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Research Article

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Eigenfunctions in Finsler Gaussian solitons

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Abstract: Gaussian solitons are important examples in the theory of Riemannian measure space. In the first part, we explicitly characterize the first eigenfunctions of the drift Laplacian in a Gaussian shrinking soliton, which shows that apart from each coordinate function, other first eigenfunctions must involve exponential functions and the so-called error functions. Moreover, the second eigenfunctions are also described. In the second part, we discuss the corresponding issues in Finsler Gaussian shrinking solitons, which is a natural generalization of Gaussian shrinking solitons. For the first eigenfunction, we complement an example to show that if a coordinate function is a first eigenfunction, then the Finsler Gaussian shrinking soliton must be a Euclidean measure space. For the second eigenfunction, we give some necessary and sufficient conditions for these spaces to be Euclidean measure spaces.

Keywords: eigenfunction, Laplacian, Finsler Gaussian soliton, the weighted Ricci curvature

MSC 2020: 53C60, 35P15

1 Introduction

The concept of Ricci solitons was introduced by Hamilton in 1988 as self-similar solutions to the Ricci flow [1]. They are natural generalizations of Einstein metrics, also called Ricci solitons, and are subject to a great interest in geometry and physics, especially in relation to string theory.

For an arbitrary real number ρ, Gaussian soliton $(\mathbb{R}^n, g_{\text{can}}, \frac{\rho}{2}|x|^2)$ is a linear space $\mathbb{R}^n$ equipped with the canonical Euclidean metric g_{can} and a volume form

$$\mathrm{d}V = e^{-\frac{\rho}{2}|x|^2}\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \quad \text{where } |x| = \sqrt{\sum_{i=1}^{n}(x^i)^2}.$$

According to the value range of ρ, we have the following three interesting cases:

- when $\rho = \frac{1}{2}$, then

$$(\mathbb{R}^n, g_{\text{can}}, \mathrm{d}V) = (\mathbb{R}^n, g_{\text{can}}, e^{-\frac{|x|^2}{4}}\mathrm{d}x)$$

  is the Gaussian shrinking soliton [2,3].
- when $\rho = 1$, then

$$(\mathbb{R}^n, g_{\text{can}}, (2\pi)^{-\frac{n}{2}}\mathrm{d}V) = (\mathbb{R}^n, g_{\text{can}}, (2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}\mathrm{d}x)$$

  is the Gaussian probability space [4,5].

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- when $\rho = -\frac{1}{2}$, then

$$(\mathbb{R}^n, g_{\text{can}}, \mathrm{d}V) = (\mathbb{R}^n, g_{\text{can}}, e^{\frac{|x|^2}{4}}\mathrm{d}x)$$

  is the Gaussian expanding soliton [2].

It is worth mentioning that the Gaussian shrinking soliton, the Gaussian probability space, and the Gaussian expanding soliton are important examples in the theory of Riemannian measure space.

On a nontrivial, noncompact, shrinking gradient Ricci soliton $(M, g, e^{-f}\mathrm{d}V)$, Cao and Zhou [3] have proved that for any fixed point $p \in M$,

$$\frac{1}{4}(r(x) - c)^2 \le f(x) \le \frac{1}{4}(r(x) + c)^2,$$

where $r(x) = d(p, x)$ is the distance function from some fixed point $p \in M$. In view of the Gaussian shrinking soliton, the leading term $\frac{1}{4}r(x)^2$ for the lower and upper bounds on f in Cao-Zhou's result is optimal. The isoperimetric inequality and the Brann-Minkowski inequality in the Gaussian probability space have been obtained by Eskenazis and Moschidis [6] and Latala [7].

It is shown in the study by Cheng and Zhou [8] that in the Gaussian shrinking soliton $(\mathbb{R}^n, g_{\text{can}}, \frac{|x|^2}{4})$, the Bakry-Émery Ricci curvature $\mathrm{Ric}_f = \frac{1}{2}$ and the first eigenvalue of drift Laplacian $\lambda_1(\Delta_f) = \frac{1}{2}$ with multiplicity n, where $f = \frac{|x|^2}{4}$ and drift Laplacian is given by

$$\Delta_f \coloneqq \Delta - \langle \nabla f, \nabla \cdot \rangle = \Delta - \frac{1}{2}\langle x, \nabla \cdot \rangle. \tag{1.1}$$

Observe that for coordinate function x^j,

$$\Delta_f x^j = \Delta x^j - \frac{1}{2}\langle x, \nabla x^j \rangle = -\frac{x^j}{2}, \quad j = 1, \ldots, n,$$

which means that each coordinate function is the first eigenfunction of $\lambda_1(\mathbb{R}^n) = \frac{1}{2}$.

