

Research Article

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New fractional integral inequalities via Euler's beta function

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Abstract: In this article, we present new fractional integral inequalities via Euler's beta function in terms of s -convex mappings. We develop some new generalizations of fractional trapezoid- and midpoint-type inequalities using the class of differentiable s -convexity. The results obtained in this study extended other related results reported in the literature.

Keywords: integral inequalities, beta function, s -convex function, fractional integral

MSC 2020: 26B25, 26D10, 26D15

1 Introduction

One important fundamental mathematical concept playing a vital role in other areas of pure and applied sciences is inequalities. The applications of inequalities – most of which are subjected to constraints – can be seen in different fields of studies to model many real-world problems, such as information theory, functional inequalities, and probability theory [1,2]. In mathematical analysis, inequalities are robust tools for comparing and analyzing functions – more especially – when establishing bounds on integrals. Thus, this study involves integral inequalities frequently used in modern mathematical analysis.

In addition, many theories of inequalities are often developed by convex functions whose findings are reported through different generalizations and extensions of convexities, such as s -convexity [3], p -convexity [4], log- h -convexity [5], harmonically convexity [6], and extended harmonically (s, m) -convexity [7]. For further studies one can refer previous studies [1,8–10]. Attracting the attention of many researchers due to their numerous applications, convexities are ubiquitous in mathematics including geometry, optimization theory, and functional analysis.

In mathematical analysis, one most useful discovery involving a convex function is the Hermite-Hadamard (H-H) inequality providing the integral mean of the function within a compact interval. This inequality is defined as follows [11]: If $v : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ is a convex mapping, with $\kappa_1 < \kappa_2$, then

$$v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} v(x) dx \leq \frac{v(\kappa_1) + v(\kappa_2)}{2}. \quad (1.1)$$

In addition to mathematics, this interesting inequality has been applied to solve many problems in different fields of studies including computer science, financial economics, and engineering [12–14]. Some interesting results and generalizations related to (1.1) can be found in [13,15–17].

In [3], Hudzik and Maligranda defined the class of s -convex functions as follows:

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Definition 1.1. A function $v : [0, \infty) \rightarrow [0, \infty)$ is called s -convex in the second sense, where $s \in (0, 1]$, if

$$v(\rho\kappa_1 + (1 - \rho)\kappa_2) \leq \rho^s v(\kappa_1) + (1 - \rho)^s v(\kappa_2) \quad (1.2)$$

holds for all $\kappa_1, \kappa_2 \in [0, \infty)$ and $\rho \in [0, 1]$.

Using (1.2), Dragomir and Fitzpatrick [18] established a variant of inequality (1.1) for s -convex mappings in the second sense as follows.

Theorem 1.2. Suppose that $v : [0, \infty) \rightarrow [0, \infty)$ is an s -convex mapping in the second sense, where $s \in (0, 1)$, and $\kappa_1, \kappa_2 \in [0, \infty)$, $\kappa_1 < \kappa_2$. If $v \in L^1([\kappa_1, \kappa_2])$, then the following inequalities hold:

$$2^{s-1} v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} v(x) dx \leq \frac{v(\kappa_1) + v(\kappa_2)}{s + 1}. \quad (1.3)$$

The constant $k = 1/(s + 1)$ is the best possible in the second inequality in (1.3).

In the last three decades, many studies in mathematical analysis massively employed fractional calculus to report interesting results through different extensions and generalizations of H-H inequalities [1,9, 19–22]. Both the fractional derivatives and fractional integrals provide variant types of potential tools for handling many special functions existing in mathematical sciences. This can be achieved by employing useful fractional operators, such as Riemann-Liouville [23,24], Katugampola [19], Caputo-Fabrizio [25], and Atangana-Baleanu [26], to study many problems of interest [3,14,27]. Consequently, the Riemann-Liouville fractional integral [24] and k -fractional integral [28] are defined as follows:

Definition 1.3. Let $v \in L_1[\kappa_1, \kappa_2]$. The Riemann-Liouville integrals $J_{\kappa_1^+}^\alpha v$ and $J_{\kappa_2^-}^\alpha v$ of order $\alpha > 0$ with $\kappa_1, \kappa_2 \geq 0$ can be defined by

$$J_{\kappa_1^+}^\alpha v(x) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^x (x - \rho)^{\alpha-1} v(\rho) d\rho, \quad x > \kappa_1$$

and

$$J_{\kappa_2^-}^\alpha v(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\kappa_2} (\rho - x)^{\alpha-1} v(\rho) d\rho, \quad x < \kappa_2,$$

whereby $\Gamma(\alpha)$ is the Gamma function and $J_{\kappa_1^+}^0 v(x) = J_{\kappa_2^-}^0 v(x) = v(x)$.

