

Research Article

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Global well-posedness of initial-boundary value problem of fifth-order KdV equation posed on finite interval

<https://doi.org/10.1515/math-2023-0158>

received February 3, 2023; accepted November 14, 2023

Abstract: We have established the existence and uniqueness of the local solution for

$$\begin{cases} \partial_t u + \partial_x^5 u - u \partial_x u = 0, & 0 < x < 1, \quad t > 0, \\ u(x, 0) = \varphi(x), & 0 < x < 1, \\ u(0, t) = h_1(t), u(1, t) = h_2(t), \quad \partial_x u(1, t) = h_3(t), \\ \partial_x u(0, t) = h_4(t), \quad \partial_x^2 u(1, t) = h_5(t), & t > 0, \end{cases} \quad (0.1)$$

in the study of Zhao and Zhang [Non-homogeneous boundary value problem of the fifth-order KdV equations posed on a bounded interval, J. Math. Anal. Appl. **470** (2019), 251–278]. A question arises naturally: *Can the local solution be extended to a global one?* This article will address this question. First, through a series of logical deductions, a global *a priori* estimate is established, and then the local solution is naturally extended to a global solution.

Keywords: fifth-order KdV equation, initial-boundary value problem, *a priori* estimate, global well-posedness**MSC 2020:** 35G16, 35Q53

1 Introduction

The Korteweg-de Vries (KdV) equation was originally derived to describe shallow water waves [1]:

$$\partial_t u + \alpha \partial_x^3 u + \partial_x(u^2) = 0.$$

The KdV equation has been extensively applied to describe the propagation of long, weakly nonlinear, dispersive waves in one spatial dimension and has led to many important developments in the field of nonlinear wave theory and integrable systems.

However, in certain specific scenarios, the third-order dispersion effect falls short in accurately capturing real physical phenomena (for instance, situations such as the angle between the propagation direction and the magneto-acoustic wave in a cold collision-free plasma, along with the influence of an external magnetic field reaching critical values [2], as well as shallow water conditions near the critical point of surface tension [3]), which demand the consideration of higher-order dispersion effects. Consequently, the fifth-order KdV equation

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$$\partial_t u + \alpha \partial_x^5 u + \beta \partial_x^5 u + \partial_x(u^2) = 0,$$

gains significance in such contexts.

The fifth-order KdV equation is almost as prominently studied in the mathematical community as the KdV equation itself. In terms of initial value problems, the pace of well-posedness research closely follows that of the KdV equation. In 2005, Cui and Tao [4] employed the oscillation integral technique to establish the Kato smoothing effect. They subsequently demonstrated that the initial value problem is locally well-posed in H^s for $s > \frac{1}{4}$. Then, in 2007, Wang et al. [5] expanded upon these findings, proving that the initial value problem can be locally solved in $H^s(\mathbb{R})$ for $s > -\frac{7}{5}$, and global solutions exist for $s > -\frac{1}{2}$ using the “I method,” which relies on conservative laws. Subsequently, there has been significant progress in further enhancing the global well-posedness of this problem, ultimately leading to the sharp results as documented in [6–9].

However, the study of initial-boundary value problems for the fifth-order KdV equation:

$$\begin{cases} \partial_t u + \partial_x^5 u - u \partial_x u = 0, & 0 < x < 1, \quad t > 0, \\ u(x, 0) = \varphi(x), & 0 < x < 1, \\ u(0, t) = h_1(t), u(1, t) = h_2(t), \partial_x u(1, t) = h_3(t), \partial_x u(0, t) = h_4(t), \partial_x^2 u(1, t) = h_5(t), & t > 0, \end{cases} \quad (1.1)$$

presents a different picture, progressing much more slowly compared to the initial-boundary value problem for the KdV equation. The significant progress mainly occurred after 2014, for instance, as referenced in [10–13], a series of studies were conducted ranging from smoothness estimates, sharp trace regularity, up to local well-posedness. For the convenience of presenting the results, we first introduce several notations.