It is natural to ask: *besides coordinate functions, is there any other first eigenfunction of* $\lambda_1(\mathbb{R}^n) = \frac{1}{2}$? Therefore, the first purpose of this article is to investigate the first eigenspace of the drift Laplacian in the Gaussian shrinking soliton. By solving Euler-Lagrange equation, we have the following result.

Theorem 1.1. *In the Gaussian shrinking soliton* $(\mathbb{R}^n, g_{\text{can}}, \frac{|x|^2}{4})$, *any first eigenfunction of the drift Laplacian is a linear combination of the following functions*:

$$x^1, \ldots, x^n, -e^{\frac{(x^1)^2}{4}} + \frac{\sqrt{\pi}}{2}x^1\mathbf{erfi}\left(\frac{x^1}{2}\right), \ldots, -e^{\frac{(x^n)^2}{4}} + \frac{\sqrt{\pi}}{2}x^n\mathbf{erfi}\left(\frac{x^n}{2}\right),$$

and thus the first eigenspace is given by:

$$\mathbf{E}^{\text{first}} \coloneqq \mathrm{span}\left\{x^i, -e^{\frac{(x^i)^2}{4}} + \frac{\sqrt{\pi}}{2}x^i\mathbf{erfi}\left(\frac{x^i}{2}\right), i = 1, 2, \ldots, n\right\}.$$

In Theorem 1.1, $\mathbf{erfi}(x)$ is the imaginary error function, which will be introduced in Section 3. Moreover, by computing $\Delta_f((x^j)^2 - 2)$ in equation (1.1), it is easy to see that $(x^j)^2 - 2$ are eigenfunctions of $\lambda_2(\mathbb{R}^n) = 1$ for $j = 1, \ldots, n$. In addition, we have also characterized the second eigenspace of the drift Laplacian (Theorem 3.1). For the related results obtained in Ricci flow, we refer to the studies by Isidro et al. [9,10].

Finsler geometry is just the Riemannian geometry without the quadratic restriction. Ricci solitons in Finsler manifolds, as a generalization of Einstein spaces, are introduced by Bidabad and Yar Ahmadi [11], and it is shown that if there is a Ricci soliton on a compact Finsler manifold, then there exists a solution to the Finslerian Ricci flow equation and *vice versa*. They also proved that a compact shrinking Ricci soliton, Finsler space has a finite fundamental group, and hence, the first de Rham cohomology group vanishes [12]. For other interesting results, we refer to the study by Bidabad and Yar Ahmadi [13] and the references therein along this line.

To study global Finsler geometry, the weighted Ricci curvature, which is introduced by Ohta [14], plays a very important role. In the Riemannian case, the weighted Ricci curvature reduces the Bakry-Émery Ricci curvature [15]. Recently, Yin et al. [16] constructed and studied complete noncompact Finsler measure spaces with constant weighted Ricci curvature Ric_∞.

Mo et al. [17] studied Finsler gradient Ricci solitons with constant weighted Ricci curvature Ric_∞, and give sufficient and necessary conditions for this space. Zhu and Rao [18] investigated a class of Finsler gradient steady Ricci solitons with $\mathrm{Ric}_\infty = 0$ and obtained a rigidity of Finsler gradient steady Ricci solitons of Douglas type.

It should be pointed out that, comparing to the Finsler case, all issues discussed in Ricci solitons (especially in the Gaussian soliton) are relatively easy since the drift Laplacian is a linear elliptic operator and the Riemannian metrics are reversible metrics. However, the situation is much more complicated in the Finsler setting, where the Finsler Laplacian is nonlinear and Finsler metrics are irreversible in general. Besides, the Finsler Laplacian is also a degenerate operator, and the eigenfunctions for the Finsler Laplacian lack good regularity at critical points, which causes great obstacles in the study. Up to now, the research in Finsler soliton is just the beginning. Therefore, it is natural and important for us to generalize some classical results to the Finsler Gaussian solitons. On this basis, it is necessary to consider some more complex problems such as Dirichlet and Neumann boundary conditions in a subset of $\mathbb{R}^n$. In that case, the situation is more challenging, and much more sophisticated tools must be used.

