only if* n *is prime.*

We now show that the triplet $(n, (n - 1)k, nk)$ is product-feasible for $k \geq 2$ and n prime. Indeed, for a prime number n, the triplet $(n, n - 1, n)$ is product-feasible by Lemma 13. On the other hand, the triplet $(1, k, k)$ satisfies the exponent triangle inequality with respect to any prime number (see the end of Section 2). Consequently, the triplet $(n, (n - 1)k, nk)$ is product-feasible by Lemma 10.

Now, assume that $n > 1$ is a composite number. It is clear that then $n \geq 4$. We need to show that the triplet $(a, b, c) = (n, (n - 1)k, nk)$ is not product-feasible. The case $k = 1$ follows from Lemma 13, so from now on we assume that $k \geq 2$.

Set

$$\Gamma_i := \{\alpha_i\beta_1, \alpha_i\beta_2, \ldots, \alpha_i\beta_b\}$$

for $i = 1, \ldots, a$. For any distinct $i, j \in A$ we have

$$c = \left|\bigcup_{s=1}^{a} \Gamma_s\right| \geq |\Gamma_i \cup \Gamma_j| = |\Gamma_i| + |\Gamma_j| - |\Gamma_i \cap \Gamma_j| = b + b - |\Gamma_i \cap \Gamma_j|,$$

which implies

$$|\Gamma_i \cap \Gamma_j| \geq 2b - c = 2(n - 1)k - nk = nk - 2k > \frac{(n - 1)k}{2} = \frac{b}{2}.$$

Thus, there is $l \in B$ for which $\alpha_i\beta_l$ and $\alpha_j\beta_l$ both lie in the intersection $\Gamma_i \cap \Gamma_j$. Since these two elements are distinct, there are $u, v \in B\backslash\{l\}$ such that $\alpha_i\beta_l = \alpha_j\beta_u$ and $\alpha_i\beta_v = \alpha_j\beta_l$. From $\beta_l = \alpha_j\beta_u/\alpha_i = \beta_u\beta_v/\beta_l$, we deduce $\beta_l^2 = \beta_u\beta_v$. If $u \neq v$ then, by Lemma 9, for some $m \in \mathbb{N}$, we obtain $\beta_l^m = \beta_u^m$. If $u = v$, then the same equality with $m = 2$ follows directly. Therefore, from $\alpha_i^m\beta_l^m = \alpha_j^m\beta_u^m$, we deduce $\alpha_i^m = \alpha_j^m$. Here, $m \in \mathbb{N}$ and $m > 1$ in view of $i \neq j$. By Corollary 12, we conclude that the conjugates of α over $\mathbb{Q}$ all have the same moduli.

We now consider two cases depending on whether the composite number $a = n$ is even or odd. If n is even and α has no real conjugates, then the product r_7 of all conjugates can be written in the form $r_7 = (\alpha\overline{\alpha})^{n/2}$. If α has a real conjugate, say α', then $-\alpha'$ is also its conjugate (as the number of nonreal conjugates is even), so $r_7 = -(\alpha_i\overline{\alpha_i})^{n/2}$ for any nonreal α_i. Therefore, in both cases, there are two distinct indices $i, j \in A$ such that $\alpha_i^{n/2} \in \mathbb{Q}(\alpha_j)$. By Corollary 12, we conclude that $\alpha_t^{n/2} = r_6 \in \mathbb{Q}$ for every $t \in A$. So α is of degree at most $n/2$ over $\mathbb{Q}$, which is not the case since α is of degree $n > n/2$ over $\mathbb{Q}$.

Likewise, if $a = n$ is odd, then α has a real conjugate α_i, and so the product r_7 of all the conjugates of α can be written as α_i^n (in both cases $\alpha_i > 0$ and $\alpha_i < 0$). Hence, $\alpha_i^n = r_7 := w \in \mathbb{Q}\setminus\{0\}$. This yields $\alpha_j^n = w$ for each $j \in A$, so that the minimal polynomial of α over $\mathbb{Q}$ is $x^n - w$. Consequently, the conjugates of α are

$$w^{1/n}\zeta^t, \quad t = 0, 1, \dots, n-1, \quad \text{where } \zeta := e^{\frac{2\pi i}{n}}.$$

By Lemma 11, the number $w^{1/n}\zeta$ has degree $n-1$ over the field $\mathbb{Q}(w^{1/n})$, because $w^{1/n}\zeta \neq w^{1/n}$ are the conjugates of α. This implies that the degree of $\zeta = \frac{w^{1/n}\zeta}{w^{1/n}}$ over the field $\mathbb{Q}(w^{1/n})$ equals $n-1$. Consequently, the degree of ζ over its subfield $\mathbb{Q}$ must be at least $n-1$. However, the degree of ζ over $\mathbb{Q}$ equals $\varphi(n)$, where φ stands for Euler's totient function [14, Theorem 3.1]. For each composite n, we have $\varphi(n) < n-1$, so the degree of ζ over $\mathbb{Q}$ is less than $n-1$, a contradiction. This completes the proof of Theorem 6.

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