

Research Article

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An application of Hayashi's inequality in numerical integration

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Abstract: This study systematically develops error estimates tailored to a specific set of general quadrature rules that exclusively incorporate first derivatives. Moreover, it introduces refined versions of select generalized Ostrowski's type inequalities, enhancing their applicability. Through the skillful application of Hayashi's celebrated inequality to specific functions, the provided proofs underpin these advancements. Notably, this approach extends its utility to approximate integrals of real functions with bounded first derivatives. Remarkably, it employs Newton-Cotes and Gauss-Legendre quadrature rules, bypassing the need for stringent requirements on higher-order derivatives.

Keywords: Hayashi's inequality, Ostrowski's inequality, quadrature rules, nonincreasing functions, differentiable functions

MSC 2020: 26D15, 26D10, 41A55, 65D30

1 Introduction

In light of our pursuit to approximate the integral $\int_p^q \Phi(u)du$, we direct our attention to a fundamental approach that involves an absolutely continuous function Φ , meticulously defined over the closed interval $[p, q]$. This method provides a systematic framework that elegantly addresses this task, yielding the following insightful representation:

$$\int_p^q \Phi(u)du = Q(\eta, \theta, \vartheta, \sigma; z) + \mathcal{E}(\eta, \theta, \vartheta, \sigma; z). \quad (1)$$

Central to this formulation, $Q(\eta, \theta, \vartheta, \sigma; z)$ embodies a versatile quadrature strategy, thoughtfully crafted as follows:

$$Q(\eta, \theta, \vartheta, \sigma; z) = \frac{q-p}{2\sigma} \left[\eta\Phi(p) + \theta\Phi(z) + 2(\vartheta - \sigma)\Phi\left(\frac{p+q}{2}\right) + \theta\Phi(p+q-z) + \eta\Phi(q) \right]. \quad (2)$$

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It is imperative to underscore that the validity of $Q(\eta, \theta, \vartheta, \sigma; z)$ hinges on the constraint that z conforms to the interval $\frac{(2\sigma - \eta)p + \eta q}{2\sigma} \leq z \leq \frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma}$. Notably, the constants η, θ, ϑ , and σ assume positive values and adhere to the pivotal relationship $\eta + \theta + \vartheta = 2\sigma$, further augmented by the condition $\vartheta \geq \sigma > 0$.

It is within this context that the error component $\mathcal{E}(\eta, \theta, \vartheta, \sigma; z)$ materializes, delineating the difference between the original integral and our approximated quadrature, as elegantly captured by the formula:

$$\mathcal{E}(\eta, \theta, \vartheta, \sigma; z) := \int_p^q \Phi(u) du - Q(\eta, \theta, \vartheta, \sigma; z). \quad (3)$$

This comprehensive framework not only provides an innovative method for approximating the desired integral but also reveals a deeper connection between continuous functions and discrete approximations, fostering a profound understanding of the interplay between mathematical analysis and practical approximation techniques.

The presented quadrature rule stands as a comprehensive formulation that orchestrates the convergence of several well-established quadrature rules, encompassing but not confined to the Midpoint, Trapezoidal, Simpson's, Maclaurin's, $\frac{3}{8}$ -Simpson's, Boole's, and some of Gauss-Legendre rules. A notable attribute of this overarching framework is the incorporation of an error term, which adeptly encapsulates specific inequalities akin to those featured in Ostrowski's inequality.

In their work [1], Alomari and Dragomir pioneered the consideration of equation (3) as a unified formula that harmonizes the representation of error intrinsic to renowned quadrature rules. Concurrently, this formulation extends to encompass a generalized rendition of Ostrowski's inequality. It is noteworthy that several prior studies have tackled various partial instances of equation (3), further highlighting its significance within the mathematical landscape. Among these instances is the inequality

$$\left| \int_p^q \Phi(u) du - (q-p) \left[\eta \frac{\Phi(p) + \Phi(q)}{2} + (1-\eta)\Phi(z) \right] \right| \leq \left[\frac{(q-p)^2}{4} (\eta^2 + (1-\eta)^2) + \left(z - \frac{p+q}{2} \right)^2 \right] \|\Phi'\|_\infty \quad (4)$$

for all $\eta \in [0, 1]$ and $p + \eta \frac{q-p}{2} \leq z \leq q - \eta \frac{q-p}{2}$, which was proved by Dragomir et al. [2]. Another related extension was proved by Ujević [3], which reads:

$$\begin{aligned} & \left| \int_p^q \Phi(u) du - (q-p) \left[(1-\eta)\Phi(z) - \eta \frac{\Phi(p) + \Phi(q)}{2} + \frac{M-m}{2} (1-\eta) \left(z - \frac{p+q}{2} \right) \right] \right| \\ & \leq \frac{M-m}{2} \cdot \left[\frac{(q-p)^2}{4} (\eta^2 + (1-\eta)^2) + \left(z - \frac{p+q}{2} \right)^2 \right]. \end{aligned} \quad (5)$$

Recently, Alomari and Bakula [4] proved the following three inequalities:

$$\left| \frac{1}{q-p} \int_p^q \Phi(u) du - \frac{(z-p)\Phi(p) + (q-z)\Phi(q)}{q-p} - \omega \left(z - \frac{p+q}{2} \right) \right| \leq \frac{\omega}{2} (q-p-\omega) \leq \frac{(q-p)^2}{8} \quad (6)$$

for all $z \in [p, q]$.

$$\left| \frac{1}{q-p} \int_p^q \Phi(u) du - \Phi(z) + \omega \left(z - \frac{p+q}{2} \right) \right| \leq \omega \frac{q-p}{2} - \omega^2 \leq \frac{(q-p)^2}{16} \quad (7)$$

for all $z \in [p, q]$, and

$$\left| \frac{1}{q-p} \int_p^q \Phi(u) du - \frac{\Phi(z) + \Phi(p+q-z)}{2} \right| \leq \omega \left[\frac{q-p}{2} - \frac{3}{2}\omega \right] \leq \frac{(q-p)^2}{24}, \quad (8)$$

for all $z \in \left[p, \frac{p+q}{2}\right]$, where $\omega = \frac{\Phi(q) - \Phi(p)}{q-p}$. Provided that $\Phi : [p, q] \rightarrow \mathbb{R}$ is an absolutely continuous function on $[p, q]$ with $0 \leq \Phi'(u) \leq (q-p)$ and Φ' is integrable on $[p, q]$.

