Research Article

Yan-Fang Li and Feng Qi*

A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing cosine#

https://doi.org/10.1515/math-2023-0159
received May 16, 2023; accepted November 19, 2023

Abstract: In this study, by virtue of a derivative formula for the ratio of two differentiable functions and with aid of a monotonicity rule, the authors expand a logarithmic expression involving the cosine function into the Maclaurin power series in terms of specific determinants and prove a decreasing property of the ratio of two logarithmic expressions containing the cosine function.

Keywords: Maclaurin power series expansion, decreasing property, logarithmic expression, cosine function, derivative formula, ratio of two differentiable functions, monotonicity rule

MSC 2020: 41A58, 26A09, 33B10

1 Motivations

In [1, pp. 42 and 55], we find the Maclaurin power series expansions

$$\cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40,320} - \cdots, \quad x \in \mathbb{R} \tag{1}$$

and

$$\ln\cos x = -\sum_{k=1}^{\infty}\frac{2^{2k-1}(2^{2k}-1)}{k(2k)!}|B_{2k}|x^{2k} = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2,520} - \cdots, \quad x^2 < \frac{\pi^2}{4},$$

where B_{2k} denotes the Bernoulli numbers, which can be generated by:

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty}B_k\frac{z^k}{k!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty}B_{2k}\frac{z^{2k}}{(2k)!} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30,240} - \frac{z^8}{1,209,600} + \cdots, \quad |z| < 2\pi.$$

Dedicated to Professor Dr. Sever Silvestru Dragomir at Victoria University in Australia.

* **Corresponding author: Feng Qi**, School of Mathematics and Physics, Hulunbuir University, Hulunbuir 021008, Inner Mongolia, China; School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, China, e-mail: honest.john.china@gmail.com
Yan-Fang Li: School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, China, e-mail: fang-98@163.com
ORCID: Yan-Fang Li 0000-0001-9555-306X; Feng Qi 0000-0001-6239-2968

For more detailed information about B_{2k}, please refer to the monograph [2] and articles [3,4].

Motivated by the recently published articles [5–7], we consider the following two problems in this study.

(1) What is the Maclaurin power series expansion of the even function

$$F(x) = \begin{cases} \ln \dfrac{2(1-\cos x)}{x^2}, & 0 < |x| < 2\pi \\ 0, & x = 0 \end{cases} \tag{2}$$

around $x = 0$?

(2) Is the even function

$$R(x) = \begin{cases} \dfrac{\ln \frac{2(1-\cos x)}{x^2}}{\ln \cos x}, & 0 < |x| < \dfrac{\pi}{2} \\ \dfrac{1}{6}, & x = 0 \\ 0, & x = \pm\dfrac{\pi}{2} \end{cases} \tag{3}$$

decreasing on the close interval $[0, \frac{\pi}{2}]$?

2 Lemmas

For smoothly solving the aforementioned two problems, we need the following lemmas.

Lemma 1. *Let $u(x)$ and $v(x) \ne 0$ be two n-time differentiable functions on an interval I for a given integer $n \ge 0$. Then, the nth derivative of the ratio $\frac{u(x)}{v(x)}$ is*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n \frac{|W_{(n+1)\times(n+1)}(x)|}{v^{n+1}(x)}, \quad n \ge 0, \tag{4}$$

where the matrix

$$W_{(n+1)\times(n+1)}(x) = \left(U_{(n+1)\times 1}(x) \quad V_{(n+1)\times n}(x)\right)_{(n+1)\times(n+1)},$$

the matrix $U_{(n+1)\times 1}(x)$ is an $(n+1)\times 1$ matrix whose elements satisfy $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, *the matrix $V_{(n+1)\times n}(x)$ is an $(n+1)\times n$ matrix whose elements are $v_{\ell,j}(x) = \binom{\ell-1}{j-1} v^{(\ell-j)}(x)$ for $1 \le \ell \le n+1$ and $1 \le j \le n$, and the notation $|W_{(n+1)\times(n+1)}(x)|$ denotes the determinant of the $(n+1)\times(n+1)$ matrix $W_{(n+1)\times(n+1)}(x)$.*

Formula (4) is a reformulation of [8, p. 40, Exercise 5] (see also the papers [4,9,10] and those papers collected at the site [11].

Lemma 2. (Monotonicity rule for the ratio of two functions [12, Theorem 1.25]) *For $a, b \in \mathbb{R}$ with $a < b$, let $\lambda(x)$ and $\mu(x)$ be continuous on $[a, b]$, differentiable on (a, b), and $\mu'(x) \ne 0$ on (a, b). If the ratio $\frac{\lambda'(x)}{\mu'(x)}$ is increasing on (a, b), then both $\frac{\lambda(x)-\lambda(a)}{\mu(x)-\mu(a)}$ and $\frac{\lambda(x)-\lambda(b)}{\mu(x)-\mu(b)}$ are increasing in $x \in (a, b)$.*

3 Maclaurin power series expansion

In this section, we give a solution to the first problem posed in the first section of this article.