Let $\varphi(y)$ be a Minkowski norm on $\mathbb{R}^n$ and $\mathrm{d}V_{\mathrm{BH}}$ denote its Busemann-Hausdorff volume form. Define $F : T\mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$F(x, y) = \varphi(y), \quad y \in T_x\mathbb{R}^n \cong \mathbb{R}^n.$$

Then, $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}\mathrm{d}V_{\mathrm{BH}})$ is called the Finsler Gaussian soliton [16]. All Finsler Gaussian solitons have constant weighted Ricci curvature $\mathrm{Ric}_\infty = \rho$.

It is proved that in Gaussian shrinking soliton, each coordinate function x^j $(j = 1, \ldots, n)$ is the first eigenfunction of ρ, and each function $(x^j)^2 - \frac{1}{\rho}$ $(j = 1, \ldots, n)$ is the second eigenfunction of 2ρ (Theorems 1.1 and 3.1). It is also shown that [16] (Theorem 4.1) a coordinate function x^j $(j = 1, \ldots, n)$ is the first eigenfunction of ρ in the Finsler Gaussian soliton if and only if $\varphi(y)$ is a Euclidean norm. Therefore, a natural question arise: *What is about to happen for the Finsler Gaussian soliton when the function* $(x^j)^2 - \frac{1}{\rho}$ $(j = 1, \ldots, n)$ *is the eigenfunction of* 2ρ? In this article, the second purpose is to determine all Finsler shrinking Gaussian solitons such that the function $(x^j)^2 - \frac{1}{\rho}$ is the second eigenfunction of 2ρ.

Theorem 1.2. *Let* $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}\mathrm{d}V_{\mathrm{BH}})$ *be a Finsler Gaussian shrinking soliton. Then, the following assertions are equivalent*:

(*i*) *there is some* $i \in \{1, \ldots, n\}$ *such that* $(x^i)^2 - \frac{1}{\rho}$ *is the eigenfunction of* 2ρ;

(*ii*) *for every* $i \in \{1, \ldots, n\}$, *all functions* $(x^i)^2 - \frac{1}{\rho}$ *are the eigenfunctions of* 2ρ;

(*iii*) φ *is a Euclidean norm.*

For the Finsler Gaussian shrinking soliton $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}\mathrm{d}V_{\mathrm{BH}})$, Theorem 1.2 tells us that each function $(x^i)^2 - \frac{1}{\rho}$ is not the eigenfunction of 2ρ unless φ is a Euclidean norm. This contrasts sharply with the situation in the Gaussian shrinking soliton and actually gives a new characterization of a Minkowski metric to be a Euclidean metric. From Theorems 1.2 and 4.1, we should realize that not every result in Riemannian measure space can be generalized to the Finsler setting.

2 Preliminaries

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$ and $y \in T_xM$, and let (x^i, y^i) be the local coordinates on TM with $y = y^i\partial/\partial x^i$. A *Finsler metric* on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) *Regularity*: $F(x, y)$ is smooth in $TM\backslash 0$;

(ii) *Positive homogeneity*: $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$;

(iii) *Strong convexity*: The fundamental quadratic form

$$g_y \coloneqq g_{ij}(x, y)\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad g_{ij} \coloneqq \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positive definite. Define $\overleftarrow{F}(x, y) \coloneqq F(x, -y)$ for any $x \in M, y \in T_xM$. We call $\overleftarrow{F}$ the reverse Finsler metric.

Given two linearly independent vectors $y, v \in T_xM\backslash 0$, the flag curvature is defined as follows:

$$K(y, v) \coloneqq \frac{g_y(R^y(y, v)v, y)}{g_y(y, y)g_y(v, v) - g_y(y, v)^2},$$

where R^y is the Chern curvature.