Notations: We express the boundary values in vector:

$$\vec{h}(t) \equiv (h_1(t), h_2(t), h_3(t), h_4(t), h_5(t)).$$

Furthermore, expressing the initial and boundary values as

$$(\phi(x), \vec{h}(t)) \equiv (\phi(x); h_1(t), h_2(t), h_3(t), h_4(t), h_5(t)).$$

Correspondingly, the function space to which the initial and boundary values belong is denoted as:

$$X_T^s \equiv H_0^s(0, 1) \times H_0^{\frac{s+2}{5}}(0, T) \times H_0^{\frac{s+2}{5}}(0, T) \times H_0^{\frac{s+1}{5}}(0, T) \times H_0^{\frac{s+1}{5}}(0, T) \times H_0^{\frac{s}{5}}(0, T).$$

Additionally, the function space to which the solution belongs is denoted as:

$$Y_T^s \equiv L^2(0, T; H^{2+s}(0, 1)) \cap C([0, T]; H^s(0, 1)).$$

Thus, local well-posedness can be formulated as follows:

Theorem A. (Local well-posedness [11]) *Let $T > 0$, $s \in [0, 5]$ (with $s \neq \frac{2j-1}{2}$, $j = 1, 2, 3, 4, 5$) be given. For any s -compatible $(\phi, \vec{h}) \in X_T^s$, there exists $T^* \in (0, T]$ such that the initial-boundary value problem (1.1) admits a unique solution $u \in Y_{T^*}^s$. Moreover, the solution depends Lipschitz continuously on $(\phi, \vec{h})$ in the corresponding space.*

The well-posedness result presented in Theorem A is local in the sense that the time interval $(0, T^*)$ on which the solution u exists depends on the size r of the initial-boundary data $(\phi, \vec{h})$ in the space X_T^s (we will rewrite it as Proposition 3.1). In general, the larger the size r , the smaller the length T^* of the time interval $(0, T^*)$. If T^* can be chosen to be T no matter how large the size r is, the initial-boundary value problem will be said to be globally well-posed.

The issue of concern in this article is whether the local solutions established in [5] can be extended to global solutions.

It follows from the standard extension argument that to show the initial-boundary value problem is globally well-posed it suffices to establish global *a priori* estimate for solutions of the initial-boundary value problem: *If $u \in C([0, T]; H^s(0, 1))$ solves the initial-boundary value problem, then for certain $s \in \mathbb{R}$*

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^s(0,1)} \leq \beta_s(\|(\phi, \vec{h})\|_{X_T^s}),$$

where $\beta_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-increasing continuous function depending only on s .

From local well-posedness and *a priori* estimates, we immediately obtain global well-posedness:

Theorem 1.1. (Global well-posedness) *Let $T > 0$, $s \in [0, 5]$ (with $s \neq \frac{2j-1}{2}$, $j = 1, 2, 3, 4, 5$). For any $(\phi, \vec{h}) \in X_T^s$, the initial-boundary value problem (1.1) admits a unique solution $u \in Y_T^s$. Moreover, the solution depends Lipschitz continuously on initial-boundary vaules $(\phi, \vec{h})$.*

The following sections are arranged as follows:

- Section 2 is the core of this article, primarily devoted to establishing *a priori* estimates. We combine the smoothing effect, trace regularity, and nonlinear estimates, and employ the Gronwall's theorem to establish the *a priori* estimates.
- Section 3 is dedicated to establishing global well-posedness. By combining local well-posedness with global *a priori* estimates, we extend the local solution to a global solution.
- In Section 4, we provide an overview of unresolved issues pertaining to the global well-posedness of the fifth-order KdV equation.

2 Global *a priori* estimate

We recall some existing estimates including smoothing effect and nonlinear estimate which will be used in the proof of global *a priori* estimate.

Lemma 2.1. (Smoothing effect) *Let $T > 0$ be given. For any compatible $\phi \in H^s(0, 1)$, $\vec{h} \in \mathcal{H}^s(\mathbb{R}^+)$, $f \in L^1(0, T; H^s(0, 1))$, the initial-boundary value problem*

$$\begin{cases} \partial_t u - \partial_x^5 u = f(x, t), & 0 < x < 1, t > 0, \\ u(x, 0) = \phi(x), & 0 < x < 1, \\ u(0, t) = h_1(t), u(1, t) = h_2(t), \partial_x u(1, t) = h_3(t), \partial_x u(0, t) = h_4(t), \partial_x^2 u(1, t) = h_5(t), & t > 0, \end{cases}$$

admits a solution $u \in C([0, T]; H^s(0, 1)) \cap L^2(0, T; H^{2+s}(0, 1))$ satisfying

$$\|u\|_{Y_T^s} + \sum_{k=0}^4 \sup_{x \in \mathbb{R}} \|\partial_x^k u\|_{H^{\frac{s+2-k}{5}}(0, T)} \leq C(\|(\phi, \vec{h})\|_{X_T^s} + \|f\|_{L^1(0, T; H^s(0, 1))}),$$

which implies

$$\|u\|_{Y_T^s} \leq C(\|(\phi, \vec{h})\|_{X_T^s} + \|f\|_{L^1(0, T; H^s(0, 1))})$$

and

$$\sum_{k=0}^4 \sup_{x \in \mathbb{R}} \|\partial_x^k u\|_{H^{\frac{s+2-k}{5}}(0, T)} \leq C(\|(\phi, \vec{h})\|_{X_T^s} + \|f\|_{L^1(0, T; H^s(0, 1))}).$$

