**Research Article**

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# Schur-power convexity of integral mean for convex functions on the coordinates

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**Abstract:** In this article, we investigate the concepts of monotonicity, Schur-geometric convexity, Schur-harmonic convexity, and Schur-power convexity for the lower and upper limits of the integral mean, focusing on convex functions on coordinate axes. Furthermore, we introduce novel and fascinating inequalities for binary means as a practical application.

**Keywords:** Schur-geometric convexity, Schur-harmonic convexity, monotonicity, Schur-power convexity, convex functions on the coordinates, inequality, binary means

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## 1 Introduction

Let $\mathbb{R}$ be a set of real numbers, $g$ be a convex function defined on the interval $I \subseteq \mathbb{R} \to \mathbb{R}$, and $c, d \in I, c < d$. Then the following double inequality

$$g\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_{c}^{d} g(t)\mathrm{d}t \leq \frac{g(c)+g(d)}{2} \tag{1}$$

is known in the literature as Hermite-Hadamard inequality for convex functions.

In 2000, Elezović and Pecccarić conducted research on the Schur-convexity of the integral arithmetic mean of a convex function, based on the Hermite-Hadamard inequality. They derived the following significant and profound theorem.

**Theorem A.** [1] *Let $I$ be an interval with non-empty interior on $\mathbb{R}$ and $g(u)$ be a continuous function on $I$. Then*

$$\Phi(u,v) = \begin{cases} \dfrac{1}{v-u}\displaystyle\int_{u}^{v} g(s)\mathrm{d}s, & u, v \in I, \quad u \neq v \\ g(u), & u, v \in I, \quad u = v \end{cases}$$

*is Schur-convex (or Schur-concave, respectively) on $I \times I$ if and only if $g(u)$ is convex (or concave, respectively) on $I$.*

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In the recent past, this result has attracted and improved by several researchers and by now there exists a considerable literature on this theorem, see the articles [2–12] and Chapter II of the book [13] and references therein.

In [14], the authors generalize the result of Theorem 1 to the case of bivariate convex functions, leading to the derivation of the following Theorem 1.

**Theorem B.** [14] *Let $I$ be an interval with non-empty interior on $\mathbb{R}$, and $g(u, v)$ be a continuous function on $I \times I$. If $g(u, v)$ is convex (or concave, respectively) on $I \times I$, then*

$$G(u, v) = \begin{cases} \dfrac{1}{(v-u)^2} \displaystyle\int_u^v \int_u^v g(s, t)\mathrm{d}s\mathrm{d}t, & (u, v) \in I \times I, \quad u \neq v \\ g(u, u), & (u, v) \in I \times I, \quad u = v \end{cases} \tag{2}$$

*is Schur-convex (or Schur-concave, respectively) on $I \times I$.*

Long et al. [3] and Sun et al. [4] proved the following theorem in different ways.

**Theorem C.** [3,4] *Suppose $g(u)$ is a continuous convex (or concave, respectively) function that is increasing (or decreasing, respectively) on $I \subseteq (0, +\infty)$. The function $\Phi(u, v)$ mentioned in Theorem A, then is both a Schur-geometrically convex (or Schur-geometrically concave, respectively) function and a Schur-harmonically convex (or Schur-harmonically concave, respectively) function on $I \times I$.*

In [15], an inequality of Hadamard's type is provided for convex functions on the coordinates defined in a plane rectangle. Subsequently, [16] extends the corresponding results. In this article, we explore the monotonicity, Schur-geometric convexity, Schur-harmonic convexity, and Schur-power convexity of $G(u, v)$ in Theorem B, considering $g(s, t)$ to be an increasing convex function on the coordinates. Our main result is stated as follows.

**Theorem 1.** *Let $I$ be an interval with non-empty interior on $\mathbb{R}$, and $g(s, t)$ be a continuous function on $I \times I$. If $g(s, t)$ is increasing convex on the coordinates on $I \times I$, then*
*(a) $G(u, v)$ is decreasing with respect to $u$ on $I$; $G(u, v)$ is increasing with respect to $v$ on $I$.*
*(b) $G(u, v)$ is Schur-geometrically convex, Schur-harmonically convex, and Schur-power convex on $I \times I \subseteq (0, +\infty) \times (0, +\infty)$.*

In this article, in Section 2, the necessary definitions and lemmas are presented. The proof of the main result is provided in Sections 3 and 4, two intriguing applications are derived based on our main result. In Section 5, we briefly discuss possible future works.