In [5], Alomari et al. proved the following refinement of equations (4) and (5).

$$\left| \frac{1}{q-p} \int_p^q \Phi(u) du - (1-\theta)\Phi(z) - \theta \frac{\Phi(p) + \Phi(q)}{2} + \omega \left(z - \frac{p+q}{2} \right) \right| \leq \left(\frac{M-m}{q-p} \right) \left(\omega \frac{q-p}{2} - \omega^2 \right) \leq \frac{(q-p)(M-m)}{16}, \quad (9)$$

for all $z \in [p, q]$, where $\omega = \frac{\Phi(q) - \Phi(p) - m(q-p)}{M-m}$.

Another related inequality of Ostrowski's type was proved in the same work [5], which reads:

$$\left| \frac{1}{q-p} \int_p^q \Phi(u) du - \eta \frac{\Phi(p) + \Phi(q)}{2} - (1-\eta) \frac{\Phi(z) + \Phi(p+q-z)}{2} \right| \leq \frac{1}{2} \omega \left(\frac{M-m}{q-p} \right) [(q-p) - 3\omega] \leq \frac{(M-m)(q-p)}{24}, \quad (10)$$

for all $z \in \left[p, \frac{p+q}{2}\right]$, where $\omega = \frac{\Phi(q) - \Phi(p) - m(q-p)}{M-m}$.

On the other hand, it should be noted that the upper bounds presented in equations (4)–(7) surpass the corresponding inequalities discussed in the prior literature, as outlined in references [6–16]. For a comprehensive understanding of these aforementioned inequalities, interested readers are advised to consult references [17–24], inclusive of the associated citations, as well as the specialized book [25] and some chapters in [25–30].

Now, let us recall the celebrated Hayashi's inequality, which reads that [31, pp. 311–312]:

Theorem 1. Let $T : [p, q] \rightarrow \mathbb{R}$ be a nonincreasing function on $[p, q]$ and $S : [p, q] \rightarrow \mathbb{R}$ an integrable mapping on $[p, q]$ with $0 \leq S(u) \leq C$ for all $u \in [p, q]$. Then, the inequality

$$C \int_{q-\omega}^q T(u) du \leq \int_p^q T(u) S(u) du \leq C \int_p^{p+\omega} T(u) du \quad (11)$$

holds, where $\omega = \frac{1}{C} \int_p^q S(u) du$.

This study focuses on the enhancement and refinement of the inequalities denoted as equations (4)–(10). Specifically, we derive improved bounds for these designated inequalities. The methodology we employ entails diverse approaches that have been previously addressed in the existing body of literature. Central to our investigation is the utilization of Hayashi's inequality, as depicted by equation (11). This key framework forms the basis for our analytical approach. Through this lens, we offer a fresh perspective on these inequalities, capitalizing on the power of Hayashi's inequality to derive enhanced bounds and refinements.

A notable feature of our work is the systematic application of Hayashi's inequality, which brings forth a novel dimension to the construction and refinement of longstanding inequalities. Our study thus both extends the understanding of the inequalities under consideration and highlights the utility of Hayashi's inequality as a versatile tool in mathematical analysis. The implications of our findings extend to practical applications that hold significant relevance. Specifically, we turn our attention to situations where the behavior of higher-order derivatives of functions becomes problematic due to their potential unbounded nature or even nonexistence. In response to these challenges, we embark on a transformative endeavor to reconfigure well-established quadrature formulas.

The crux of our approach lies in the strategic decision to focus exclusively on functions that are differentiable, leveraging only their first derivatives. This strategic shift not only simplifies the computational

landscape but also accommodates scenarios where the computation or consideration of higher derivatives is either infeasible or lacks physical meaning. By revisiting and redefining classical quadrature formulas within the context of first derivatives, we forge a novel path that proves especially beneficial in scenarios involving functions with intricate, nonstandard characteristics. This adaptation allows us to effectively capture the essence of these functions while sidestepping the potential complexities associated with higher derivatives.

2 The results

Let us begin with the following generalization of equation (4).

Theorem 2. Let $\Phi : [p, q] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[p, q]$ with $0 \leq \Phi'(u) \leq (q - p)$ and Φ' is integrable on $[p, q]$. Then

$$\left| \frac{1}{q-p} \int_p^q \Phi(u) du - \frac{1}{2\sigma} \left[\eta\Phi(p) + \theta\Phi(z) + 2(\vartheta - \sigma)\Phi\left(\frac{p+q}{2}\right) + \theta\Phi(p+q-z) + \eta\Phi(q) \right] \right| \leq \omega \frac{(q-p)}{2} - 2\omega^2 \leq \frac{(q-p)^2}{32}, \quad (12)$$

where $\omega = \frac{\Phi(q) - \Phi(p)}{q-p}$, for all $\frac{(2\sigma - \eta)p + \eta q}{2\sigma} \leq z \leq \frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma}$, and η, θ, ϑ , and σ are positive constants such that $\eta + \theta + \vartheta = 2\sigma$ with $\vartheta \geq \sigma > 0$.

Proof. Consider a fixed value z within the interval $[p, q]$. In pursuit of a more streamlined exposition that facilitates the attainment of our objective, we have structured the proof into four distinct steps, enumerated as follows.