Definition 1.4. Let $v \in L_1[\kappa_1, \kappa_2]$. Then the k -Riemann-Liouville fractional integrals of order $\alpha > 0$ can be defined by

$$J_{\kappa_1^+,k}^\alpha v(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{\kappa_1}^x (x - \rho)^{\frac{\alpha}{k}-1} v(\rho) d\rho, \quad x > \kappa_1$$

and

$$J_{\kappa_2^-,k}^\alpha v(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{\kappa_2} (\rho - x)^{\frac{\alpha}{k}-1} v(\rho) d\rho, \quad x < \kappa_2,$$

where

$$\Gamma_k(\alpha) = \int_0^\infty \rho^{\alpha-1} e^{-\frac{\rho^k}{k}} d\rho, \quad \mathcal{R}(\alpha) > 0.$$

In the following theorem, Sarikaya et al. [29] established a new version of inequality (1.1) through Riemann-Liouville fractional integral.

Theorem 1.5. Let $v : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a function with $\kappa_1 < \kappa_2$ and $v \in L_1([\kappa_1, \kappa_2])$. If v is a convex function on $[\kappa_1, \kappa_2]$, then the following holds:

$$v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\kappa_2 - \kappa_1)^\alpha} [J_{\alpha+}^\alpha v(\kappa_2) + J_{\alpha-}^\alpha v(\kappa_1)] \leq \frac{v(\kappa_1) + v(\kappa_2)}{2}.$$

Many generalizations of integral inequalities exist in the literature for different functions, such as Mittag-Leffler [30], beta function [27], and Bessel functions [31].

In [32], the extension of Euler's beta function is presented as follows:

$$\beta(\gamma_1, \gamma_2; \sigma) = \int_0^1 \rho^{\gamma_1-1} (1-\rho)^{\gamma_2-1} e^{\frac{-\sigma}{\rho(1-\rho)}} d\rho, \quad \text{for } \gamma_1, \gamma_2, \sigma > 0.$$

Recently, Sarikaya and Kozan [27] proved the generalization of fractional integral inequalities for Euler's beta function via convex and differentiable convex mappings along with the following lemma.

Lemma 1.6. Let $v : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (κ_1, κ_2) with $\kappa_1 < \kappa_2$. If $v' \in L[\kappa_1, \kappa_2]$, then the following equality holds:

$$\begin{aligned} & \frac{v(\kappa_1) + v(\kappa_2)}{2} \beta(\gamma_1, \gamma_2; \sigma) - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \\ &= \frac{(\kappa_2 - \kappa_1)}{2} \int_0^1 \beta_\rho(\gamma_1, \gamma_2; \sigma) [v'(\rho\kappa_2 + (1-\rho)\kappa_1) - v'(\rho\kappa_1 + (1-\rho)\kappa_2)] d\rho, \end{aligned} \quad (1.4)$$

whereby $\beta_\rho(\gamma_1, \gamma_2; \sigma)$ is an incomplete Euler's beta mapping given by

$$\beta_\rho(\gamma_1, \gamma_2; \sigma) = \int_0^\rho \lambda^{\gamma_1-1} (1-\lambda)^{\gamma_2-1} e^{\frac{-\sigma}{\lambda(1-\lambda)}} d\lambda, \quad 0 < \rho \leq 1$$

for γ_1, γ_2 , and $\sigma > 0$.

While the above lemma was the estimates of the right-hand side of (1.1), the following results give the left estimates.