## 2 Definitions and lemmas

To prove Theorem 1, we give the following lemmas and definitions.

**Definition 1.** Let $(x_1, x_2)$ and $(y_1, y_2) \in \mathbb{R} \times \mathbb{R}$.
(1) $(x_1, x_2) \leq (y_1, y_2)$ means $x_1 \leq y_1, x_2 \leq y_2$.
(2) Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}$, $\psi: \Omega \to \mathbb{R}$ said to be increasing if $(x_1, x_2) \leq (y_1, y_2)$ implies $\psi(x_1, x_2) \leq \psi(y_1, y_2)$. $\psi$ is said to be decreasing if and only if $-\psi$ is increasing.
(3) A set $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ is said to be convex if $(x_1, x_2), (y_1, y_2) \in \Omega$, and $0 \leq \beta \leq 1$ implies

$$(\beta x_1 + (1-\beta)y_1, \beta x_2 + (1-\beta)y_2) \in \Omega.$$

(4) Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ be a convex set. A function $\psi\colon \Omega \to \mathbb{R}$ is said to be a convex function on $\Omega$ if for all $\beta \in [0,1]$ and all $(x_1, x_2), (y_1, y_2) \in \Omega$, inequality

$$\psi(\beta x_1 + (1-\beta)y_1, \beta x_2 + (1-\beta)y_2) \le \beta\psi(x_1, x_2) + (1-\beta)\psi(y_1, y_2) \tag{3}$$

holds. If for all $\beta \in [0,1]$ and all $(x_1, x_2), (y_1, y_2) \in \Omega$, the strict inequality in (3) holds, then $\psi$ is said to be strictly convex. $\psi$ is called concave (or strictly concave, respectively) if and only if $-\psi$ is convex (or strictly convex, respectively).

**Lemma 1.** [17, p. 39, Proposition 4.5], [18, p. 644, B.3.d] *Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ be an open convex set and let $\psi(x,y) : \Omega \to \mathbb{R}$ be twice differentiable. Then $\psi$ is convex on $\Omega$ if and only if the Hessian matrix*

$$H(x,y) = \begin{pmatrix} \dfrac{\partial^2\psi}{\partial x^2} & \dfrac{\partial^2\psi}{\partial x\partial y} \\ \dfrac{\partial^2\psi}{\partial y\partial x} & \dfrac{\partial^2\psi}{\partial y^2} \end{pmatrix}$$

*is nonnegative definite on $\Omega$. If $H(\boldsymbol{x})$ is positive definite on $\Omega$, then $\psi$ is strictly convex on $\Omega$.*

**Definition 2.** [15] Let the bidimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R} \times \mathbb{R}$ with $a < b$ and $c < d$. A function $f\colon \Delta \to \mathbb{R}$ will be called convex on the coordinates, if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) := f(u,y)$, and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) := f(u,v)$, are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$.

The following lemma holds:

**Lemma 2.** [15] *Every convex mapping $f\colon \Delta \to \mathbb{R}$ is convex on the coordinates, but the converse is not generally true.*

For example obviously, for $x > 0, y > 0, m \ge 2$, the function $f(x,y) = x^m y^m$ is convex on the coordinates, but is not convex. In fact, the Hessian matrix of $f(x,y)$

$$H(x,y) = \begin{pmatrix} m(m-1)x^{m-2}y^m & m^2x^{m-1}y^{m-1} \\ m^2x^{m-1}y^{m-1} & m(m-1)x^my^{m-2} \end{pmatrix}$$

$\det H(x,y) = m^2(m-1)^2x^{2m-2}y^{2m-2} - m^4x^{2m-2}y^{2m-2} < 0$, but $m(m-1)x^{m-2}y^m > 0$, so $f(x,y)$ is not convex.

**Lemma 3.** [15] *Suppose that $g : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the coordinates on $\Delta$. Then one has the inequalities*

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b g(s,t)\mathrm{d}s\mathrm{d}t \le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b (g(s,c) + g(s,d))\mathrm{d}s + \frac{1}{d-c}\int_c^d (g(a,t) + g(b,t))\mathrm{d}t\right]. \tag{4}$$

**Definition 3.** [17,18] Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}$, $(x_1, x_2)$ and $(y_1, y_2) \in \Omega$, and let $\psi : \Omega \to \mathbb{R}$.

(1) $(x_1, x_2)$ is said to be majorized by $(y_1, y_2)$ (in symbols $(x_1, x_2) \prec (y_1, y_2)$) if $\max\{x_1, x_2\} \le \max\{y_1, y_2\}$ and $x_1 + x_2 = y_1 + y_2$.