Theorem 1. *Let the real numbers*

$$\omega_k = \frac{(-1)^k}{(k+1)(2k+1)}, \quad k \ge 0$$

and the determinants

$$E_{2n} = -\begin{vmatrix} A_{2n-1,1} & B_{2n-1,2n-1} \\ \omega_n & C_{1,2n+1} \end{vmatrix}, \quad n \ge 1,$$

where the matrices $A_{2n-1,1}$, $B_{2n-1,2n-1}$, and $C_{1,2n-1}$ for $n \ge 1$ are defined by:

$$A_{2n-1,1} = \begin{pmatrix} 0 \\ \omega_1 \\ 0 \\ \omega_2 \\ \vdots \\ 0 \\ \omega_{n-1} \\ 0 \end{pmatrix} = (a_{i,j})_{\substack{1 \le i \le 2n-1 \\ j=1}},$$

$$a_{i,1} = \begin{cases} 0, & 1 \le i = 2k-1 \le 2n-1; \\ \omega_k, & 2 \le i = 2k \le 2n-2, \end{cases}$$

$$B_{2n-1,2n-1} = \begin{vmatrix} \binom{0}{0}\omega_0 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{1}\omega_0 & 0 & \cdots & 0 \\ \binom{2}{0}\omega_1 & 0 & \binom{2}{2}\omega_0 & \cdots & 0 \\ 0 & \binom{3}{1}\omega_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{2n-4}{0}\omega_{n-2} & 0 & \binom{2n-4}{2}\omega_{n-3} & \cdots & 0 \\ 0 & \binom{2n-3}{1}\omega_{n-2} & 0 & \cdots & 0 \\ \binom{2n-2}{0}\omega_{n-1} & 0 & \binom{2n-2}{2}\omega_{n-1} & \cdots & \binom{2n-2}{2n-2}\omega_0 \end{vmatrix}$$

$$= (b_{i,j})_{1 \le i,j \le 2n-1},$$

$$b_{i,j} = \begin{cases} 0, & 1 \le i < j \le 2n-1; \\ \binom{i}{j}\omega_k, & 0 \le i-j = 2k \le 2n-2; \\ 0, & 1 \le i-j = 2k-1 \le 2n-3, \end{cases}$$

$$C_{1,2n-1} = \begin{pmatrix} 0 & \binom{2n-1}{1}\omega_{n-1} & 0 & \binom{2n-1}{3}\omega_{n-2} & \cdots & 0 & \binom{2n-1}{2n-3}\omega_1 & 0 \end{pmatrix}$$

$$= (c_{i,j})_{\substack{i=1 \\ 1 \le j \le 2n-1}},$$

$$c_{1,j} = \begin{cases} 0, & 1 \le j = 2k-1 \le 2n-1; \\ \binom{2n-1}{2k-1}\omega_{n-k}, & 2 \le j = 2k \le 2n-2. \end{cases}$$

Then, the function $F(x)$ *defined by* (2) *can be expanded into the Maclaurin power series expansion*:

$$F(x) = -\sum_{n=1}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n} = -\frac{x^2}{12} - \frac{x^4}{1{,}440} - \frac{x^6}{90{,}720} - \frac{x^8}{4{,}838{,}400} - \cdots \tag{5}$$

for $|x| < 2\pi$.

Proof. On the interval $(0, \pi)$, directly differentiating yields

$$F'(x) = \frac{x \sin x + 2\cos x - 2}{x(1 - \cos x)} = \frac{\frac{x \sin x + 2\cos x - 2}{x^3}}{\frac{1-\cos x}{x^2}} \triangleq \frac{u(x)}{v(x)},$$

where

$$u(x) = \begin{cases} \dfrac{x \sin x + 2\cos x - 2}{x^3}, & x \ne 0 \\ 0, & x = 0 \end{cases} \quad \text{and} \quad v(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & x \ne 0 \\ \dfrac{1}{2}, & x = 0 \end{cases}$$

have the series expansions

$$u(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+2)(2k+3)} \frac{x^{2k+1}}{(2k+1)!} = \frac{1}{2} \sum_{k=0}^{\infty} \omega_{k+1} \frac{x^{2k+1}}{(2k+1)!}$$

and

$$v(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(k+1)(2k+1)} \frac{x^{2k}}{(2k)!} = \frac{1}{2} \sum_{k=0}^{\infty} \omega_{k} \frac{x^{2k}}{(2k)!}.$$