Step 1: Let $G(u) = \frac{(2\sigma - \eta)p + \eta q}{2\sigma} - u$, $u \in [p, z]$. Applying the Hayashi's inequality (11) by setting $T(u) = G(u)$ and $S(u) = \Phi'(u)$, we obtain

$$\begin{aligned} (q-p) \int_{z-\omega}^z \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - u \right) du &\leq \int_p^z \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - u \right) \Phi'(u) du \\ &\leq (q-p) \int_p^{p+\omega} \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - u \right) du, \end{aligned} \quad (13)$$

where

$$\omega = \frac{1}{q-p} \int_p^q \Phi'(u) du = \frac{\Phi(q) - \Phi(p)}{q-p}.$$

Each term in equation (13) can be simplified as follows:

$$\begin{aligned} \int_{z-\omega}^z \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - u \right) du &= \frac{1}{2} \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - z + \omega \right)^2 - \frac{1}{2} \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - z \right)^2 \\ &= \omega \left(\frac{(2\sigma - \eta)p + \eta q}{2} - z + \frac{1}{2}\omega \right), \\ \int_p^z \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - u \right) \Phi'(u) du &= - \left(z - \frac{(2\sigma - \eta)p + \eta q}{2\sigma} \right) \Phi(z) - \frac{\eta}{2\sigma} (q-p) \Phi(p) + \int_p^z \Phi(u) du, \end{aligned}$$

and

$$\begin{aligned} \int_p^{p+\omega} \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - u \right) du &= \frac{1}{2} \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - p \right)^2 - \frac{1}{2} \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - p - \omega \right)^2 \\ &= \omega \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - p - \frac{1}{2}\omega \right). \end{aligned}$$

Then by substituting in equation (13), we have

$$\begin{aligned} (q - p)\omega \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - z + \frac{1}{2}\omega \right) &\leq - \left(z - \frac{(2\sigma - \eta)p + \eta q}{2\sigma} \right) \Phi(z) - \frac{\eta}{2\sigma} (q - p) \Phi(p) + \int_p^z \Phi(u) du \\ &\leq (q - p)\omega \left(\frac{(2\sigma - \eta)p + \eta q}{2\sigma} - p - \frac{1}{2}\omega \right). \end{aligned} \quad (14)$$

Step 2: Let $G(u) = \frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - u$, $u \in (z, \frac{p+q}{2}]$. By employing (11) again, we obtain

$$\begin{aligned} (q - p) \int_{\frac{p+q}{2} - \omega}^{\frac{p+q}{2}} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - u \right) du &\leq \int_z^{\frac{p+q}{2}} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - u \right) \Phi'(u) du \\ &\leq (q - p) \int_z^{z+\omega} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - u \right) du. \end{aligned} \quad (15)$$

Each term in equation (15) can be simplified as follows:

$$\begin{aligned} \int_{\frac{p+q}{2} - \omega}^{\frac{p+q}{2}} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - u \right) du &= \frac{1}{2} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - \frac{p+q}{2} + \omega \right)^2 \\ &\quad - \frac{1}{2} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - \frac{p+q}{2} \right)^2 \\ &= \omega \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - \frac{p+q}{2} + \frac{1}{2}\omega \right), \\ \int_z^{\frac{p+q}{2}} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - u \right) \Phi'(u) du &= - \left(\frac{p+q}{2} - \frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} \right) \Phi \left(\frac{p+q}{2} \right) \\ &\quad + \left(z - \frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} \right) \Phi(z) + \int_z^{\frac{p+q}{2}} \Phi(u) du, \end{aligned}$$

and

$$\begin{aligned} \int_z^{z+\omega} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - u \right) du &= \frac{1}{2} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - z \right)^2 \\ &\quad - \frac{1}{2} \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - z - \omega \right)^2 \\ &= \omega \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - z - \frac{1}{2}\omega \right). \end{aligned}$$

By substituting in equation (15), we have

$$\begin{aligned}
 & (q-p)\omega \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - \frac{p+q}{2} + \frac{1}{2}\omega \right) \\
 & \leq - \left(\frac{p+q}{2} - \frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} \right) \Phi \left(\frac{p+q}{2} \right) + \left(z - \frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} \right) \Phi(z) + \int_z^{\frac{p+q}{2}} \Phi(u) du \quad (16) \\
 & \leq (q-p)\omega \left(\frac{(2\sigma - \eta - \theta)p + (\eta + \theta)q}{2\sigma} - z - \frac{1}{2}\omega \right).
 \end{aligned}$$

Step 3: Let $G(u) = \frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - u$, $u \in \left[\frac{p+q}{2}, p+q-z \right]$. By employing (11) again we obtain

$$\begin{aligned}
 (q-p) \int_{p+q-z-\omega}^{p+q-z} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - u \right) du & \leq \int_{\frac{p+q}{2}}^{p+q-z} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - u \right) \Phi'(u) du \\
 & \leq (q-p) \int_{\frac{p+q}{2}}^{\frac{p+q}{2} + \omega} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - u \right) du. \quad (17)
 \end{aligned}$$

Each term in equation (17) can be simplified as follows:

$$\begin{aligned}
 & \int_{p+q-z-\omega}^{p+q-z} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - u \right) du = \frac{1}{2} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - p - q + z + \omega \right)^2 \\
 & \quad - \frac{1}{2} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - p - q + z \right)^2 \\
 & = \omega \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - p - q + z + \frac{1}{2}\omega \right), \\
 & \int_{\frac{p+q}{2}}^{p+q-z} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - u \right) \Phi'(u) du = - \left(p + q - z - \frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} \right) \Phi(p+q-z) \\
 & \quad + \left(\frac{p+q}{2} - \frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} \right) \Phi \left(\frac{p+q}{2} \right) + \int_{\frac{p+q}{2}}^{p+q-z} \Phi(u) du,
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\frac{p+q}{2}}^{\frac{p+q}{2} + \omega} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - u \right) du = \frac{1}{2} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - \frac{p+q}{2} \right)^2 \\
 & \quad - \frac{1}{2} \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - \frac{p+q}{2} - \omega \right)^2 \\
 & = \omega \left(\frac{(\eta + \theta)p + (2\sigma - \eta - \theta)q}{2\sigma} - \frac{p+q}{2} - \frac{1}{2}\omega \right).
 \end{aligned}$$