(2) $\psi$ is said to be a Schur-convex function on $\Omega$ if $(x_1, x_2) \prec (y_1, y_2)$ on $\Omega$ implies $\psi(x_1, x_2) \le \psi(y_1, y_2)$, $\psi$ is said to be a Schur-concave function on $\Omega$ if and only if $-\psi$ is Schur-convex function.

**Definition 4.** [19]

(i) $\Omega \subseteq (0,+\infty) \times (0,+\infty)$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta) \in \Omega$ for any $(x_1, x_2)$ and $(y_1, y_2) \in \Omega$, where $\alpha$ and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(ii) Let $\Omega \subseteq (0, +\infty) \times (0, +\infty)$ be a geometrically convex set, $\psi : \Omega \to [0, +\infty)$ is said to be a Schur-geometrically convex function on $\Omega$ if $(\log x_1, \log x_2) \prec (\log y_1, \log y_2)$ implies $\psi(x_1, x_2) \le \psi(y_1, y_2)$ for any $(x_1, x_2)$ and $(y_1, y_2) \in \Omega$. $\psi$ is said to be a Schur-geometrically concave function on $\Omega$ if and only if $-\psi$ is Schur-geometrically convex function.

**Definition 5.** [20,21] Let $\Omega \subseteq (0, +\infty) \times (0, +\infty)$.

(i) A set $\Omega$ is said to be a harmonically convex set if

$$\left(\frac{x_1 y_1}{\lambda x_1 + (1-\lambda) y_1}, \frac{x_2 y_2}{\lambda x_2 + (1-\lambda) y_2}\right) \in \Omega$$

for any $(x_1, x_2), (y_1, y_2) \in \Omega$ and $\lambda \in [0, 1]$.

(ii) Let $\Omega$ be a harmonically convex set, a function $\psi : \Omega \to [0, +\infty)$ is said to be a Schur-harmonically convex function on $\Omega$ if $(\frac{1}{x_1}, \frac{1}{x_2}) \prec (\frac{1}{y_1}, \frac{1}{y_2})$ implies $\psi(x_1, x_2) \le \psi(y_1, y_2)$ for any $(x_1, x_2), (y_1, y_2) \in \Omega$. A function $\psi$ is said to be a Schur-harmonically concave function on $\Omega$ if and only if $-\psi$ is a Schur-harmonically convex function.

**Definition 6.** [22] Let $f : (0, +\infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \neq 0; \\ \ln x, & m = 0. \end{cases} \tag{5}$$

Then a function $\psi : \Omega \subseteq (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ is said to be Schur-$m$-power convex on $\Omega$ if

$$(f(x_1), f(x_2)) \prec (f(y_1), f(y_2))$$

implies $\psi(x_1, x_2) \le \psi(y_1, y_2)$ for all $(x_1, x_2) \in \Omega$ and $(y_1, y_2) \in \Omega$. If $-\psi$ is Schur-$m$-power convex, then we say that $\psi$ is Schur-$m$-power concave.

If making $f(x) = x, \log x, \frac{1}{x}$ in Definition 6, then definitions of the Schur-convex, Schur-geometrically convex, and Schur-harmonically convex functions can be deduced, respectively.

**Lemma 4.** [17, p. 5] *Let* $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$. *Then*

$$\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right) \prec (x_1, x_2).$$

**Lemma 5.** [17, p. 57] *Let* $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ *be a symmetric convex set with a non-empty interior* $\Omega^\circ$. $\psi : \Omega \to \mathbb{R}$ *is continuous on* $\Omega$ *and differentiable in* $\Omega^\circ$. *Then the function* $\psi$ *is the Schur-convex (or Schur-concave, respectively), if and only if* $\psi$ *is symmetric on* $\Omega$ *and*

$$(x_1 - x_2)\left(\frac{\partial \psi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}\right) \ge 0 \ (or \ \le 0, respectively)$$

*holds for any* $(x_1, x_2) \in \Omega^\circ$.

**Lemma 6.** [19] *Let* $\Omega \subseteq (0, +\infty) \times (0, +\infty)$ *be geometrically symmetric convex set, and has a non-empty interior set* $\Omega^\circ$. *Let* $\psi : \Omega \to [0, +\infty)$ *be continuous on* $\Omega$ *and differentiable in* $\Omega^\circ$. *Then* $\psi$ *is the Schur-geometrically convex (or Schur-geometrically concave, respectively) function, if and only if it is symmetric on* $\Omega$ *and*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \psi}{\partial x_1} - x_2 \frac{\partial \psi}{\partial x_2}\right) \ge 0 \ (or \ \le 0, \ respectively) \tag{6}$$

*holds for any* $(x_1, x_2) \in \Omega^\circ$.