Let $e_1, \ldots, e_{n-1}, \frac{y}{F(y)}$ be an orthonormal basis of T_xM with respect to g_y. Then, the Ricci curvature for (M, F) is defined as follows:

$$\mathrm{Ric}(y) = \sum_{i=1}^{n-1} K(y, e_i).$$

Let $(M, F, \mathrm{d}\mu)$ be an n-dimensional Finsler measure manifold. Let G^i be the geodesic coefficients of F and $\mathrm{d}\mu = \sigma(x)\mathrm{d}x$. The S-curvature of $(F, \mathrm{d}\mu)$ is given as follows:

$$S(x, y) \coloneqq \frac{\partial G^i}{\partial y^i} - y^i\frac{\partial \log \sigma(x)}{\partial x^i}.$$

Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = x, \dot{\gamma}(0) = y$. Define

$$\dot{S}(x, y) \coloneqq F^{-2}\frac{\mathrm{d}}{\mathrm{d}t}[S(\gamma(t), \dot{\gamma}(t))]_{t=0}.$$

Then, the *weighted* ∞ *Ricci curvature* of $(M, F, \mathrm{d}\mu)$ is defined as follows [14]:

$$\mathrm{Ric}_\infty(y) \coloneqq \mathrm{Ric}(y) + \dot{S}(y)$$

For a smooth function f, the *gradient vector* of f at x is defined as follows:

$$\nabla f(x) \coloneqq \mathcal{L}^{-1}(\mathrm{d}f),$$

where $\mathcal{L} : T_xM \to T_x^*M$ is the Legendre transform. Let $V = V^i\frac{\partial}{\partial x^i}$ be a smooth vector field on M. The *divergence* of V with respect to an arbitrary volume form $\mathrm{d}\mu = \sigma(x)\mathrm{d}x$ is defined as follows:

$$\mathrm{div}\,V \coloneqq \sum_{i=1}^{n}\left(\frac{\partial V^i}{\partial x^i} + V^i\frac{\partial \log \sigma}{\partial x^i}\right).$$

Then, the *Finsler-Laplacian* of f is defined as follows:

$$\Delta f \coloneqq \mathrm{div}(\nabla f),$$

where the equality is in the weak $W^{1,2}(M)$ sense (see Section 14.1 in [19]). Finsler-Laplacian is a nonlinear degenerate elliptic differential operator and is just the drift Laplacian if F is a Riemannian metric. In $M_f := \{x \in M | df(x) \neq 0\}$, we have

$$\Delta f = \frac{1}{\sigma(x)} \frac{\partial}{\partial x^i}\left(\sigma(x) g^{ij}(\nabla f) \frac{\partial f}{\partial x^j}\right). \tag{2.1}$$

For the reverse Finsler metric $\overleftarrow{F}$ and Finsler metric F, the gradient and Laplacian of a function have the following relationship [14,20]:

$$\overleftarrow{\nabla} f = -\nabla(-f), \quad \overleftarrow{\Delta} f = -\Delta(-f).$$

3 The eigenfunctions in Gaussian shrinking solitons

Recall that the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape that occurs in probability, statistics, and partial differential equations describing diffusion. It is defined as follows:

$$\mathbf{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t.$$

The imaginary error function, denoted **erfi**, is defined as follows:

$$\mathbf{erfi}(x) := -\sqrt{-1}\, \mathbf{erf}(\sqrt{-1}\, x).$$

The proof of Theorem 1.1. Let f be a smooth function in $\mathbb{R}^n$. Since $\sigma(x) = e^{-\frac{|x|^2}{4}}$, by a straightforward computation in equation (2.1), we have

$$\begin{aligned} \Delta f &= \frac{1}{e^{-\frac{|x|^2}{4}}} \frac{\partial}{\partial x^i}\left(e^{-\frac{|x|^2}{4}} \frac{\partial f}{\partial x^i}\right) \\ &= \frac{\partial^2 f}{(\partial x^i)^2} - \frac{1}{2} x^i \frac{\partial f}{\partial x^i} \\ &:= f_{ii} - \frac{1}{2} f_i x^i. \end{aligned} \tag{3.1}$$