By substituting in equation (17), we have

$$\begin{aligned}
 & (q-p)\omega \left(\frac{(\eta+\theta)p + (2\sigma-\eta-\theta)q}{2\sigma} - p - q + z + \frac{1}{2}\omega \right) \\
 & \leq - \left(p + q - z - \frac{(\eta+\theta)p + (2\sigma-\eta-\theta)q}{2\sigma} \right) \Phi(p+q-z) \\
 & \quad + \left(\frac{p+q}{2} - \frac{(\eta+\theta)p + (2\sigma-\eta-\theta)q}{2\sigma} \right) \Phi\left(\frac{p+q}{2}\right) + \int_{\frac{p+q}{2}}^{p+q-z} \Phi(u) du \\
 & \leq (q-p)\omega \left(\frac{(\eta+\theta)p + (2\sigma-\eta-\theta)q}{2\sigma} - \frac{p+q}{2} - \frac{1}{2}\omega \right).
 \end{aligned} \tag{18}$$

Step 4: Let $G(u) = \frac{\eta p + (2\sigma - \eta)q}{2\delta} - u$, $u \in (p+q-z, q]$. By employing (11) again, we obtain

$$\begin{aligned}
 (q-p) \int_{q-\omega}^q \left(\frac{\eta p + (2\sigma - \eta)q}{2\delta} - u \right) du & \leq \int_{p+q-z}^q \left(\frac{\eta p + (2\sigma - \eta)q}{2\delta} - u \right) \Phi'(u) du \\
 & \leq (q-p) \int_{p+q-z}^{p+q-z+\omega} \left(\frac{\eta p + (2\sigma - \eta)q}{2\delta} - u \right) du.
 \end{aligned} \tag{19}$$

Each term in equation (19) can be simplified as follows:

$$\begin{aligned}
 \int_{q-\omega}^q \left(\frac{\eta p + (2\sigma - \eta)q}{2\delta} - u \right) du & = \frac{1}{2} \left(\frac{\eta p + (2\sigma - \eta)q}{2\sigma} - q + \omega \right)^2 - \frac{1}{2} \left(\frac{\eta p + (2\sigma - \eta)q}{2\sigma} - q \right)^2 \\
 & = \omega \left(\frac{\eta p + (2\sigma - \eta)q}{2\sigma} - q + \frac{1}{2}\omega \right) \\
 \int_{p+q-z}^q \left(\frac{\eta p + (2\sigma - \eta)q}{2\delta} - u \right) \Phi'(u) du & = - \left(q - \frac{\eta p + (2\sigma - \eta)q}{2\sigma} \right) \Phi(q) + \left(p + q - z - \frac{\eta p + (2\sigma - \eta)q}{2\sigma} \right) \Phi(p+q-z) \\
 & \quad + \int_{p+q-z}^q \Phi(u) du
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{p+q-z}^{p+q-z+\omega} \left(\frac{\eta p + (2\sigma - \eta)q}{2\delta} - u \right) du & = \frac{1}{2} \left(\frac{\eta p + (2\sigma - \eta)q}{2\sigma} - p - q + z \right)^2 - \frac{1}{2} \left(\frac{\eta p + (2\sigma - \eta)q}{2\sigma} - p - q + z - \omega \right)^2 \\
 & = \omega \left(\frac{\eta p + (2\sigma - \eta)q}{2\sigma} - p - q + z - \frac{1}{2}\omega \right).
 \end{aligned}$$

By substituting in equation (19), we have

$$\begin{aligned}
 (q-p)\omega \left(\frac{\eta p + (2\sigma - \eta)q}{2\sigma} - q + \frac{1}{2}\omega \right) & \leq - \left(q - \frac{\eta p + (2\sigma - \eta)q}{2\sigma} \right) \Phi(q) \\
 & \quad + \left(p + q - z - \frac{\eta p + (2\sigma - \eta)q}{2\sigma} \right) \Phi(p+q-z) + \int_{p+q-z}^q \Phi(u) du \\
 & \leq (q-p)\omega \left(\frac{\eta p + (2\sigma - \eta)q}{2\sigma} - p - q + z - \frac{1}{2}\omega \right).
 \end{aligned} \tag{20}$$

By adding the inequalities (14), (16), (18), and (20), we reach an inequality of the form

$$-N \leq v \leq N,$$

i.e., $|v| \leq N$. Namely, we obtain

$$\begin{aligned}
-(q-p)\left(\omega\frac{q-p}{2} - 2\omega^2\right) &\leq \int_p^q \Phi(u)du - \frac{(q-p)}{2\sigma}\left[\eta\Phi(p) + \theta\Phi(z) + 2(\vartheta - \sigma)\Phi\left(\frac{p+q}{2}\right) + \theta\Phi(p+q-z) + \eta\Phi(q)\right] \\
&\leq (q-p)\left(\omega\frac{q-p}{2} - 2\omega^2\right).
\end{aligned}$$

Upon meticulous evaluation, the application of this approach yields the initial inequality as delineated in formula (12). The subsequent step involves establishing the validity of the second inequality, which requires a thoughtful consideration of the function $\lambda(u) = -2u^2 + \frac{q-p}{2}u$.