Lemma 1.7. [27] Let $v : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (κ_1, κ_2) with $\kappa_1 < \kappa_2$. If $v' \in L[\kappa_1, \kappa_2]$, then the following equality holds:

$$v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \beta(\gamma_1, \gamma_2; \sigma) - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx = \frac{(\kappa_2 - \kappa_1)}{2} \sum_{\vartheta=1}^4 \Lambda_{\vartheta}, \quad (1.5)$$

where

$$\begin{aligned} \Lambda_1 &= \int_0^{\frac{1}{2}} \beta_\rho(\gamma_1, \gamma_2; \sigma) v'(\rho\kappa_2 + (1-\rho)\kappa_1) d\rho, \\ \Lambda_2 &= \int_0^{\frac{1}{2}} (-\beta_\rho(\gamma_1, \gamma_2; \sigma)) v'(\rho\kappa_1 + (1-\rho)\kappa_2) d\rho, \\ \Lambda_3 &= \int_{\frac{1}{2}}^1 (-\beta_{1-\rho}(\gamma_1, \gamma_2; \sigma)) v'(\rho\kappa_2 + (1-\rho)\kappa_1) d\rho, \\ \Lambda_4 &= \int_{\frac{1}{2}}^1 \beta_{1-\rho}(\gamma_1, \gamma_2; \sigma) v'(\rho\kappa_1 + (1-\rho)\kappa_2) d\rho \end{aligned}$$

for $\gamma_1, \gamma_2, \sigma > 0$.

Changing the variable of identity (1.5), we have the following remark:

Remark 1.8. [27] From the assumption of Lemma 1.7, the following identity holds:

$$\begin{aligned} & v\left(\frac{\kappa_1 + \kappa_2}{2}\right)\beta(\gamma_1, \gamma_2; \sigma) - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \\ &= \frac{(\kappa_2 - \kappa_1)}{2} \int_0^{\frac{1}{2}} [\beta_\rho(\gamma_1, \gamma_2; \sigma) + \beta_\rho(\gamma_2, \gamma_1; \sigma)] [v'(\rho\kappa_2 + (1 - \rho)\kappa_1) - v'(\rho\kappa_1 + (1 - \rho)\kappa_2)] d\rho. \end{aligned} \quad (1.6)$$

Proposition 1.9. [27] When the order of the integrals is changed, one can obtain the following:

$$\begin{aligned} & \int_0^{\frac{1}{2}} [\beta_\rho(\gamma_1, \gamma_2; \sigma) + \beta_\rho(\gamma_2, \gamma_1; \sigma)] d\rho \\ &= \int_0^{\frac{1}{2}} \int_0^\rho \lambda^{\kappa_1 - 1} (1 - \lambda)^{\kappa_2 - 1} e^{\frac{-\sigma}{\lambda(1 - \lambda)}} d\lambda d\rho + \int_0^{\frac{1}{2}} \int_0^\rho (1 - \lambda)^{\kappa_1 - 1} \lambda^{\kappa_2 - 1} e^{\frac{-\sigma}{\lambda(1 - \lambda)}} d\lambda d\rho \\ &= \frac{1}{2} \beta(\kappa_1, \kappa_2; \sigma) - \beta_{\frac{1}{2}}(\kappa_1 + 1, \kappa_2; \sigma) - \beta_{\frac{1}{2}}(\kappa_2 + 1, \kappa_1; \sigma). \end{aligned} \quad (1.7)$$

Motivated by the work of Sarikaya and Kozan [27], who generalized integral inequalities connected with (1.1) via classical convexity involving Euler's beta function, we opt to use the class of s -convexity to establish new fractional inequalities which generalized results in [27]. We also obtained some new bounds for trapezoid- and midpoint-type inequalities via differentiable s -convexity. The results presented in this study provide extensions of some earlier works including the inequalities of Riemann-Liouville fractional integral and k -Riemann-Liouville fractional integral.

2 Main results

In this section, we present new results of H-H type using Euler's beta function.

Theorem 2.1. Suppose that $v : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ is an s -convex function on $[\kappa_1, \kappa_2]$, where $s \in (0, 1]$ and $\kappa_1 < \kappa_2$. Thus, the following inequality is satisfied for $\gamma_1, \gamma_2, \sigma > 0$ and beta mapping $\beta(\gamma_1, \gamma_2)$

$$\begin{aligned} v\left(\frac{\kappa_1 + \kappa_2}{2}\right)\beta(\gamma_1, \gamma_2; \sigma) &\leq \frac{1}{2^s(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \\ &\leq [\beta(\gamma_1 + s, \gamma_2; \sigma) + \beta(\gamma_1, \gamma_2 + s; \sigma)] \frac{v(\kappa_1) + v(\kappa_2)}{2^s}, \end{aligned} \quad (2.1)$$

where

$$\Theta(x) = (\kappa_2 - x)^{\gamma_1 - 1} (x - \kappa_1)^{\gamma_2 - 1} + (\kappa_2 - x)^{\gamma_2 - 1} (x - \kappa_1)^{\gamma_1 - 1}.$$