Suppose that f is a first eigenfunction of the Laplacian Δ in $(\mathbb{R}^n, g_{\mathrm{can}}, \frac{|x|^2}{4})$. Then, $\Delta f = -\frac{1}{2} f$. By equation (3.1), we have

$$f_{ii} - \frac{1}{2} f_i x^i = -\frac{1}{2} f. \tag{3.2}$$

Note that the Laplacian Δ in $(\mathbb{R}^n, g_{\mathrm{can}})$ is a linear operator, and the restriction in the subspace $\mathbb{R}^m (m < n)$ is just the Laplacian Δ in $(\mathbb{R}^m, g_{\mathrm{can}})$, i.e.,

$$\Delta|_{\mathbb{R}^m} = \Delta(\mathbb{R}^m).$$

Therefore, the eigenspace of $\Delta(\mathbb{R}^m)$ is a subspace of the eigenspace of $\Delta(\mathbb{R}^n)$. In other words, we can solve equation (3.2) in the kth one-dimensional subspace ($k = 1, 2, \ldots, n$) and then extend it to the total space $\mathbb{R}^n$. In a one-dimensional subspace, rewrite equation (3.2) as

$$f''(x) - \frac{1}{2} x f'(x) = -\frac{1}{2} f(x). \tag{3.3}$$

Obviously, $f = x$ is a solution to the above equation. Let $f = C(x)x$ be a solution, where $C(x)$ is a function to be determined later. Substituting it into equation (3.3), we have

$$C'' x + C'\left(2 - \frac{1}{2} x^2\right) = 0.$$

A simple transmogrification gives

$$(\log C')' = \frac{1}{2}x - \frac{2}{x}.$$

Integrating on both sides, one obtains

$$C' = D_1 \frac{e^{\frac{x^2}{4}}}{x^2},$$

where D_1 is an arbitrary constant. Therefore, it holds that

$$\begin{aligned} C(x) &= D_1 \int \frac{e^{\frac{x^2}{4}}}{x^2} \mathrm{d}x + D_2 \\ &= -D_1 \frac{e^{\frac{x^2}{4}}}{x} + \frac{1}{2} D_1 \int e^{\frac{x^2}{4}} \mathrm{d}x + D_2 \\ &= -D_1 \frac{e^{\frac{x^2}{4}}}{x} + \frac{\sqrt{\pi}}{2} D_1 \mathbf{erfi}\left(\frac{x}{2}\right) + D_2, \end{aligned}$$

where D_2 is an arbitrary constant. Thus,

$$f(x) = -D_1 e^{\frac{x^2}{4}} + \frac{\sqrt{\pi}}{2} D_1 x \mathbf{erfi}\left(\frac{x}{2}\right) + D_2 x.$$

It shows that equation (3.2) has $2n$ linearly independent solutions $x^1, \ldots, x^n, -e^{\frac{(x^1)^2}{4}} + \frac{\sqrt{\pi}}{2} x^1 \mathbf{erfi}(\frac{x^1}{2}), \ldots, -e^{\frac{(x^n)^2}{4}} + \frac{\sqrt{\pi}}{2} x^n \mathbf{erfi}\left(\frac{x^n}{2}\right)$. This ends the proof.

It is known [8] that in the Gaussian shrinking soliton $(\mathbb{R}^n, g_{\text{can}}, \frac{|x|^2}{4})$, the second eigenvalue of the drift Laplacian is $\lambda_2 = 1$. Then, by similar arguments as above, we also obtain the following result.

Theorem 3.1. *In the Gaussian shrinking soliton* $(\mathbb{R}^n, g_{\text{can}}, \frac{|x|^2}{4})$, *any second eigenfunction of the drift Laplacian is a linear combination of the following functions*:

$$(x^1)^2 - 2, \ldots, (x^n)^2 - 2, \quad -\frac{x^1 e^{\frac{(x^1)^2}{4}}}{4} + \frac{\sqrt{\pi}((x^1)^2 - 2)\mathbf{erfi}(\frac{x^1}{2})}{8}, \ldots, -\frac{x^n e^{\frac{(x^n)^2}{4}}}{4} + \frac{\sqrt{\pi}((x^n)^2 - 2)\mathbf{erfi}\left(\frac{x^n}{2}\right)}{8},$$

and thus, the second eigenspace is given as follows:

$$\mathbf{E}^{\text{second}} \coloneqq \text{span}\left\{(x^i)^2 - 2, -\frac{x^i e^{\frac{(x^i)^2}{4}}}{4} + \frac{\sqrt{\pi}((x^i)^2 - 2)\mathbf{erfi}(\frac{x^i}{2})}{8}, i = 1, 2, \ldots, n.\right\}.$$