Proof. See [11]. □

Lemma 2.2. (Nonlinear estimate) *For $s \geq 0$, there is a $C > 0$ such that for any $T > 0$ and $u, v \in Y_T^s$,*

$$\int_0^T \|u \partial_x v\|_{H^s(0,1)} d\tau \leq C \left(T^{\frac{1}{2}} + T^{\frac{1}{4}} \right) \|u\|_{Y_T^s} \|v\|_{Y_T^s}.$$

Proof. See [14]. □

Now we state the global *a priori* estimate precisely:

Proposition 2.3. (Global *a priori* estimate) Let $T > 0$, $s \in [0, 5]$ (with $s \neq \frac{2j-1}{2}$, $j = 1, 2, 3, 4, 5$) be given. Let $u \in C([0, T]; H^s(0, 1))$ solves the initial-boundary value problem

$$\begin{cases} \partial_t u + \partial_x^5 u - u \partial_x u = 0, & 0 < x < 1, \quad t > 0, \\ u(x, 0) = \varphi(x), & 0 < x < 1, \\ u(0, t) = h_1(t), u(1, t) = h_2(t), \partial_x u(1, t) = h_3(t), \partial_x u(0, t) = h_4(t), \partial_x^2 u(1, t) = h_5(t), & t > 0, \end{cases} \quad (2.1)$$

then

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^s(0,1)} \leq \beta_s(\|(\varphi, \vec{h})\|_{X_T^s}),$$

where $\beta_s : R^+ \rightarrow R^+$ is a non-increasing continuous function.

Proof. The proof is divided into three steps:

Step 1. $s = 0$. The initial-boundary value problem (2.1) can be decomposed into

$$\begin{cases} \partial_t v - \partial_x^5 v = 0, & 0 < x < 1, \quad t > 0, \\ v(x, 0) = 0, & 0 < x < 1, \\ v(0, t) = h_1(t), v(1, t) = h_2(t), \partial_x v(1, t) = h_3(t), \partial_x v(0, t) = h_4(t), \partial_x^2 v(1, t) = h_5(t), & t > 0 \end{cases} \quad (2.2)$$

and

$$\begin{cases} \partial_t w - \partial_x^5 w = w \partial_x w + \partial_x(wv) + v \partial_x v, & 0 < x < 1, \quad t > 0, \\ w(x, 0) = \phi(x), & 0 < x < 1, \\ w(0, t) = 0, w(1, t) = 0, \partial_x w(1, t) = 0, \partial_x w(0, t) = 0, \partial_x^2 w(1, t) = 0, & t > 0. \end{cases} \quad (2.3)$$

According to Lemma 2.1, for (2.2), we have the following estimate:

$$\|v\|_{Y_T^0} + \sum_{k=0}^4 \sup_{x \in \mathbb{R}} \|\partial_x^k v\|_{H^{\frac{2-k}{5}}(0,T)} \leq \|(\varphi, \vec{h})\|_{X_T^0},$$

which implies that

$$\|v\|_{L^2(0,T; H^2(0,1))} \leq \|(\varphi, \vec{h})\|_{X_T^0}. \quad (2.4)$$

Multiplying both sides of the equation in (2.3) by w and then integrating with respect to x over the interval $(0, 1)$, and after performing integration by parts, we obtain:

$$\frac{d}{dt} \int_0^1 w^2 dx \leq C \int_0^1 |\partial_x v| w^2 dx + C \int_0^1 |v \partial_x v w| dx.$$