Proof. From s -convexity of a function v , we obtain

$$v\left(\frac{\tau_1 + \tau_2}{2}\right) \leq \frac{v(\tau_1) + v(\tau_2)}{2^s}.$$

Taking $\tau_1 = \rho\kappa_1 + (1 - \rho)\kappa_2$ and $\tau_2 = \rho\kappa_2 + (1 - \rho)\kappa_1$, we obtain

$$v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{v(\rho\kappa_1 + (1-\rho)\kappa_2) + v(\rho\kappa_2 + (1-\rho)\kappa_1)}{2^s}. \quad (2.2)$$

Both sides of equality (2.2) can be multiplied by $\rho^{m-1}(1-\rho)^{n-1}e^{-\frac{\sigma}{\rho(1-\rho)}}$ and the result obtained can be integrated with respect to ρ over $[0, 1]$ as follows:

$$\begin{aligned} & 2^s v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \int_0^1 \rho^{m-1}(1-\rho)^{n-1} e^{-\frac{\sigma}{\rho(1-\rho)}} d\rho \\ & \leq \int_0^1 \rho^{m-1}(1-\rho)^{n-1} v(\rho\kappa_1 + (1-\rho)\kappa_2) e^{-\frac{\sigma}{\rho(1-\rho)}} d\rho + \int_0^1 \rho^{m-1}(1-\rho)^{n-1} v((1-\rho)\kappa_1 + \rho\kappa_2) e^{-\frac{\sigma}{\rho(1-\rho)}} d\rho. \end{aligned}$$

Changing the variables gives the following:

$$\begin{aligned} v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \beta(\gamma_1, \gamma_2; \sigma) & \leq \frac{1}{2^s(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} [(\kappa_2 - x)^{\gamma_1 - 1} (x - \kappa_1)^{\gamma_2 - 1} + (\kappa_2 - x)^{\gamma_2 - 1} (x - \kappa_1)^{\gamma_1 - 1}] \\ & \times v(x) e^{-\frac{\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx. \end{aligned}$$

Thus, the first part of (2.1) is proved. In order to prove the second part, we use the definition of s -convexity

$$v(\rho\kappa_1 + (1-\rho)\kappa_2) + v((1-\rho)\kappa_1 + \rho\kappa_2) \leq [\rho^s + (1-\rho)^s](v(\kappa_1) + v(\kappa_2)) \quad (2.3)$$

for every $\rho \in [0, 1]$ and $s \in (0, 1]$.

We now multiply both sides of (2.3) by $\rho^{\gamma_1-1}(1-\rho)^{\gamma_2-1}e^{-\frac{\sigma}{\rho(1-\rho)}}$ and integrate the result therein with respect to ρ over $[0, 1]$ as follows:

$$\begin{aligned} & \frac{1}{2^s(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} [(\kappa_2 - x)^{\gamma_1 - 1} (x - \kappa_1)^{\gamma_2 - 1} + (\kappa_2 - x)^{\gamma_2 - 1} (x - \kappa_1)^{\gamma_1 - 1}] v(x) e^{-\frac{\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \\ & \leq \frac{v(\kappa_1) + v(\kappa_2)}{2^s} [\beta(\gamma_1 + s, \gamma_2; \sigma) + \beta(\gamma_1, \gamma_2 + s; \sigma)]. \quad \square \end{aligned}$$

Corollary 2.2.

(1) Taking $\gamma_1 = \gamma_2 = 1$ in Theorem 2.1, we obtain

$$2^{s-1} v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{(\kappa_2 - \kappa_1)\Omega(\sigma)} \int_{\kappa_1}^{\kappa_2} v(x) e^{-\frac{\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \leq \frac{v(\kappa_1) + v(\kappa_2)}{(s+1)}, \quad (2.4)$$

where $\Omega(\sigma) = \int_0^1 e^{-\frac{\sigma}{\rho(1-\rho)}} d\rho$, $\sigma > 0$.

(2) Choosing $\sigma \rightarrow 0$ in inequality (2.4), we then obtain inequality (1.3).

Corollary 2.3.