These two series expansions imply

$$u^{(n)}(0) = \begin{cases} 0, & n = 2k \\ \dfrac{\omega_{k+1}}{2}, & n = 2k+1 \end{cases} \quad \text{and} \quad v^{(n)}(0) = \begin{cases} \dfrac{\omega_k}{2}, & n = 2k \\ 0, & n = 2k+1 \end{cases}$$

for $k, n \ge 0$. Accordingly, making use of Formula (4) results in

$$F^{(2n+2)}(0) = \lim_{x\to 0} \left[\frac{u(x)}{v(x)}\right]^{(2n+1)}$$

$$= \frac{(-1)^{2n+1}}{v^{2n+2}(0)} \begin{vmatrix} u(0) & v(0) & 0 & \cdots & 0 \\ u'(0) & v'(0) & \binom{1}{1} v(0) & \cdots & 0 \\ u''(0) & v''(0) & \binom{2}{1} v'(0) & \cdots & 0 \\ u^{(3)}(0) & v^{(3)}(0) & \binom{3}{1} v''(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u^{(2n-2)}(0) & v^{(2n-2)}(0) & \binom{2n-2}{1} v^{(2n-3)}(0) & \cdots & 0 \\ u^{(2n-1)}(0) & v^{(2n-1)}(0) & \binom{2n-1}{1} v^{(2n-2)}(0) & \cdots & 0 \\ u^{(2n)}(0) & v^{(2n)}(0) & \binom{2n}{1} v^{(2n-1)}(0) & \cdots & \binom{2n}{2n} v(0) \\ u^{(2n+1)}(0) & v^{(2n+1)}(0) & \binom{2n+1}{1} v^{(2n)}(0) & \cdots & \binom{2n+1}{2n} v'(0) \end{vmatrix}$$

$$
\begin{aligned}
&= \frac{-1}{\omega_0^{2n+2}} \begin{vmatrix}
0 & \omega_0 & 0 & 0 & \cdots & 0 \\
\omega_1 & 0 & \binom{1}{1}\omega_0 & 0 & \cdots & 0 \\
0 & \omega_1 & 0 & \binom{2}{2}\omega_0 & \cdots & 0 \\
\omega_2 & 0 & \binom{3}{1}\omega_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \omega_{n-1} & 0 & \binom{2n-2}{2}\omega_{n-2} & \cdots & 0 \\
\omega_n & 0 & \binom{2n-1}{1}\omega_{n-1} & 0 & \cdots & 0 \\
0 & \omega_n & 0 & \binom{2n}{2}\omega_n & \cdots & \binom{2n}{2n}\omega_0 \\
\omega_{n+1} & 0 & \binom{2n+1}{1}\omega_n & 0 & \cdots & 0
\end{vmatrix} \\
&= - \begin{vmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
\omega_1 & 0 & \binom{1}{1} & 0 & \cdots & 0 \\
0 & \omega_1 & 0 & \binom{2}{2} & \cdots & 0 \\
\omega_2 & 0 & \binom{3}{1}\omega_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \omega_{n-1} & 0 & \binom{2n-2}{2}\omega_{n-2} & \cdots & 0 \\
\omega_n & 0 & \binom{2n-1}{1}\omega_{n-1} & 0 & \cdots & 0 \\
0 & \omega_n & 0 & \binom{2n}{2}\omega_n & \cdots & \binom{2n}{2n} \\
\omega_{n+1} & 0 & \binom{2n+1}{1}\omega_n & 0 & \cdots & 0
\end{vmatrix} \\
&= - \begin{vmatrix} A_{2n+1,1} & B_{2n+1,2n+1} \\ \omega_{n+1} & C_{1,2n+1} \end{vmatrix} \\
&= -E_{2n+2}
\end{aligned}
$$

for $n \geq 0$. Consequently, we obtain

$$
F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{F^{(2n)}(0)}{(2n)!} x^{2n} = -\sum_{n=1}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n}.
$$

The required proof is thus complete. □

Remark 1. For $n = 3$, the determinant E_6 is

$$E_6 = \begin{vmatrix} 0 & \binom{0}{0}\omega_0 & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \binom{1}{1}\omega_0 & 0 & 0 & 0 \\ 0 & \binom{2}{0}\omega_1 & 0 & \binom{2}{2}\omega_0 & 0 & 0 \\ \omega_2 & 0 & \binom{3}{1}\omega_1 & 0 & \binom{3}{3}\omega_0 & 0 \\ 0 & \binom{4}{0}\omega_2 & 0 & \binom{4}{2}\omega_2 & 0 & \binom{4}{4}\omega_0 \\ \omega_3 & 0 & \binom{5}{1}\omega_2 & 0 & \binom{5}{3}\omega_1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 & 1 & 0 & 0 \\ \frac{1}{15} & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{1}{15} & 0 & -1 & 0 & 1 \\ -\frac{1}{28} & 0 & \frac{1}{3} & 0 & -\frac{5}{3} & 0 \end{vmatrix} = \frac{1}{126}$$

and $-\frac{E_6}{6!} = -\frac{1}{90,720}$. This coincides with the coefficient of the term x^6 in the Maclaurin power series expansion (5).