Observe that

$$\int_0^1 |\partial_x v| w^2 dx \leq \sup_{x \in (0,1)} |\partial_x v| \|w\|_{L^2(0,1)}^2 \leq C_\varepsilon \|v\|_{H^{\frac{3+\varepsilon}{2}}(0,1)}^2 \|w\|_{L^2(0,1)}^2$$

and

$$\int_0^1 |v \partial_x v w| dx \leq \sup_{x \in (0,1)} |\partial_x v| \|v\|_{L^2(0,1)} \|w\|_{L^2(0,1)} \leq C_\varepsilon \|v\|_{H^{\frac{3+\varepsilon}{2}}(0,1)}^2 \|w\|_{L^2(0,1)},$$

where ε is any fixed positive constant, we deduce that

$$\frac{d}{dt} \|w\|_{L^2(0,1)}^2 = \frac{d}{dt} \int_0^1 w^2 dx \leq C_\varepsilon \|v\|_{H^{\frac{3+\varepsilon}{2}}(0,1)} \|w\|_{L^2(0,1)}^2 + C_\varepsilon \|v\|_{H^{\frac{3+\varepsilon}{2}}(0,1)}^2 \|w\|_{L^2(0,1)}$$

for any $t \geq 0$. Thus, we have

$$\frac{d}{dt} \|w\|_{L^2(0,1)} = \frac{d}{dt} \int_0^1 w^2 dx \leq C_\varepsilon \|v\|_{H^{\frac{3+\varepsilon}{2}}(0,1)} \|w\|_{L^2(0,1)} + C_\varepsilon \|v\|_{H^{\frac{3+\varepsilon}{2}}(0,1)}^2.$$

Applying the Gronwall inequality, Hölder's inequality, and combining with (2.4), we obtain:

$$\begin{aligned} \|w(\cdot, t)\|_{L^2(0,1)} &= e^{C_\varepsilon \int_0^t \|v(\cdot, t)\|_{H^{\frac{3+\varepsilon}{2}}(0,1)} dt} \left(\|w(\cdot, 0)\|_{L^2(0,1)} + C_\varepsilon \int_0^t \|v(\cdot, t)\|_{H^{\frac{3+\varepsilon}{2}}(0,1)}^2 dt \right) \\ &\leq e^{C_\varepsilon T \|v\|_{L^2(0,T; H^{\frac{3+\varepsilon}{2}}(0,1))}} \left(\|\phi\|_{L^2(0,1)} + C_\varepsilon \|v\|_{L^2(0,T; H^{\frac{3+\varepsilon}{2}}(0,1))}^2 \right) \\ &\leq e^{C_\varepsilon T \|v\|_{L^2(0,T; H^2(0,1))}} (\|\phi\|_{L^2(0,1)} + C_\varepsilon \|v\|_{L^2(0,T; H^2(0,1))}^2) \\ &\leq C \|(\phi, \vec{h})\|_{X_T^0}. \end{aligned}$$

Step 2. $s = 5$. For a smooth solution u of (2.2), $v = \partial_t u$ solves

$$\begin{cases} \partial_t v - \partial_x^5 v = \partial_x(uv), & 0 < x < 1, t > 0, \\ v(x, 0) = \phi^*(x), & 0 < x < 1, \\ v(1, t) = h_1'(t), v(0, t) = h_2'(t), \partial_x v(1, t) = h_3'(t), \partial_x v(0, t) = h_4'(t), \partial_x^2 v(1, t) = h_5'(t), & t > 0, \end{cases} \quad (2.5)$$

where

$$\phi^*(x) = \phi' \phi + \phi^{(5)}.$$

By Lemmas 2.1 and 2.2, there exists a constant $C > 0$ such that for any $T' \leq T$,

$$\|v\|_{Y_{T'}^0} \leq \|(\phi^*, \vec{h}')\|_{X_T^0} + C(T'^{\frac{1}{2}} + T'^{\frac{1}{4}}) \|u\|_{Y_T^0} \|v\|_{Y_T^0}.$$

Choose $T' < T$ such that $C(T'^{\frac{1}{2}} + T'^{\frac{1}{4}}) \|u\|_{Y_T^0} < \frac{1}{2}$, with such a choice,

$$\|v\|_{Y_{T'}^0} \leq 2 \|(\phi^*, \vec{h}')\|_{X_T^0}.$$

Note that T' only depends on $\|u\|_{Y_T^0}$, and therefore depends only on $\|(\phi, \vec{h})\|_{X_T^0}$, by the estimate proved in step 1. By a standard extension argument, we have

$$\|v\|_{Y_{T'}^0} \leq C_1 \|(\phi, \vec{h})\|_{X_T^0},$$

where C_1 depends only on T and $\|(\phi, \vec{h})\|_{X_T^0}$. The estimate (2.1) with $s = 5$ then follows from

$$v = \partial_t u = \partial_x^5 u + u \partial_x u.$$

Step 3. $0 < s < 5$. Through nonlinear interpolation, we demonstrate the validity of equation (2.1) for $0 < s < 5$. We omit the proof in this context, for a comprehensive understanding, we direct readers to the referenced source [14,15]. $\square$