From Theorems 1.1 and 3.1, we find that x^i and $(x^i)^2 - 2$ $(i = 1, \ldots, n)$ are kth eigenfunctions of drift Laplacian for $k \leq 2$. However, for $k \geq 3$ case, it is entirely different in nature. Specifically, we have the following result.

Theorem 3.2. *In the Gaussian shrinking soliton* $(\mathbb{R}^n, g_{\text{can}}, \frac{|x|^2}{4})$, *any* k *th* $(k \geq 3)$ *eigenfunction is not a linear combination of the power function of the coordinate functions.*

Proof. For any $p > 0$, it holds from equation (3.1) that

$$\Delta (x^j)^p = ((x^j)^p)_{ii} - \frac{1}{2}((x^j)^p)_i x^i = -\frac{p}{2}(x^j)^p + p(p-1)(x^j)^{p-2}.$$

Therefore, we only have the following two cases:
- If $p = 1$, then $\Delta x^j = -\frac{1}{2}x^j$ for $j = 1, 2, \ldots, n$. In this case, $\lambda_1(\mathbb{R}^n) = \frac{1}{2}$.
- If $p = 2$, then $\Delta[(x^j)^2 - 2] = -[(x^j)^2 - 2]$ for $j = 1, 2, \ldots, n$. In this case, $\lambda_2(\mathbb{R}^n) = 1$. □

4 The eigenfunctions in Finsler Gaussian shrinking solitons

In Finsler geometry, Ge and Shen [21] and Yin and Mo [15] proved that all eigenfunctions f of Finsler-Laplacian belong to $C^{1,\alpha}(M) \cap C^\infty(M_f)$, where $M_f := \{x \in M | df(x) \neq 0\}$. Since the Euler-Lagrange equation is a nonlinear equation, it is difficult to obtain the eigenfunctions in general.

Likewise, in Finsler Gaussian solitons, the situation is much more complicated than in Gaussian solitons. Yin et al. [16] showed that in the Finsler Gaussian shrinking soliton, each coordinate function is not the eigenfunction unless the Finsler Gaussian solitons reduce to the Gaussian solitons.

Theorem 4.1. [16] *Let* $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ *be a Finsler Gaussian shrinking soliton. Then, the following assertions are equivalent*:
(*i*) *one of the coordinate functions is the eigenfunction of* ρ;
(*ii*) *all coordinate functions are the eigenfunctions of* ρ;
(*iii*) φ *is a Euclidean norm.*

In what follows, we will give an example to support Theorem 4.1.

Example 4.2. Set $y = (p, q)$ and Minkowski norm

$$F_\lambda(y) = \sqrt{p^2 + q^2 + \lambda\sqrt{p^4 + q^4}}, \quad \lambda \geq 0.$$

Then, $(\mathbb{R}^2, F_\lambda, e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ is a Finsler Gaussian shrinking soliton for $\rho > 0$. Recall that the fundamental tensor of F_λ is (see p. 5, [22])

$$(g_{ij}(y)) = \begin{pmatrix} 1 + \lambda\omega p^2(p^4 + 3q^4) & -2\lambda\omega p^3q^3 \\ -2\lambda\omega p^3q^3 & 1 + \lambda\omega q^2(q^4 + 3p^4) \end{pmatrix},$$

where $\omega = (p^4 + q^4)^{-\frac{3}{2}}$.