Remark 2. On 11 September 2023, Gradimir V. Milovanović (Serbian Academy of Sciences and Arts) pointed out that

$$E_{2n} = \frac{|B_{2n}|}{n}, \quad n \ge 1. \tag{6}$$

This intrinsic observation can be derived from the interesting, but not-easily-guessed, relation:

$$(-z\mathrm{i})F'(-z\mathrm{i}) = 2\left(\frac{z}{\mathrm{e}^z - 1} - 1 + \frac{z}{2}\right), \quad |z| < 2\pi,$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit in the theory of complex numbers. Then, the series expansion (5) can be reformulated as:

$$F(x) = -\sum_{n=1}^{\infty} \frac{|B_{2n}|}{n} \frac{x^{2n}}{(2n)!}, \quad |x| < 2\pi. \tag{7}$$

We can also regard Relation (6) as a determinantal expression of the Bernoulli numbers B_{2n} for $n \ge 1$. For known results of determinantal expressions of the Bernoulli numbers B_{2n}, please refer to the literature [7,13–15], for example.

Remark 3. From the series expansion (7), we construct a positive and even function:

$$H_m(x) = \begin{cases} -\frac{m+1}{|B_{2m+2}|} \frac{(2m+2)!}{x^{2m+2}} \left[F(x) + \sum_{n=1}^{m} \frac{|B_{2n}|}{n} \frac{x^{2n}}{(2n)!}\right], & 0 < |x| < 2\pi \\ 1, & x = 0 \end{cases}$$

for $m \ge 1$. We now propose the following two problems.

(1) Discuss the logarithmic convexity or logarithmic concavity of the even and positive function $H_m(x)$ on $(0, 2\pi)$.
(2) Expand the function $H_m(x)$ into a Maclaurin power series at $x = 0$.

We believe that, making use of the series expansion

$$H_m(x) = \sum_{n=0}^{\infty} \frac{m+1}{n+m+1} \frac{(2m+2)!}{(2n+2m+2)!} \frac{|B_{2n+2m+2}|}{|B_{2m+2}|} x^{2n}, \quad |x| < 2\pi,$$

employing the derivative Formula (4), and with the help of a monotonicity rule in the articles [5,6] for the quotient of two power series, these two problems can be possibly solved.

4 Decreasing property

In this section, we solve the second problem posed in the first section of this article.

Theorem 2. *The function* $R(x)$ *defined by* (3) *decreasingly maps* $[0, \frac{\pi}{2}]$ *onto* $[0, \frac{1}{6}]$.

First proof. Straightforward computation yields

$$\frac{F'(x)}{(\ln\cos x)'} = \frac{\cot x(x\sin x + 2\cos x - 2)}{x(\cos x - 1)}$$

and

$$\left[\frac{F'(x)}{(\ln\cos x)'}\right]' = \frac{\csc^2 x}{x^2(1-\cos x)^2}Y(x),$$

where

$$Y(x) = x^2\sin x - x^2\sin x\cos^2 x + 4\sin x\cos^2 x - 2x\cos^2 x + 4x\cos x - 4\sin x\cos x + 2\cos x\sin^3 x - 2x. \tag{8}$$

It is well known that

$$\sin x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}. \tag{9}$$

In [1, p. 43], we find

$$\cos^2 x = 1 - \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k-1}}{(2k)!}x^{2k}, \quad |x| < \infty \tag{10}$$

and

$$\cos^3 x = \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k\frac{3^{2k}+3}{(2k)!}x^{2k}, \quad |x| < \infty. \tag{11}$$

Differentiating results in

$$\sin x\cos x = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(2k+1)!}x^{2k+1}, \quad |x| < \infty \tag{12}$$

and

$$\sin x\cos^2 x = \frac{1}{12}\sum_{k=0}^{\infty}(-1)^k\frac{3^{2k+2}+3}{(2k+1)!}x^{2k+1}, \quad |x| < \infty. \tag{13}$$