3 Proof of the main theorem

Proposition 3.1. (Local well-posedness) Let $T > 0$, $s \in [0, 5]$ (with $s \neq \frac{2j-1}{2}$, $j = 1, 2, 3, 4, 5$) be given. For any $r > 0$, there exists a $T^* \in (0, T]$ depending only on r such that for any $(\varphi, \vec{h}) \in X_T^s$ with

$$\|(\varphi, \vec{h})\|_{X_T^s} \leq r,$$

the initial-boundary value problem

$$\begin{cases} \partial_t u + \partial_x^5 u - u \partial_x u = 0, & 0 < x < 1, \quad t > 0, \\ u(x, 0) = \varphi(x), & 0 < x < 1, \\ u(0, t) = h_1(t), u(1, t) = h_2(t), \partial_x u(1, t) = h_3(t), \partial_x u(0, t) = h_4(t), \partial_x^2 u(1, t) = h_5(t), & t > 0 \end{cases} \quad (3.1)$$

admits a unique solution $u \in Y_{T^*}^s$. Moreover, the solution depends Lipschitz continuously on $(\varphi, \vec{h})$ in the corresponding spaces.

Finally, we give the **proof of Theorem 1.1**.

Proof. By Proposition 3.1, for any s -compatible $(\varphi, \vec{h}) \in X_T^s$, there exists $T_1 \in (0, T]$ such that the initial-boundary value problem

$$\begin{cases} \partial_t u + \partial_x^5 u - u \partial_x u = 0, & 0 < x < 1, \quad t > 0, \\ u(x, 0) = \varphi(x), & 0 < x < 1, \\ u(0, t) = h_1(t), u(1, t) = h_2(t), \partial_x u(1, t) = h_3(t), \partial_x u(0, t) = h_4(t), \partial_x^2 u(1, t) = h_5(t) & t > 0, \end{cases}$$

admits a unique solution $u \in Y_{T_1}^s$.

By Proposition 2.3,

$$\sup_{0 \leq t \leq T_1} \|u\|_{H^s(0,1)} \leq \beta_s(\|(\varphi, \vec{h})\|_{X_T^s}),$$

which implies that

$$\|u\|_{C([0, T_1], H^s(0,1))} \leq \beta_s(\|(\varphi, \vec{h})\|_{X_T^s}).$$

Taking $(u(T_1, x), \vec{h})$ as data, applying Proposition 3.1 to determine $T_2 \in (T_1, T]$ and the solution $u(t, x) \in Y_{T_2}^s$. This procedure can be repeated until $T_n = T$, where we arrive the global solution. $\square$

4 Concluding comments

In 2019, we show in [11] a general local well-posedness of initial-boundary value problem of fifth-order KdV equation

$$\begin{cases} \partial_t u - \partial_x^5 u = u \partial_x u, & 0 < x < 1, \quad t > 0, \\ u(x, 0) = \varphi(x), & 0 < x < 1, \\ \text{boundary value,} & t > 0, \end{cases} \quad (4.1)$$

with the following 16 possible admissible boundary values:

- a** $u(0, t) = h_1(t), u(1, t) = h_2(t), \partial_x u(0, t) = h_3(t), \partial_x u(1, t) = h_3(t), \partial_x^2 u(1, t) = h_5(t);$
- b** $\partial_x u(0, t) = h_1(t), \partial_x u(1, t) = h_2(t), \partial_x^2 u(1, t) = h_3(t), \partial_x^4 u(0, t) = h_4(t), \partial_x^4 u(1, t) = h_5(t);$
- c** $u(0, t) = h_1(t), u(1, t) = h_2(t), \partial_x^2 u(1, t) = h_3(t), \partial_x^3 u(0, t) = h_4(t), \partial_x^3 u(1, t) = h_5(t);$
- d** $\partial_x^2 u(1, t) = h_1(t), \partial_x^3 u(0, t) = h_2(t), \partial_x^3 u(1, t) = h_3(t), \partial_x^4 u(0, t) = h_4(t), \partial_x^4 u(1, t) = h_5(t);$
- e** $u(0, t) = h_1(t), \partial_x u(0, t) = h_2(t), \partial_x u(1, t) = h_3(t), \partial_x^2 u(1, t) = h_4(t), \partial_x^4 u(1, t) = h_5(t);$
- f** $u(1, t) = h_1(t), \partial_x u(0, t) = h_2(t), \partial_x u(1, t) = h_3(t), \partial_x^2 u(1, t) = h_4(t), \partial_x^4 u(0, t) = h_5(t);$
- g** $u(0, t) = h_1(t), \partial_x^2 u(1, t) = h_2(t), \partial_x^3 u(0, t) = h_3(t), \partial_x^3 u(1, t) = h_4(t), \partial_x^4 u(1, t) = h_5(t);$
- h** $u(1, t) = h_1(t), \partial_x^2 u(1, t) = h_2(t), \partial_x^3 u(0, t) = h_3(t), \partial_x^3 u(1, t) = h_4(t), \partial_x^4 u(0, t) = h_5(t);$
- i** $u(0, t) = h_1(t), u(1, t) = h_2(t), \partial_x u(0, t) = h_3(t), \partial_x^2 u(1, t) = h_4(t), \partial_x^3 u(1, t) = h_5(t);$
- j** $u(0, t) = h_1(t), u(0, t) = h_2(t), \partial_x u(1, t) = h_3(t), \partial_x^2 u(1, t) = h_4(t), \partial_x^3 u(0, t) = h_5(t);$
- k** $\partial_x u(0, t) = h_1(t), \partial_x^2 u(1, t) = h_2(t), \partial_x^3 u(1, t) = h_3(t), \partial_x^4 u(0, t) = h_4(t), \partial_x^4 u(1, t) = h_5(t);$
- l** $\partial_x u(1, t) = h_1(t), \partial_x^2 u(1, t) = h_2(t), \partial_x^3 u(0, t) = h_3(t), \partial_x^4 u(0, t) = h_4(t), \partial_x^4 u(1, t) = h_5(t);$
- m** $u(0, t) = h_1(t), \partial_x u(0, t) = h_2(t), \partial_x^2 u(1, t) = h_3(t), \partial_x^3 u(1, t) = h_4(t), \partial_x^4 u(1, t) = h_5(t);$
- n** $u(0, t) = h_1(t), \partial_x u(1, t) = h_2(t), \partial_x^2 u(1, t) = h_3(t), \partial_x^3 u(0, t) = h_4(t), \partial_x^4 u(1, t) = h_5(t);$
- o** $u(1, t) = h_1(t), \partial_x u(0, t) = h_2(t), \partial_x^2 u(1, t) = h_3(t), \partial_x^3 u(1, t) = h_4(t), \partial_x^4 u(0, t) = h_5(t);$
- p** $u(1, t) = h_1(t), \partial_x u(1, t) = h_2(t), \partial_x^2 u(1, t) = h_3(t), \partial_x^3 u(0, t) = h_4(t), \partial_x^4 u(0, t) = h_5(t).$

However, not all of the mentioned cases are suitable for the method proposed in this article. What we are interested in is which other cases can employ the method presented in this article to extend their local solutions into global solutions?

The boundary conditions (a), (c), (i), and (j) share a common characteristic: when the boundary conditions are homogeneous, the corresponding solutions exhibit global L^2 -boundedness. In fact,

$$\begin{aligned} \frac{d}{dt} \int_0^1 u^2 dx &= 2 \int_0^1 u(\partial_x^5 u + u \partial_x u) dx \\ &= 2u \partial_x^4 u|_0^1 - 2\partial_x u \partial_x^3 u|_0^1 + (\partial_x^2 u)^2|_0^1 + \frac{2}{3} u^3 \Big|_0^1 \leq 0, \end{aligned}$$

when boundary value takes (a) or (c) or (i) or (j). Thus, we have

$$\|u(x, t)\|_{L^2(0,1)} = \int_0^1 u^2 dx \leq \int_0^1 \varphi^2 dx = \|\varphi(x)\|_{L^2(0,1)}. \quad (4.2)$$

It is not difficult to see that the method presented in this article is also applicable to (a), (c), (i), and (j). Indeed, making minor adjustments to the proofs presented in this article is all that is needed to achieve the goal.

Finally, are the remaining cases globally well-posed? How can this be proven? The answer lies in adding nonlinear feedback. We will discuss this issue elsewhere.

Acknowledgement: The first author is sponsored by Qing-Lan project. The authors are deeply grateful for the suggestions made by the anonymous reviewer.

Conflict of interest: The authors state no conflict of interest.

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