**Lemma 7.** [20,21] *Let* $\Omega \subseteq (0, +\infty) \times (0, +\infty)$ *be a harmonically symmetric convex set with a non-empty interior* $\Omega^\circ$. *Let* $\psi : \Omega \to [0, +\infty)$ *be continuous on* $\Omega$ *and differentiable on* $\Omega$. *Then* $\psi$ *is a Schur-harmonically convex (or Schur-harmonically concave, respectively) function if and only if* $\psi$ *is symmetric on* $\Omega$ *and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial\psi}{\partial x_1} - x_2^2\frac{\partial\psi}{\partial x_2}\right) \geq 0 \quad (or\ \leq 0,\ respectively) \tag{7}$$

*holds for any* $(x_1, x_2) \in \Omega^\circ$.

**Lemma 8.** [22] *Let* $\Omega \subseteq (0, +\infty) \times (0, +\infty)$ *be a symmetric convex set with a non-empty interior* $\Omega^\circ$ *and* $\psi : \Omega \to [0, +\infty)$ *be continuous on* $\Omega$ *and differentiable in* $\Omega^\circ$. *Then* $\psi$ *is a Schur-m-power convex (or Schur-m-power concave, respectively) function if and only if* $\psi$ *is symmetric on* $\Omega$ *and*

$$(x_1 - x_2)\left(x_1^{1-m}\frac{\partial\psi}{\partial x_1} - x_2^{1-m}\frac{\partial\psi}{\partial x_2}\right) \geq 0 \quad (or\ \leq 0,\ respectively) \tag{8}$$

*holds for any* $(x_1, x_2) \in \Omega^\circ$.

The concept of Schur-$m$-power convex was initially introduced by Z.-H. Yang, who established the aforementioned decision theorem (Lemma 8). The inequality (8) within this theorem adopts a simplified form recommended by Ming Li. Li conveyed this form to Yang, who, in turn, shared it with the author (J. Zhang).

Obviously, Lemma 8 contains Lemmas 5–7.

**Lemma 9.** [14] *Let* $I$ *be an interval with non-empty interior on* $\mathbb{R}$ *and* $g(s,t)$ *be a continuous function on* $I \times I$. *For* $(u, v) \in I \times I$, $u \neq v$, *let* $G(u,v) = \int_u^v\int_u^v g(s,t)\mathrm{d}s\mathrm{d}t$. *Then*

$$\frac{\partial G}{\partial v} = \int_u^v g(s,v)\mathrm{d}s + \int_u^v g(v,t)\mathrm{d}t, \tag{9}$$

$$\frac{\partial G}{\partial u} = -\left(\int_u^v g(s,u)\mathrm{d}s + \int_u^v g(u,t)\mathrm{d}t\right). \tag{10}$$

# 3 Proofs of main results

**Proof of Theorem 1.** Since the function $g(s,t)$ is coordinate convex, so for $c = a = u, d = b = v$, from (4), we have

$$\begin{aligned} G(u,v) = \frac{\int_u^v\int_u^v g(s,t)\mathrm{d}s\mathrm{d}t}{(v-u)^2} &\leq \frac{1}{4}\left[\frac{1}{v-u}\int_u^v g(s,u)\mathrm{d}s + \frac{1}{v-u}\int_u^v g(s,v)\mathrm{d}s\right] \\ &+ \frac{1}{4}\left[\frac{1}{v-u}\int_u^v g(u,t)\mathrm{d}t + \frac{1}{v-u}\int_u^v g(v,t)\mathrm{d}t\right]. \end{aligned} \tag{11}$$

$G(u,v)$ is evidently symmetric, without loss of generality, we may assume that $u \leq v$. Since $g(s,t)$ is increasing, $g(s,u) \leq g(s,v)$ and $g(u,t) \leq g(v,t)$, from inequality (11), it follows that

$$G(u,v) = \frac{\int_u^v\int_u^v g(s,t)\mathrm{d}s\mathrm{d}t}{(v-u)^2} \leq \frac{1}{2}\left[\frac{1}{v-u}\left(\int_u^v g(s,v)\mathrm{d}s + \int_u^v g(v,t)\mathrm{d}t\right)\right]. \tag{12}$$