Theorem 2.1 in the study [16] states that

$$\left(\frac{\sin x}{x}\right)^{\ell} = 1 + \sum_{j=1}^{\infty}(-1)^j\frac{T(\ell+2j,\ell)}{\binom{\ell+2j}{\ell}}\frac{(2x)^{2j}}{(2j)!} \tag{14}$$

for $\ell \ge 0$ and $x \in \mathbb{C}$, where

$$T(n,\ell) = \frac{1}{\ell!}\sum_{m=0}^{\ell}(-1)^m\binom{\ell}{m}\left(\frac{\ell}{2}-m\right)^n. \tag{15}$$

Taking $n = 4 + 2j$ and $\ell = 4$ in (16) gives

$$T(4+2j,4) = \frac{1}{4!}\sum_{m=0}^{4}(-1)^m\binom{4}{m}(2-m)^{4+2j} = \frac{4^{j+1}-1}{3}.$$

Setting $\ell = 4$ in (14) leads to

$$\left(\frac{\sin x}{x}\right)^4 = 1 + \sum_{j=1}^{\infty}(-1)^j\frac{T(4+2j,4)}{\binom{4+2j}{4}}\frac{(2x)^{2j}}{(2j)!} = \sum_{j=0}^{\infty}(-1)^j\frac{2^{2j+3}(4^{j+1}-1)}{(2j+4)!}x^{2j},$$

which can be rearranged as:

$$\sin^4 x = \sum_{j=0}^{\infty}(-1)^j\frac{2^{2j+3}(2^{2j+2}-1)}{(2j+4)!}x^{2j+4}, \quad |x| < \infty.$$

Differentiating gives

$$\cos x \sin^3 x = \sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}(2^{2k+2}-1)}{(2k+3)!}x^{2k+3}, \quad |x| < \infty. \tag{16}$$

Making use of the Maclaurin power series expansions (1), (9), (10), (11), (12), (13), and (16), we can expand the function $Y(x)$ as:

$$\begin{aligned} Y(x) &= \frac{x^9}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\frac{\begin{bmatrix} 2^{4k+17} - \left(2k^2+17k+54\right)3^{2k+7} \\ + (k+6)4^{k+5} + 6k^2 + 35k + 34 \end{bmatrix}}{(2k+9)!}x^{2k} \\ &\triangleq \frac{x^9}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\Psi_k x^{2k} \\ &= -\frac{x^9}{2}\sum_{k=0}^{\infty}\Psi_{2k+1}\left(\frac{\Psi_{2k}}{\Psi_{2k+1}} - x^2\right)x^{4k}. \end{aligned}$$

By induction, we can verify that

$$2^{4k+17} - (2k^2+17k+54)3^{2k+7} > 0, \quad k \ge 0. \tag{17}$$

Hence, the sequence

$$\Psi_k = \frac{\begin{bmatrix} 2^{4k+17} - \left(2k^2+17k+54\right)3^{2k+7} \\ + (k+6)4^{k+5} + 6k^2 + 35k + 34 \end{bmatrix}}{(2k+9)!} > 0, \quad k \ge 0.$$

The inequality

$$\frac{\Psi_{2k}}{\Psi_{2k+1}} > 3 > \left(\frac{\pi}{4}\right)^2, \quad k \ge 0$$

is equivalent to

$$\Psi_{2k} > 3\Psi_{2k+1}, \quad k \ge 0, \tag{18}$$

i.e.,

$$\begin{aligned} &(8k^2+42k+31)2^{8k+18} - (128k^4+1216k^3+4384k^2+7142k+3969)3^{4k+7} + (8k^3+66k^2+175k+144)2^{4k+12} \\ &\quad + 384k^4 + 3136k^3 + 8992k^2 + 10274k + 3515 > 0 \end{aligned}$$

for $k \ge 0$. By induction, we can verify that the sequence

$$\begin{aligned} &(8k^2+42k+31)2^{8k+18} - (128k^4+1216k^3+4384k^2+7142k+3969)3^{4k+7} \\ &= 16(8k^2+42k+31)3^{4k+7}\left[\frac{4^{4k+7}}{3^{4k+7}} - \frac{128k^4+1216k^3+4384k^2+7142k+3969}{16(8k^2+42k+31)}\right] > 0 \end{aligned} \tag{19}$$

for $k \ge 1$. This means that Inequality (18) is valid for $k \ge 1$. Moreover, it is easy to see that

$$\Psi_0 = \frac{19}{360} = 0.0527 \cdots > 3\Psi_1 = \frac{29}{560} = 0.0517 \cdots .$$

Hence, Inequality (18) is valid for $k \ge 0$.

Combining the aforementioned results, we conclude that $Y(x) < 0$ on $[0, \frac{\pi}{2}]$. This means that the derivative ratio $\frac{F'(x)}{(\ln\cos x)'}$ is decreasing on $(0, \frac{\pi}{2}]$. Applying Lemma 2 leads to the decreasing property of the function $R(x)$ on $(0, \frac{\pi}{2})$. The first proof of Theorem 2 is complete.