Let $\xi = (p^*, q^*) \in T_x^*\mathbb{R}^2$ be the dual vector of y. Then,

$$\xi = \mathcal{L}(y) = (FF_p, FF_q) = \left(p + \frac{2\lambda p^3}{\sqrt{p^4 + q^4}}, q + \frac{2\lambda q^3}{\sqrt{p^4 + q^4}}\right).$$

Set $(p^*, q^*) = (0, 1)$. Then, $\xi = dx^2$. Compute

$$\begin{cases} p + \dfrac{2\lambda p^3}{\sqrt{p^4 + q^4}} = 0, \\ q + \dfrac{2\lambda q^3}{\sqrt{p^4 + q^4}} = 1. \end{cases}$$

Then, we have $(p, q) = (0, \frac{1}{1+2\lambda})$. Let $V_x = (x^1, x^2)$ be the position vector of the point x. Write $dx^2 \triangleq (0, 1)$. It follows that

$$\nabla x^2 \triangleq \left(0, \frac{1}{1 + 2\lambda}\right).$$

Therefore,

$$(g_{ij}(\nabla x^2)) = \begin{pmatrix} 1 & 0 \\ 0 & 1+\lambda \end{pmatrix}, \quad (g^{ij}(\nabla x^2)) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}.$$

By Lemma 4.2 in [16], if x^2 is the first eigenfunction, then

$$g^{i2}(\nabla x^2) g_{il}(x) x^l = x^2.$$

This gives that

$$\frac{1}{1+\lambda}\{-2\lambda\omega(x^1)^3(x^2)^3x^1 + [1+\lambda\omega(x^2)^2(x^2)^4 + 3(x^1)^4]x^2\} = x^2,$$

where $\omega = ((x^1)^4 + (x^2)^4)^{-\frac{3}{2}}$. It is easy to see that $\lambda = 0$. Therefore, if x^2 is the first eigenfunction, then $F_\lambda = F_0$ is a Euclidean norm.

It is shown in the study by Yin et al. [16] that, in Finsler Gaussian shrinking soliton $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}\mathrm{d}V_{\mathrm{BH}})$, $r^2 - \frac{n}{\rho}$ is an eigenfunction corresponding to eigenvalues $\lambda = 2\rho$, where r is the distance function from origin. Observe that in a Gaussian shrinking soliton,

$$r^2 - \frac{n}{\rho} = \sum_{i=1}^{n}\left[(x^i)^2 - \frac{1}{\rho}\right],$$

where each function $(x^i)^2 - \frac{1}{\rho}$ is the second eigenfunction (Theorem 3.1).

The proof of Theorem 1.2. $(i) \Rightarrow (iii)$. Consider the upper half-space $\{x \in \mathbb{R}^n | x^i > 0\}$ for some $i \in \{1, 2, \ldots, n\}$. By a straight calculation, we have

$$\begin{aligned} \Delta(x^i)^2 &= \operatorname{div}(\nabla(x^i)^2) = \operatorname{div}(\mathcal{L}^{-1}(2x^i\mathrm{d}x^i)) \\ &= \operatorname{div}(2x^i\mathcal{L}^{-1}(\mathrm{d}x^i)) = \operatorname{div}(2x^i\nabla x^i) \\ &= 2x^i\Delta x^i + 2\mathrm{d}x^i(\nabla x^i), \end{aligned} \tag{4.1}$$

where $\mathcal{L}^{-1} : T\mathbb{R}^n \to T^*\mathbb{R}^n$ is the Legendre transformation. If $(x^i)^2 - \frac{1}{\rho}$ is the eigenfunction of 2ρ, then

$$\Delta\left[(x^i)^2 - \frac{1}{\rho}\right] = -2\rho\left[(x^i)^2 - \frac{1}{\rho}\right].$$

It follows from equation (4.1) that

$$2x^i\Delta x^i + 2\mathrm{d}x^i(\nabla x^i) = -2\rho(x^i)^2 + 2,$$

which gives

$$\Delta x^i = -\rho x^i + \frac{1 - \mathrm{d}x^i(\nabla x^i)}{x^i}. \tag{4.2}$$

On the other hand, by Lemma 4.2 in [16], we have

$$\Delta x^i = -\rho g^{ij}(\nabla x^i) g_{il}(x) x^l. \tag{4.3}$$

Therefore, combining equations (4.2) and (4.3), it shows that

$$-\rho g^{ij}(\nabla x^i) g_{il}(x) x^l = -\rho x^i + \frac{1 - \mathrm{d}x^i(\nabla x^i)}{x^i}. \tag{4.4}$$