To proceed, let us investigate the behavior of $\lambda(u)$. It is notable that $\lambda(u)$ reaches its maximum value when $u = \frac{q-p}{8}$. Substituting this critical point into the function yields $\lambda\left(\frac{q-p}{8}\right) = \frac{(q-p)^2}{32}$. This analysis conclusively establishes the upper limit of $\lambda(u)$.

With this crucial insight in mind, we proceed to examine the behavior of $\lambda(\omega)$. By virtue of the earlier-established bound on $\lambda(u)$, i.e., $\max\lambda(u) = \frac{(q-p)^2}{32}$, we deduce that $\lambda(\omega) = -\omega^2 + \frac{q-p}{2}\omega$ remains consistently below or equal to $\frac{(q-p)^2}{32}$.

This compelling demonstration thus completes the theorem's validation process, effectively affirming the truth of both the inequalities articulated within its framework. Through this comprehensive analysis, we not only establish the inequalities but also provide a deeper understanding of the underlying mathematical concepts and their interplay, further strengthening the theorem's integrity. $\square$

A generalization of equation (12) is incorporated in the following corollary.

Corollary 1. Let $\Phi : [p, q] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[p, q]$ with $m \leq \Phi'(u) \leq M$ and Φ' is integrable on $[p, q]$. Then

$$\begin{aligned}
&\left| \frac{1}{q-p} \int_p^q \Phi(u)du - \frac{1}{2\sigma} \left[\eta\Phi(p) + \theta\Phi(z) + 2(\vartheta - \sigma)\Phi\left(\frac{p+q}{2}\right) + \theta\Phi(p+q-z) + \eta\Phi(q) \right] \right| \\
&\leq \frac{M-m}{2} \cdot \omega \cdot \frac{(q-p-2\omega)}{q-p} \leq \frac{(M-m)(q-p)}{32},
\end{aligned} \tag{21}$$

where $\omega = \frac{\Phi(q) - \Phi(c) - m(q-p)}{M-m}$. For all, $\frac{(2\sigma-\eta)p + \eta q}{2\sigma} \leq z \leq \frac{(2\sigma-\eta-\theta)p + (\eta+\theta)q}{2\sigma}$, and η, θ, ϑ , and σ are positive constants such that $\eta + \theta + \vartheta = 2\sigma$ with $\vartheta \geq \sigma > 0$.

Proof. The proof can be established by repeating the proof of Theorem 3, with $k(u) = \Phi'(u) - m, u \in [p, q]$. $\square$

Corollary 2. Let Φ as in Corollary 1. Then

$$\left| \frac{(q-p)}{2\sigma} \left[\theta\Phi(z) + 2(\vartheta - \sigma)\Phi\left(\frac{p+q}{2}\right) + \theta\Phi(p+q-z) \right] - \int_p^q \Phi(u)du \right| \leq \frac{(M-m)(q-p)}{32}, \tag{22}$$

for all $p \leq z \leq \frac{(2\sigma-\theta)p + \theta q}{2\sigma}$.

Proof. Choose $\eta = 0$ in equation (21), then we obtain the required inequality. $\square$

Corollary 3. Let Φ as in Corollary 1. Then

$$\left| (q-p) \left[t \cdot \frac{\Phi(z) + \Phi(p+q-z)}{2} + (1-t) \cdot \Phi\left(\frac{p+q}{2}\right) \right] - \int_p^q \Phi(u)du \right| \leq \frac{(M-m)(q-p)}{32}, \tag{23}$$

for all $p \leq z \leq \frac{p+q}{2}$ and all $t \in [0, 1]$.

Proof. Choose $\eta = 0$, $\theta = t \in [0, 1]$, $\vartheta = 2 - t$, and $\sigma = 1$ in equation (22), then the desired result follows. $\square$

Remark 1. Let Φ as in Corollary 1. Setting $p = -1$, $q = 1$, $z = -\frac{1}{\sqrt{3}}$, $\theta = 1$, and $\vartheta = \sigma$. Then

$$\left| \Phi\left(-\frac{1}{\sqrt{3}}\right) + \Phi\left(\frac{1}{\sqrt{3}}\right) - \int_{-1}^1 \Phi(u) du \right| \leq \frac{M - m}{16}, \quad (24)$$

which represents the error of the two-point Gauss-Legendre quadrature rule.

Another consequence follows by setting $p = -1$, $q = 1$, $z = -\sqrt{\frac{3}{5}}$, $\theta = 5$, $\vartheta = 13$, and $\sigma = 9$. Then

$$\left| \frac{5}{9} \Phi\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} \Phi(0) + \frac{5}{9} \Phi\left(\sqrt{\frac{3}{5}}\right) - \int_{-1}^1 \Phi(u) du \right| \leq \frac{M - m}{16}, \quad (25)$$

which represents the error of the three-point Gauss-Legendre quadrature rule.

Corollary 4. Let f be mentioned earlier, if we choose

(1) $\eta = \theta = 0$, $\mu = \frac{3}{2}$, and $\sigma = 1$ with $z = p$, this leads us to the ensuing inequality, known as midpoint inequality

$$\left| \Phi\left(\frac{p+q}{2}\right) - \frac{1}{q-p} \int_p^q \Phi(u) du \right| \leq \frac{(M-m)(q-p)}{32}. \quad (26)$$

(2) $\eta = 1$, $\theta = 0$, and $\sigma = \vartheta = 1$ with $z = \frac{p+q}{2}$, this leads us to the ensuing inequality, known as trapezoid inequality

$$\left| \frac{\Phi(p) + \Phi(q)}{2} - \frac{1}{q-p} \int_p^q \Phi(u) du \right| \leq \frac{(M-m)(q-p)}{32}. \quad (27)$$

(3) $\eta = 1$, $\theta = 0$, $\vartheta = 5$, and $\sigma = 3$ with $z = \frac{5p+q}{6}$, this leads us to the ensuing inequality, known as Simpson's inequality

$$\left| \frac{1}{6} \left[\Phi(p) + 4\Phi\left(\frac{p+q}{2}\right) + \Phi(q) \right] - \frac{1}{q-p} \int_p^q \Phi(u) du \right| \leq \frac{(M-m)(q-p)}{32}. \quad (28)$$

(4) $\eta = 0$, $\theta = 1$, and $\vartheta = \sigma = 1$, this leads us to the ensuing inequality, known as two-point inequality

$$\left| \frac{\Phi(z) + \Phi(p+q-z)}{2} - \frac{1}{q-p} \int_p^q \Phi(u) du \right| \leq \frac{(M-m)(q-p)}{32}. \quad (29)$$

for all $p \leq z \leq \frac{p+q}{2}$.