Second proof. We start out this proof from considering the function $Y(x)$ defined by (8). Straightforward differentiating and expanding give

$$\begin{aligned}
Y'(x) &= [(6x^2+4)\cos x + (3x^2-4)\cos(2x) + 4\cos(3x) - 4x\sin x + 2x\sin(2x) + 3x^2 - 4]\sin^2\frac{x}{2}\\
&= \frac{1}{2}\sin^2\frac{x}{2}\sum_{k=3}^{\infty}(-1)^k\frac{8\times 3^{2k} - (6k^2+k+8)2^{2k} - 8(k-1)(6k+1)}{(2k)!}x^{2k}\\
&\triangleq \frac{1}{2}\sin^2\frac{x}{2}\sum_{k=3}^{\infty}(-1)^k W_k x^{2k}\\
&= \frac{1}{2}\sin^2\frac{x}{2}\sum_{k=1}^{\infty}[(-1)^{2k+1}W_{2k+1}x^{4k+2} + (-1)^{2k+2}W_{2k+2}x^{4k+4}]\\
&= \frac{1}{2}\sin^2\frac{x}{2}\sum_{k=1}^{\infty}(W_{2k+2}x^2 - W_{2k+1})x^{4k+2}\\
&= \frac{1}{2}\sin^2\frac{x}{2}\sum_{k=1}^{\infty}W_{2k+2}\left(x^2 - \frac{W_{2k+1}}{W_{2k+2}}\right)x^{4k+2}
\end{aligned}$$

for $x \in [0, \frac{\pi}{2}]$.

By induction, we obtain

$$7\times 3^{2k} - (6k^2+k+8)2^{2k} = 7\times 2^{2k}\left[\left(\frac{3}{2}\right)^{2k} - \frac{6k^2+k+8}{7}\right] > 0 \tag{20}$$

and

$$3^{2k} - 8(k-1)(6k+1) > 0 \tag{21}$$

for $k \ge 3$. This means that $W_k > 0$ for $k \ge 3$.

In order to prove the inequality $x^2 - \frac{W_{2k+1}}{W_{2k+2}} < 0$ for $k \ge 1$ and $x \in [0, \frac{\pi}{2}]$, it is sufficient to show

$$\frac{W_{2k+1}}{W_{2k+2}} > 3 > \left(\frac{\pi}{2}\right)^2 = 2.467\ldots, \quad k \ge 1. \tag{22}$$

The inequality $W_{2k+1} > 3W_{2k+2}$ for $k \ge 1$ is equivalent to

$$\begin{aligned}
&9(16k^2+28k-15)3^{4k} - (96k^4+272k^3+242k^2+33k-57)2^{4k+1} - (384k^4+896k^3+608k^2+54k-39)\\
&\quad > 0, \quad k \ge 1.
\end{aligned}$$

By induction, we can verify that

$$\begin{aligned}
&8(16k^2+28k-15)3^{4k} - (96k^4+272k^3+242k^2+33k-57)2^{4k+1}\\
&\quad = (16k^2+28k-15)2^{4k+3}\left[\left(\frac{3}{2}\right)^{4k} - \frac{96k^4+272k^3+242k^2+33k-57}{4(16k^2+28k-15)}\right] > 0
\end{aligned} \tag{23}$$

and

$$\begin{aligned}(16k^2+28k-15)3^{4k}&-(384k^4+896k^3+608k^2+54k-39)\\&=(16k^2+28k-15)\left[3^{4k}-\frac{384k^4+896k^3+608k^2+54k-39}{16k^2+28k-15}\right]>0\end{aligned}\tag{24}$$

for $k \ge 1$. Consequently, Inequality (22) is valid for $k \ge 1$. This implies that the derivative $Y'(x)$ is negative on $(0, \frac{\pi}{2}]$ and then that the function $Y(x)$ is decreasing on $[0, \frac{\pi}{2}]$. Due to $Y(0) = 0$, the function $Y(x)$ is negative on $(0, \frac{\pi}{2}]$. Then, the derivative ratio $\frac{F'(x)}{(\ln\cos x)'}$ is decreasing on $(0, \frac{\pi}{2}]$. Furthermore, using Lemma 2 leads to the decreasing property of the function $R(x)$ on $(0, \frac{\pi}{2})$. The second proof of Theorem 2 is complete.