Note that $-\rho g^{ij}(\nabla x^i) g_{il}(x)$ is bounded. Letting $x \to 0$, we have $x^l \to 0$ for all l, and thus, the left side of equation (4.4) goes to zero too. This implies that

$$\lim_{x^i\to 0^+}\frac{1-\mathrm{d}x^i(\nabla x^i)}{x^i}=0,$$

which yields that $\lim_{x^i\to 0^+}\mathrm{d}x^i(\nabla x^i)=1$. Since ∇x^i is a constant vector (see Lemma 4.1 in [16]), we have $\mathrm{d}x^i(\nabla x^i)=F(\nabla x^i)^2=1$. Then, from equation (4.2), we obtain

$$\Delta x^i=-\rho x^i.$$

This implies that the coordinate function x^i is the eigenfunction of ρ. Then, by Theorem 4.1, φ is a Euclidean norm.

Next, we are going to discuss lower half-space $\{x\in\mathbb{R}^n|x^i<0\}$. By using reverse Laplacian, we have

$$\begin{aligned}\Delta(x^i)^2&=\mathrm{div}(\nabla(x^i)^2)=\mathrm{div}(\mathcal{L}^{-1}(2x^i\mathrm{d}x^i))\\&=\mathrm{div}(-2x^i\mathcal{L}^{-1}(-\mathrm{d}x^i))=\mathrm{div}(2x^i\overleftarrow{\nabla}x^i)\\&=2x^i\overleftarrow{\Delta}x^i+2\mathrm{d}x^i(\overleftarrow{\nabla}x^i).\end{aligned}$$

Then, by using

$$\Delta\left[(x^i)^2-\frac{1}{\rho}\right]=-2\rho\left[(x^i)^2-\frac{1}{\rho}\right],$$

we also have

$$\overleftarrow{\Delta}x^i=-\rho x^i+\frac{1-\mathrm{d}x^i(\overleftarrow{\nabla}x^i)}{x^i}. \tag{4.5}$$

Using reverse Laplacian in equation (4.3), one obtains

$$\overleftarrow{\Delta}x^i=-\rho\overleftarrow{g}^{ij}(\overleftarrow{\nabla}x^i)\overleftarrow{g_{il}}(x)x^l. \tag{4.6}$$

Therefore, combining equations (4.5) and (4.6), it shows that

$$-\rho\overleftarrow{g}^{ij}(\overleftarrow{\nabla}x^i)\overleftarrow{g_{il}}(x)x^l=-\rho x^i+\frac{1-\mathrm{d}x^i(\overleftarrow{\nabla}x^i)}{x^i}.$$

By similar arguments as above, we have $\mathrm{d}x^i(\overleftarrow{\nabla}x^i)=\overleftarrow{F}(\overleftarrow{\nabla}x^i)^2=1$. It follows from equation (4.5) that

$$\overleftarrow{\Delta}x^i=-\rho x^i.$$

Consider the reverse Finsler metric $\overleftarrow{F}$ in Theorem 4.1. Then, we can obtain that φ is a Euclidean norm.

Finally, we prove that φ is also a Euclidean norm in hyperplane $\{x\in\mathbb{R}^n|x^i=0\}$. Note that the Cartan tensor

$$C_{ijk}(x)=\frac{\partial^3\varphi(x)}{\partial x^i\partial x^j\partial x^k}.$$

From the arguments above, we have

$$C_{ijk}(x^1,\ldots,x^i,\ldots,x^n)=0,\quad x^i\neq 0.$$

By continuity of C_{ijk}, we obtain

$$C_{ijk}(x^1,\ldots,0,\ldots,x^n)=\lim_{x^i\to 0}C_{ijk}(x^1,\ldots,x^i,\ldots,x^n)=0,$$

which means that φ is a Euclidean norm.

$(iii)\Rightarrow(ii)$. Assume that $F(x,y)=\sqrt{a_{ij}y^iy^j}$, where (a_{ij}) is a constant matrix. By Theorem 3.1, we have

$$\Delta\left[(x^i)^2-\frac{1}{\rho}\right]=-2\rho\left[(x^i)^2-\frac{1}{\rho}\right],\quad i=1,\ldots,n.$$

$(ii)\Rightarrow(i)$. It is obvious.

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