(5) If we choose $\eta = 0$, $\theta = 3$, $\vartheta = 5$, and $\sigma = 4$ with $z = \frac{5p+q}{6}$, this leads us to the ensuing inequality, known as Maclaurin's inequality

$$\left| \frac{1}{8} \left[3\Phi\left(\frac{5p+q}{6}\right) + 2\Phi\left(\frac{p+q}{2}\right) + 3\Phi\left(\frac{p+5q}{6}\right) \right] - \frac{1}{q-p} \int_p^q \Phi(u) du \right| \leq \frac{(M-m)(q-p)}{32}. \quad (30)$$

(6) $\eta = 1$, $\theta = 3$, and $\vartheta = \sigma = 4$ with $z = \frac{2p+q}{3}$, this leads us to the ensuing inequality, known as 3/8-Simpson's inequality

$$\left| \frac{1}{8} \left[\Phi(p) + 3\Phi\left(\frac{2p+q}{3}\right) + 3\Phi\left(\frac{p+2q}{3}\right) + \Phi(q) \right] - \frac{1}{q-p} \int_p^q \Phi(u) du \right| \leq \frac{(M-m)(q-p)}{32}. \quad (31)$$

(7) $\eta = 7$, $\theta = 32$, $\vartheta = 51$, and $\sigma = 45$ with $z = \frac{3p+q}{4}$, this leads us to the ensuing inequality, known as Boole's inequality

$$\left| \frac{1}{90} \left[7\Phi(p) + 32\Phi\left(\frac{3p+q}{4}\right) + 12\Phi\left(\frac{p+q}{2}\right) + 32\Phi\left(\frac{p+3q}{4}\right) + 7\Phi(q) \right] - \frac{1}{q-p} \int_p^q \Phi(u) du \right| \leq \frac{(M-m)(q-p)}{32}. \quad (32)$$

3 Applications in numerical integration

Let us recall that the Gauss-Legendre quadratures are described as follows [32]:

$$\int_{-1}^1 \Phi(u) du = \sum_{j=1}^n A_j \Phi(v_j) + \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} \Phi^{(2n)}(\mu), \quad -1 < \mu < 1, \quad (33)$$

where A_j are the weights given by the formula

$$A_j = \frac{2}{(1-u_j^2)[P'_n(u_j)]^2}.$$

$P_n(t)$ is the Legendre polynomials, which are defined by its generating functions:

$$\frac{1}{\sqrt{1-2us+s^2}} = \sum_{n=0}^{\infty} P_n(u)s^n$$

or the Rodrigues' formula

$$P_n(u) = \frac{1}{2^n n!} \frac{d^n}{du^n} \{(u^2-1)^n\}.$$

Also, v_j in equation (33) are the zeros of the normalized Legendre polynomials $P_n(u)$.

It is remarkable that Gauss-Legendre quadratures require $(2n)$ -times continuously differentiable functions with bounded $(2n)$ -derivatives.

In this section, we embark on the presentation of more intricate quadrature formulas, stemming from the foundational structure delineated in equation (1). Specifically, our focus lies on the derivation and elucidation of advanced Gauss-Legendre formulas of higher order. These formulas materialize through the deliberate selection of nodes and their corresponding weights, a meticulous process that stands as a testament to the precision and adaptability of numerical integration.

An integral facet of our approach is the realization that these advanced formulas deviate from their classical counterparts in terms of their prerequisites. Unlike their traditional formulations, the high-order Gauss-Legendre and Newton-Cotes formulas presented here do not demand higher-order derivatives for their establishment. Instead, the key requirement revolves around the presence of bounded and continuous first derivatives. This shift in criteria not only simplifies the applicability of these formulas but also extends their utility to a broader range of functions. By unraveling these formulas within the context of bounded and continuous first derivatives, we bridge the gap between theoretical complexity and practical feasibility. This synthesis of mathematical rigor and pragmatic application encapsulates the essence of our endeavor. The presented formulas, while advancing the boundaries of numerical integration, simultaneously underscore the elegance that can be achieved by harnessing the simplicity of well-behaved first derivatives.

Let $d_\ell : p = z_0 < z_1 < \dots < z_{\ell-1} < z_\ell = q$ represent a division of the interval $[p, q]$, and denote $\Delta_j = z_{j+1} - z_j$ as the respective subintervals. In the subsequent discourse, our focus centers on the delineation of upper bounds applicable to the error approximation intrinsic to the general Gauss-Legendre quadrature formulas.

Theorem 3. Assume that the assumptions of Corollary 1 hold. Then, we have

$$\int_{-1}^1 \Phi(u) du = S(\Phi, d_\ell) + R(\Phi, d_\ell), \quad (34)$$

where $S(\Phi, d_\ell)$ is given in the formula

$$S(\Phi, d_\ell) = \frac{1}{\sigma} \sum_{j=0}^{\ell-1} [\theta_j \Phi(-\xi_j) + \theta_j \Phi(\xi_j)] + \frac{2\ell}{\sigma} (\vartheta - \sigma) \Phi(0) \quad (35)$$

and the remainder $R(\Phi, d_\ell)$ satisfies the bound

$$|R(\Phi, d_\ell)| \leq \frac{M - m}{16}$$

for all $-1 \leq \xi_j \leq \frac{\theta_j - \sigma}{\sigma} \leq 1$, and θ_j, ϑ , and σ are positive constants such that $\sum_{j=0}^{\ell-1} \theta_j + \vartheta = 2\sigma$ with $\vartheta \geq \sigma > 0$.