Remark 4. For $t \ge 0$, let

$$G_1(t) = \frac{2^{4t+17}}{3^{2t+7}} - (2t^2 + 17t + 54) > 0, \quad t \ge 0.$$

It is clear that

$$G_1^{(3)}(t) = \frac{1{,}048{,}576}{2{,}187}\left(\frac{4}{3}\right)^{2t}\left(\ln\frac{4}{3}\right)^3 > 0, \quad t \ge 0.$$

This means that the second derivative $G_1''(t)$ is increasing in $t \ge 0$. From

$$G_1''(0) = \frac{524{,}288}{2{,}187}\left(\frac{4}{3}\right)^{2t}\left(\ln\frac{4}{3}\right)^2 - 4 = 15.840\ldots,$$

it follows that $G_1''(t) > 15$ for $t \ge 0$. This means that the first derivative $G_1'(t)$ is increasing in $t \ge 0$. Since

$$G_1'(0) = \frac{262{,}144}{2{,}187}\ln\frac{4}{3} - 17 = 17.482\ldots,$$

we deduce that the first derivative $G_1'(t) > 17$ for $t \ge 0$. Hence, the function $G_1(t)$ is increasing in $t \ge 0$. From the fact that

$$G_1(0) = \frac{2^{17}}{3^7} - 54 = 5.932\ldots,$$

we see that $G_1(t) > 5$ for $t \ge 0$. Consequently, Inequality (17) is alternatively proved.

Similarly, we can also prove Inequalities (20) and (21) alternatively.

Remark 5. For $t \ge 1$, let

$$G_2(t) = (8t^2 + 42t + 31)\frac{2^{8t+18}}{3^{4t+7}} - (128t^4 + 1{,}216t^3 + 4{,}384t^2 + 7{,}142t + 3{,}969).$$

By calculus, we arrive at

$$\begin{aligned}G_2(1) &= \frac{262{,}144}{27}\left(\frac{4}{3}\right)^4 - 16{,}839\\&= 13846.351\ldots,\\G_2'(1) &= \frac{15{,}204{,}352}{2{,}187}\left(\frac{4}{3}\right)^4 + \frac{1{,}048{,}576}{27}\left(\frac{4}{3}\right)^4\ln\frac{4}{3} - 20070\\&= 37212.729\ldots,\\G_2''(1) &= \frac{64}{2{,}187}\left[65{,}536\left(\frac{4}{3}\right)^4 + 5{,}308{,}416\left(\frac{4}{3}\right)^4\left(\ln\frac{4}{3}\right)^2 + 1{,}900{,}544\left(\frac{4}{3}\right)^4\ln\frac{4}{3} - 601{,}425\right]\\&= 79662.224\ldots,\end{aligned}$$

$$G_2^{(3)}(1) = \frac{128}{729}\left[3{,}538{,}944\left(\frac{4}{3}\right)^4\left(\ln\frac{4}{3}\right)^3 + 1{,}900{,}544\left(\frac{4}{3}\right)^4\left(\ln\frac{4}{3}\right)^2 + 131{,}072\left(\frac{4}{3}\right)^4\ln\frac{4}{3} - 59{,}049\right],$$
$$= 144599.308\ldots,$$
$$G_2^{(4)}(1) = \frac{1{,}024}{2{,}187}\left[5{,}308{,}416\left(\frac{4}{3}\right)^4\left(\ln\frac{4}{3}\right)^4 + 3{,}801{,}088\left(\frac{4}{3}\right)^4\left(\ln\frac{4}{3}\right)^3 + 393{,}216\left(\frac{4}{3}\right)^4\left(\ln\frac{4}{3}\right)^2 - 6{,}561\right]$$
$$= 232812.497\ldots,$$

and

$$G_2^{(5)}(t) = \frac{2^{18}}{3^7}\sum_{k=0}^{5}\binom{5}{k}(8t^2 + 42t + 31)^{(k)}\left[\left(\frac{4}{3}\right)^{4t}\right]^{(5-k)} > 0, \quad t \ge 1.$$

Discussing as done in Remark 4, we conclude that the function $G_2(t)$ is positive in $t \ge 1$. Inequality (19) is thus alternatively proved.

Similarly, we can also prove Inequalities (23) and (24) alternatively.

5 Conclusion

Let $f(x)$ be an even, positive, and analytic function on $(-r, r)$ such that $f(0) = 1$ and $f^{(2m)}(0) \neq 0$ for $m \ge 1$. Then,

$$f(x) = \sum_{k=0}^{\infty} f^{(2k)}(0)\frac{x^{2k}}{(2k)!}, \quad x \in (-r, r).$$

What are the Maclaurin power series expansions of the logarithmic expressions $\ln f(x)$ and $\ln\frac{2[f(x)-1]}{f''(0)x^2}$? What is the monotonicity of the even function $\frac{\ln\frac{2[f(x)-1]}{f''(0)x^2}}{\ln f(x)}$ on the interval $(0, r)$? Generally, what about the properties for the logarithms of the normalized remainders

$$F_n(x) = \begin{cases} 0, & x = 0 \\ \ln\left\{\dfrac{(2n)!}{f^{(2n)}(0)}\dfrac{1}{x^{2n}}\left[f(x) - \displaystyle\sum_{k=0}^{n-1} f^{(2k)}(0)\frac{x^{2k}}{(2k)!}\right]\right\}, & x \neq 0 \end{cases}$$

for $n \ge 0$ and their ratios $R_{m,n}(x) = \frac{F_n(x)}{F_m(x)}$ for $n > m \ge 0$?