(1) In Theorem 2.1, choosing $\kappa_1 = 1$, $\kappa_2 = \alpha$ (or $\kappa_1 = \alpha$, $\kappa_2 = 1$), we obtain

$$\begin{aligned} v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \Omega(\alpha, \sigma) & \leq \frac{1}{2^s(\kappa_2 - \kappa_1)^\alpha} \int_{\kappa_1}^{\kappa_2} [(x - \kappa_1)^{\alpha-1} + (\kappa_2 - x)^{\alpha-1}] v(x) e^{-\frac{\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \\ & \leq [\beta(1+s, \alpha; \sigma) + \Omega(\alpha+s, \sigma)] \frac{v(\kappa_1) + v(\kappa_2)}{2^s}, \end{aligned}$$

where $\Omega(\alpha+s, \sigma) = \int_0^1 (1-\rho)^{\alpha+s} e^{-\frac{\sigma}{\rho(1-\rho)}} d\rho$ and $\Omega(\alpha, \sigma) = \int_0^1 (1-\rho)^\alpha e^{-\frac{\sigma}{\rho(1-\rho)}} d\rho$, $\sigma > 0$.

(2) If we take $s = 1$ in inequality (2.1), then we obtain inequality (2.10) of Theorem 2.5 in [27].

Remark 2.4. In Theorem 2.1, one can set $\kappa_1 = 1$, $\kappa_2 = \frac{\alpha}{k}$ (or $\kappa_1 = \frac{\alpha}{k}$, $\kappa_2 = 1$) with $\sigma \rightarrow 0$ to obtain some interesting inequalities for generalization of Riemann-Liouville fractional integrals of order α , which is k -Riemann-Liouville integrals.

The next results extend some estimates for the right-hand side of H-H-type inequalities via Euler's beta mapping involving differentiable s -convexity.

Theorem 2.5. Let $v : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (κ_1, κ_2) with $\kappa_1 < \kappa_2$ satisfying that $v' \in L[\kappa_1, \kappa_2]$. If $|v'|$ is s -convexity on $[\kappa_1, \kappa_2]$, then we obtain

$$\left| \beta(\gamma_1, \gamma_2; \sigma) \frac{v(\kappa_1) + v(\kappa_2)}{2} - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \right|$$

$$\leq \frac{(\kappa_2 - \kappa_1)}{s + 1} (|v'(\kappa_1)| + |v'(\kappa_2)|) \int_0^{\frac{1}{2}} [\beta_{1-\rho}(\gamma_1, \gamma_2; \sigma) - \beta_\rho(\gamma_1, \gamma_2; \sigma)] d\rho, \quad (2.5)$$

where $\gamma_1, \gamma_2, \sigma > 0$.

Proof. From identity (1.4) and the s -convexity of $|v'|$, we have

$$\left| \beta(\gamma_1, \gamma_2; \sigma) \frac{v(\kappa_1) + v(\kappa_2)}{2} - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \right|$$

$$\leq \frac{(\kappa_2 - \kappa_1)}{2} \int_0^1 |\beta_\rho(\gamma_1, \gamma_2; \sigma) - \beta_{1-\rho}(\gamma_1, \gamma_2; \sigma)| |v'((1 - \rho)\kappa_1 + \rho\kappa_2)| d\rho$$

$$\leq \frac{(\kappa_2 - \kappa_1)}{2} \int_0^{\frac{1}{2}} [\beta_{1-\rho}(\gamma_1, \gamma_2; \sigma) - \beta_\rho(\gamma_1, \gamma_2; \sigma)] [(1 - \rho)^s |v'(\kappa_1)| + \rho^s |v'(\kappa_2)|] d\rho$$

$$+ \frac{(\kappa_2 - \kappa_1)}{2} \int_{\frac{1}{2}}^1 [\beta_\rho(\gamma_1, \gamma_2; \sigma) - \beta_{1-\rho}(\gamma_1, \gamma_2; \sigma)] [(1 - \rho)^s |v'(\kappa_1)| + \rho^s |v'(\kappa_2)|] d\rho$$

$$= \frac{(\kappa_2 - \kappa_1)}{s + 1} (|v'(\kappa_1)| + |v'(\kappa_2)|) \int_0^{\frac{1}{2}} [\beta_{1-\rho}(\gamma_1, \gamma_2; \sigma) - \beta_\rho(\gamma_1, \gamma_2; \sigma)] d\rho. \quad \square$$

Remark 2.6. In inequality (2.5) of Theorem 2.5, we have the following:

- (1) If we choose $s = 1$, then we obtain Theorem 3.6 in [27].
- (2) If we choose $\gamma_1 = \gamma_2 = s = 1$ and $\sigma \rightarrow 0$, then one can obtain Theorem 2.2 in [16].
- (3) If we take $\gamma_1 = s = 1$, $\gamma_2 = \alpha$, $\sigma \rightarrow 0$ (or $\gamma_1 = \alpha$, $\gamma_2 = s = 1$, $\sigma \rightarrow 0$), then we obtain Theorem 3 in [29].
- (4) If we choose $\gamma_1 = s = 1$, $\gamma_2 = \frac{\alpha}{k}$, $\sigma \rightarrow 0$ (or $\gamma_1 = \frac{\alpha}{k}$, $\gamma_2 = s = 1$, $\sigma \rightarrow 0$), then we obtain the following inequality related to k -fractional integral given in [27]

$$\left| \frac{v(\kappa_1) + v(\kappa_2)}{2} - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} [J_{\kappa_1^+, k}^\alpha f(\kappa_2) + J_{\kappa_2^-, k}^\alpha f(\kappa_1)] \right|$$

$$\leq \frac{(\kappa_2 - \kappa_1)}{\left(\frac{\alpha}{k} + 1\right)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right) \left(\frac{|v'(\kappa_1)| + |v'(\kappa_2)|}{2}\right).$$

Theorem 2.7. Let $v : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (κ_1, κ_2) with $\kappa_1 < \kappa_2$ and $v' \in L[\kappa_1, \kappa_2]$. If $|v'|^q$ is s -convex on $[\kappa_1, \kappa_2]$ for some $q > 1$, then the following inequality is satisfied for $\gamma_1, \gamma_2, \sigma > 0$

$$\left| \beta(\gamma_1, \gamma_2; \sigma) \frac{v(\kappa_1) + v(\kappa_2)}{2} - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \right| \leq \frac{(\kappa_2 - \kappa_1)}{2} \left(\frac{|v'(\kappa_1)|^q + |v'(\kappa_2)|^q}{s + 1} \right)^{\frac{1}{q}} \left(\int_0^1 |\beta_\rho(\gamma_1, \gamma_2; \sigma) - \beta_{1-\rho}(\gamma_1, \gamma_2; \sigma)|^p dt \right)^{\frac{1}{p}}, \quad (2.6)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using identity (1.4), Hölder's inequality, and the s -convexity of $|v'|^q$, we obtain the following:

$$\begin{aligned} & \left| \beta(\gamma_1, \gamma_2; \sigma) \frac{v(\kappa_1) + v(\kappa_2)}{2} - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \right| \\ & \leq \frac{(\kappa_2 - \kappa_1)}{2} \left(\int_0^1 |v'((1 - \rho)\kappa_1 + \rho\kappa_2)|^q d\rho \right)^{\frac{1}{q}} \left(\int_0^1 |\beta_\rho(\gamma_1, \gamma_2; \sigma) - \beta_{1-\rho}(\gamma_1, \gamma_2; \sigma)|^p dt \right)^{\frac{1}{p}} \\ & \leq \frac{(\kappa_2 - \kappa_1)}{2} \left(\frac{|v'(\kappa_1)|^q + |v'(\kappa_2)|^q}{s + 1} \right)^{\frac{1}{q}} \left(\int_0^1 |\beta_\rho(\gamma_1, \gamma_2; \sigma) - \beta_{1-\rho}(\gamma_1, \gamma_2; \sigma)|^p dt \right)^{\frac{1}{p}}. \quad \square \end{aligned}$$

As special cases of Theorem 2.7, we derive the following:

Remark 2.8.

- (1) Setting $s = 1$ in inequality (2.6), we have Theorem 3.7 in [27].
- (2) Choosing $\gamma_1 = \gamma_2 = s = 1$ and $\sigma \rightarrow 0$, we obtain Theorem 2.3 in [16].
- (3) Taking $\gamma_1 = s = 1$, $\gamma_2 = \alpha$, $\sigma \rightarrow 0$ (or $\gamma_1 = \alpha$, $\gamma_2 = s = 1$, $\sigma \rightarrow 0$), we then have Theorem 8 in [33].