Proof. By employing Theorem 2 with $\eta = 0$, $p = -1$ and $q = 1$ to the subintervals $[z_j, z_{j+1}]$, followed by the aggregation of the resultant inequalities over the range of indices $j = 0, 1, \dots, \ell - 1$, the desired outcome is attained. The intricacies of this process shall be omitted. $\square$

Corollary 5. (Gauss-Legendre four-point quadrature rule) Assume that the assumptions of Corollary 1 hold. Then, we have

$$\int_{-1}^1 \Phi(u) du = S(\Phi, d_\ell) + R(\Phi, d_\ell),$$

where $S(\Phi, d_\ell)$ is given in the formula

$$S(\Phi, d_\ell) = \frac{1}{\sigma} [\theta_0 \Phi(\xi_0) + \theta_1 \Phi(\xi_1) + \theta_2 \Phi(\xi_2) + \theta_3 \Phi(\xi_3)],$$

and the remainder $R(\Phi, d_\ell)$ satisfies the bound

$$|R(\Phi, d_\ell)| \leq \frac{M - m}{16},$$

where

$$\begin{aligned} \xi_0 &= -\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}, \xi_1 = -\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}, \xi_2 = \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}, \xi_3 = \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} \\ \theta_0 &= \theta_3 = 18 - \sqrt{30}, \theta_1 = \theta_2 = 18 + \sqrt{30}, \quad \text{and} \quad \vartheta = \sigma = 36. \end{aligned}$$

Proof. This is a direct consequence of Theorem 3. We need to note that by setting $\ell = 4$, $\vartheta = \sigma = 36$. Also, choosing $\theta_j (j = 0, 1, 2, 3)$, and $\xi_j (j = 0, 1, 2, 3)$ as mentioned earlier, we obtain the desired result. $\square$

Corollary 6. (Gauss-Legendre five-point quadrature rule) Assume that the assumptions of Corollary 1 hold. Then, we have

$$\int_{-1}^1 \Phi(u) du = S(\Phi, d_\ell) + R(\Phi, d_\ell), \quad (36)$$

where $S(\Phi, d_\ell)$ is given in the formula

$$S(\Phi, d_\ell) = \frac{1}{\sigma} [\theta_0 \Phi(\xi_0) + \theta_1 \Phi(\xi_1) + 2(\vartheta - \sigma) \Phi(0) + \theta_2 \Phi(\xi_2) + \theta_3 \Phi(\xi_3)], \quad (37)$$

and the remainder $R(\Phi, d_\ell)$ satisfies the bound

$$|R(\Phi, d_\ell)| \leq \frac{M - m}{16}, \quad (38)$$

where

$$\xi_0 = -\frac{1}{3}\sqrt{5 + 2\sqrt{\frac{10}{7}}}, \xi_1 = -\frac{1}{3}\sqrt{5 - 2\sqrt{\frac{10}{7}}}, \xi_2 = \frac{1}{3}\sqrt{5 - 2\sqrt{\frac{10}{7}}}, \xi_3 = \frac{1}{3}\sqrt{5 + 2\sqrt{\frac{10}{7}}}$$

$$\theta_0 = \theta_3 = 5(322 - 13\sqrt{70}), \theta_1 = \theta_2 = 5(322 + 13\sqrt{70}), \vartheta = 5,780, \quad \text{and} \quad \sigma = 4,500.$$

Proof. This is a direct consequence of Theorem 3. We need to note that by setting $\ell = 5$, $\vartheta = 5,780$, and $\sigma = 4,500$. Also, choosing θ_j ($j = 0, 1, 2, 3$), and ξ_j ($j = 0, 1, 2, 3$) as mentioned earlier, we obtain the desired result. $\square$

Remark 2. Clearly, we can generate several quadrature rules of Gauss-Legendre type of any order as we wish by choosing a certain value of the nodes ξ_j and together with corresponding weights ℓ , θ_j , ϑ , and σ .

In the forthcoming numerical experiment, we embark on a comprehensive exploration by leveraging the efficacy of our proposed quadrature rule, succinctly articulated in formula (35). Our primary objective centers around rigorously assessing the performance and accuracy of this novel rule across a range of functions meticulously selected for this investigation. By subjecting the listed functions to the application of our quadrature rule, we aim to discern how effectively our approach captures the underlying integral values. This experiment offers a unique opportunity to quantitatively measure the accuracy of our rule in approximating the definite integrals associated with each function. The significance of this endeavor goes beyond mere validation; it also serves to illuminate the potential superiority of our approach in producing precise results across a range of diverse functions and integral contexts.

Example 1. In the subsequent numerical experiment, we employ our designated quadrature rule, as expressed in formula (35), to evaluate the specified functions within the confines of the interval $[-1, 1]$.

$\Phi(t)$	E.V.	Boole	3/8-Simpson	4-Gauss	5-Gauss	A.E.
$t^4 \sin\left(\frac{1}{t^2}\right)$	0.32827	0.09725	0.21418	0.38533	0.27547	0.05279
$t^4 \cos\left(\frac{1}{t}\right)$	0.12658	0.06555	0.12590	0.13525	0.121198	0.00067
$\exp(t t)$	2.20947	1.10681	1.14040	2.20893	2.20946	0.00001
$\sin(\exp(t t))$	1.55223	0.76290	0.72597	1.55474	1.55164	0.00058
$\exp(t^2 t)$	1.74682	1.00583	1.09633	1.74666	1.74683	0.000007

E.V. := The exact value of $\int_1^1 \Phi(t)dt$.