In [5], the special case $f(x) = \frac{\tan x}{x}$ was discussed.

In [6], the special case $f(x) = \frac{\sin x}{x}$ was investigated.

In this article, we studied the special case $f(x) = \cos x$ and $R_{0,1}(x)$.

In subsequent articles, we will investigate more general cases.

Acknowledgement: The authors are grateful to anonymous referees for their careful corrections, valuable comments, helpful suggestions, and intrinsic observations on the original version of this article.

Funding information: The authors state no funding involved.

Author contributions: All authors contributed equally to the manuscript and read and approved the final manuscript.

Conflict of interest: The authors state no conflicts of interest.

Data availability statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

[1] D. Zwillinger, V. Moll, I.S. Gradshteyn, and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Eighth eds., Academic Press, Amsterdam, 2014, DOI: https://doi.org/10.1016/B978-0-12-384933-5.00013-8.

[2] N. M. Temme, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, John Wiley & Sons, Inc., New York, 1996, DOI: https://doi.org/10.1002/9781118032572.

[3] F. Qi, *A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers*, J. Comput. Appl. Math. **351** (2019), 1–5, DOI: https://doi.org/10.1016/j.cam.2018.10.049.

[4] F. Qi, *On signs of certain Toeplitz-Hessenberg determinants whose elements involve Bernoulli numbers*, Contrib. Discrete Math. **19** (2024), no. 1.

[5] Y.-W. Li, F. Qi, and W.-S. Du, *Two forms for Maclaurin power series expansion of logarithmic expression involving tangent function*, Symmetry **15** (2023), no. 9, 1686, DOI: https://doi.org/10.3390/sym15091686.

[6] X.-L. Liu, H.-X. Long, and F. Qi, *A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing sine*, Mathematics **11** (2023), no. 14, 3107, DOI: https://doi.org/10.3390/math11143107.

[7] Z.-Y. Sun, B.-N. Guo, and F. Qi, *Determinantal expressions, identities, concavity, Maclaurin power series expansions for van der Pol numbers, Bernoulli numbers, and cotangent*, Axioms **12** (2023), no. 7, 665, DOI: https://doi.org/10.3390/axioms12070665.

[8] N. Bourbaki, *Functions of a Real Variable: Elementary Theory*, Springer-Verlag, Berlin, 2004, DOI: https://doi.org/10.1007/978-3-642-59315-4.

[9] Y.-F. Li, D. Lim, and F. Qi, *Closed-form formulas, determinantal expressions, recursive relations, power series, and special values of several functions used in Clark-Ismail's two conjectures*, Appl. Comput. Math. **22** (2023), no. 4, 442–464, DOI: https://doi.org/10.30546/1683-6154.22.4.2023.442.

[10] F. Qi, *Determinantal expressions and recursive relations of Delannoy polynomials and generalized Fibonacci polynomials*, J. Nonlinear Convex Anal. **22** (2021), no. 7, 1225–1239.

[11] F. Qi, Dr. Prof. Feng Qi's Academic Homepage & Blogs, https://qifeng618.wordpress.com/2020/03/22/some-papers-authored-by-dr-prof-feng-qi-andutilizing-a-general-derivative-formula-for-the-ratio-of-two-differentiable-functions https://qifeng618.wordpress.com, Accessed on December 3, 2023.

[12] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[13] J. Cao, J. L. López-Bonilla, and F. Qi, *Three identities and a determinantal formula for differences between Bernoulli polynomials and numbers*, Electron. Res. Arch. **32** (2024), no. 1.

[14] F. Qi and R. J. Chapman, *Two closed forms for the Bernoulli polynomials*, J. Number Theory **159** (2016), 89–100, DOI: https://doi.org/10.1016/j.jnt.2015.07.021.

[15] F. Qi and B.-N. Guo, *Some determinantal expressions and recurrence relations of the Bernoulli polynomials*, Mathematics **4** (2016), no. 4, 65, DOI: https://doi.org/10.3390/math4040065.

[16] F. Qi and P. Taylor, *Series expansions for powers of sinc function and closed-form expressions for specific partial Bell polynomials*, Appl. Anal. Discrete Math. **18** (2024), no. 1, DOI: https://doi.org/10.2298/AADM230902020Q.