Corollary 2.9. In inequality (2.6), if we set $\gamma_1 = s = 1$, $\gamma_2 = \frac{\alpha}{k}$, $\sigma \rightarrow 0$ (or $\gamma_1 = \frac{\alpha}{k}$, $\gamma_2 = s = 1$, $\sigma \rightarrow 0$), we obtain the following inequality for k -fractional integral which is proved in [27]

$$\begin{aligned} & \left| \frac{v(\kappa_1) + v(\kappa_2)}{2} - \frac{\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}} [J_{\kappa_1^+, k}^\alpha f(\kappa_2) + J_{\kappa_2^-, k}^\alpha f(\kappa_1)] \right| \\ & \leq \frac{(\kappa_2 - \kappa_1)}{2} \left(\frac{|v'(\kappa_1)|^q + |v'(\kappa_2)|^q}{2} \right)^{\frac{1}{q}} \left(\frac{\frac{\alpha}{pk} + 1}{2} \right)^{\frac{1}{p}}, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{p} = 1$ and $\frac{\alpha}{k} \in [0, 1]$.

The following theorem presents new midpoint-type inequality Euler's beta function via differentiable s -convexity.

Theorem 2.10. Let $v : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (κ_1, κ_2) with $\kappa_1 < \kappa_2$ and $v' \in L[\kappa_1, \kappa_2]$. If $|v'|^q$ is s -convex on $[\kappa_1, \kappa_2]$ for some $q > 1$, then the following inequality is satisfied for $\gamma_1, \gamma_2, \sigma > 0$

$$\left| v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \beta(\gamma_1, \gamma_2; \sigma) - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \right|$$

$$\leq s^2 \left(\frac{1}{2} \beta(\gamma_1, \gamma_2; \sigma) - \beta_{\frac{\gamma_1}{2}}(\gamma_1 + 1, \gamma_2; \sigma) - \beta_{\frac{\gamma_2}{2}}(\gamma_2 + 1, \gamma_1; \sigma) \right) \frac{(\kappa_2 - \kappa_1)[|v'(\kappa_1)| + |v'(\kappa_2)|]}{2}. \quad (2.7)$$

Proof. Employing s -convexity of $|v'|^q$ together with identities (1.6) and (1.7), we obtain the following:

$$\left| v\left(\frac{\kappa_1 + \kappa_2}{2}\right) \beta(\gamma_1, \gamma_2; \sigma) - \frac{1}{2(\kappa_2 - \kappa_1)^{\gamma_1 + \gamma_2 - 1}} \int_{\kappa_1}^{\kappa_2} \Theta(x) v(x) e^{\frac{-\sigma(\kappa_2 - \kappa_1)^2}{(\kappa_2 - x)(x - \kappa_1)}} dx \right|$$

$$\leq \frac{s^2(\kappa_2 - \kappa_1)[|v'(\kappa_1)| + |v'(\kappa_2)|]}{2} \int_0^1 (\beta_\rho(\gamma_1, \gamma_2; \sigma) + \beta_\rho(\gamma_1, \gamma_2; \sigma)) d\rho$$

$$\leq s^2 \left(\frac{1}{2} \beta(\gamma_1, \gamma_2; \sigma) - \beta_{\frac{\gamma_1}{2}}(\gamma_1 + 1, \gamma_2; \sigma) - \beta_{\frac{\gamma_2}{2}}(\gamma_2 + 1, \gamma_1; \sigma) \right) \frac{(\kappa_2 - \kappa_1)[|v'(\kappa_1)| + |v'(\kappa_2)|]}{2}. \quad \square$$

Remark 2.11. In inequality (2.7) of Theorem 2.10, we have the following:

- (1) If we take $s = 1$, then we obtain Theorem 4.8 in [27].
- (2) If we choose $s = \gamma_1 = \gamma_2 = 1$ and $\sigma \rightarrow 0$, then one can obtain Theorem 2.3 in [15].
- (3) If we take $\gamma_1 = s = 1$, $\gamma_2 = \alpha$, $\sigma \rightarrow 0$ (or $\gamma_1 = \alpha$, $\gamma_2 = s = 1$, $\sigma \rightarrow 0$), then we obtain Theorem 2 in [34].

3 Conclusion

This study is concerned with the generalization and extension of two different fractional integrals inequalities: Riemann-Liouville fractional integral and k -Riemann-Liouville fractional integral. The new fractional integral inequalities for s -convexity are established via Euler's beta function. The new estimations of trapezoid- and midpoint-type inequalities involving s -convex differentiable functions are equally reported in the study. In the future work, our results can be generalized through different classes of convexity, such as p -convexity, log- h -convexity, harmonically convexity, extended harmonically (s, m) -convexity to obtain new fractional integral inequalities.

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