A.E. := The absolute error of the highly accurate quadrature rule (35) relative to the exact value.

We begin by recognizing that traditional quadrature rules encounter limitations when applied to such functions, where the presence of higher-order derivatives is either nonexistent or unbounded; for example the functions $t^4 \sin\left(\frac{1}{t^2}\right)$ and $t^4 \cos\left(\frac{1}{t}\right)$ do not have bounded second derivatives. Also, the function

$$\Phi(t) = \exp(t^2|t|) = \begin{cases} \exp(t^3), & -1 \leq t < 0, \\ \exp(-t^3), & 0 \leq t < 1, \end{cases}$$

which clearly does not have higher-order derivatives. But

$$\Phi'(t) = \begin{cases} -3t^2 \exp(-t^3), & -1 < t < 0, \\ 3t^2 \exp(t^3), & 0 < t < 1. \end{cases}$$

Noting that, $t = 0$ is a removable discontinuity point of $\Phi'(t)$. To see this, we note that $\lim_{t \rightarrow 0^+} \Phi'(t) = 0 = \lim_{t \rightarrow 0^-} \Phi'(t)$, so that we can redefine $\Phi'(0)$ to be 0. Similarly, the function

$$\Phi(t) = \exp(t^2 \lfloor t \rfloor) = \begin{cases} \exp(-t^2), & -1 \leq t < 0, \\ 1, & 0 \leq t < 1, \end{cases}$$

which clearly does not have higher-order derivatives. But

$$\Phi'(t) = \begin{cases} -2t \exp(-t^2), & -1 < t < 0, \\ 0, & 0 < t < 1. \end{cases}$$

Noting that, $t = 0$ is a removable discontinuity point of $\Phi'(t)$. To see this, we note that $\lim_{t \rightarrow 0^+} \Phi'(t) = 0 = \lim_{t \rightarrow 0^-} \Phi'(t)$, so that we can redefine $\Phi'(0)$ to be 0.

The selection of these particular quadrature rules is underpinned by their remarkable ability to yield high-precision approximations. The investigation clearly indicates that the application of the quadrature rule denoted as equation (35) (the value in bold) yields significantly improved approximations when compared to other classical rules, which may not be applied in this case. This superiority becomes even more pronounced when we meticulously scrutinize the absolute error associated with these quadrature rules in relation to the true or exact value of the integral under consideration.

Upon closer examination, it is clear that the five-point Gauss-Legendre quadrature rule (37) consistently and significantly outperforms other rules. For example, when applied to the first function, the absolute error of equation (37) is only 0.05279, showcasing its precision. In contrast, other rules display notably larger absolute errors for the same function. This difference in error magnitudes underscores the heightened accuracy embedded in the quadrature rule (37). However, for the second function, the 3/8-Simpson rule gives a more accurate approximation with the absolute error of 0.00067, while other rules exhibit a noticeably larger absolute error when applied to the same function. Moreover, this trend extends to other functions as well. In the context of the third, fourth, and fifth functions, the absolute errors associated with (37) are 0.00001, 0.00058, and 0.000007, respectively. In sharp contrast, the Boole, 3/8-Simpson, 2-Gauss-Legendre, 3-Gauss-Legendre, 4-Gauss-Legendre rules yield larger absolute errors for the same functions. This divergence in error values underscores the consistent precision advantage offered by the quadrature rule (37) across various functions with distinct characteristics.

The significance of this observation cannot be overstated. The empirical evidence strongly suggests that the quadrature rule (37) excels in achieving higher accuracy compared to other rules, particularly when dealing with functions that might pose challenges for precise integration. In practical terms, this outcome holds immense value, especially in scenarios where precision is of paramount importance. Whether in scientific simulations, engineering calculations, or any other context where accurate integral approximations are required, the adoption of the quadrature rule (37) over other rules can lead to more reliable and robust results. As such, this investigation serves as a valuable guideline for practitioners seeking the most accurate approach to integral approximation, highlighting the effectiveness of the quadrature rule (37) in achieving this goal.

Remark 3. Our method employed Hayashi's Inequality to develop a set of symmetrical quadrature rules encompassing both Newton-Cotes and Gauss-Legendre types. These rules are tailored for functions with integrable and bounded first derivatives. Traditionally, the applicability of such quadrature rules necessitates higher-order derivatives of the functions under consideration. However, our findings demonstrate that these quadratures can be effectively employed without imposing additional constraints on the target function slated for approximation. Conversely, the compilation of referenced sources aimed to alleviate constraints on the derivatives of functions slated for approximation, albeit through an alternative approach. Our outcomes offer

two distinct advantages: first, leveraging Hayashi's Inequality enabled us to obtain more precise upper bounds for error estimation. Second, our findings facilitated the construction of a series of Gauss-Legendre quadratures, a benefit not previously explored, achieved through an alternative methodology, see, for example, [8,12–14,17,24].

4 Conclusions

This study introduces enhanced versions of generalized Ostrowski's type inequalities, expanding their applicability in mathematical analysis. Through the strategic application of Hayashi's inequality to specific functions, we establish the validity of these improvements through rigorous proofs.

The core of our methodology revolves around estimating integrals for functions with unique characteristics. Employing diverse quadrature rules yields estimations surpassing those derived from precise function evaluations. Notably, our approach necessitates only the primary derivative to approximate values, obviating the need for additional bounded derivatives due to their potential nonexistence or unbounded behavior. Moreover, we encapsulate a computationally efficient solution in the formulation (35), streamlining intricate calculations involving unbounded functions.

In summary, our work not only unveils a novel approach to quadrature formulas but also demonstrates its practicality through its relevance to situations where higher derivatives present challenges. This application-driven extension of established mathematical tools exemplifies the adaptability of our approach and its potential to address real-world complexities.

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