By Lemma 9, we have

$$\frac{\partial G(u,v)}{\partial v} = \frac{-2\int_u^v\int_u^v g(s,t)\mathrm{d}s\mathrm{d}t}{(v-u)^3} + \frac{1}{(v-u)^2}\left(\int_u^v g(s,v)\mathrm{d}s + \int_u^v g(v,t)\mathrm{d}t\right)$$
$$= \frac{1}{v-u}\left[\frac{1}{v-u}\left(\int_u^v g(s,v)\mathrm{d}s + \int_u^v g(v,t)\mathrm{d}t\right) - \frac{2\int_u^v\int_u^v g(s,t)\mathrm{d}s\mathrm{d}t}{(v-u)^2}\right].$$

From $u \le v$ and (12), it follows that $\frac{\partial G(u,v)}{\partial v} \ge 0$, this means that $G(u,v)$ is increasing with respect to $v$ on $I$.

$$\frac{\partial G(u,v)}{\partial u} = \frac{2\int_u^v\int_u^v g(s,t)\mathrm{d}s\mathrm{d}t}{(v-u)^3} - \frac{1}{(v-u)^2}\left(\int_u^v g(s,u)\mathrm{d}s + \int_u^v g(u,t)\mathrm{d}t\right)$$
$$= \frac{1}{v-u}\left[\frac{2\int_u^v\int_u^v g(s,t)\mathrm{d}s\mathrm{d}t}{(v-u)^2} - \frac{1}{v-u}\left(\int_u^v g(s,u)\mathrm{d}s + \int_u^v g(u,t)\mathrm{d}t\right)\right].$$

From $u \le v$ and (12), it follows that $\frac{\partial G(u,v)}{\partial u} \le 0$, this means that $G(u,v)$ is decreasing with respect to $u$ on $I$. Thus, $(a)$ is proved.

From $\frac{\partial G}{\partial u} \le 0$, $\frac{\partial G}{\partial v} \ge 0$ and $0 < u \le v$, it follows that

$$(\log u - \log v)\left(u\frac{\partial G}{\partial u} - v\frac{\partial G}{\partial v}\right) \ge 0,$$

$$(u-v)\left(u^2\frac{\partial G}{\partial u} - v^2\frac{\partial G}{\partial v}\right) \ge 0,$$

and

$$(u-v)\left(u^{1-m}\frac{\partial G}{\partial u} - v^{1-m}\frac{\partial G}{\partial v}\right) \ge 0.$$

Thus, $(b)$ is proved. □

The proof of Theorem 1 is complete.

## 4 Application on binary mean

**Theorem 2.** *Let $c > 0$, $d > 0$, and $m \neq 0$. Then*

$$H_e(c^2, d^2) \ge (M_m(c,d))^2, \tag{13}$$

*where $H_e(c,d) = \frac{c+\sqrt{cd}+d}{3}$ and $M_m(c,d) = \sqrt[m]{\frac{c^m+d^m}{2}}$ are the Heronian mean and power mean of positive numbers c and d, respectively.*

**Proof.** From [23], we know that the function of two variables

$$\psi(x,y) = \frac{x^2}{2r^2} + \frac{y^2}{2s^2}$$

is a convex function on $(0,+\infty)\times(0,+\infty)$, where $s>0$, $r>0$. By Lemma 2, convex function must be convex on the coordinates. The function $\psi(x,y)$ is obviously increasing, by Theorem 1, $G(c,d)$ is Schur-power convex function. For $c>0$, $d>0$, and $c\neq d$, from

$$\left(\frac{\frac{c^m-1}{m}+\frac{d^m-1}{m}}{2},\frac{\frac{c^m-1}{m}+\frac{d^m-1}{m}}{2}\right)\prec\left(\frac{c^m-1}{m},\frac{d^m-1}{m}\right),$$

i.e.,

$$\left(\frac{(M_m(c,d))^m-1}{m},\frac{(M_m(c,d))^m-1}{m}\right)\prec\left(\frac{c^m-1}{m},\frac{d^m-1}{m}\right),$$

it follows that

$$\begin{aligned}G(c,d)&=\frac{1}{(d-c)^2}\int_c^d\int_c^d\left(\frac{x^2}{2r^2}+\frac{y^2}{2s^2}\right)\mathrm{d}x\mathrm{d}y\\&=\frac{1}{(d-c)^2}\int_c^d\left(\frac{d^3-c^3}{6r^2}+\frac{y^2(d-c)}{2s^2}\right)\mathrm{d}y\\&=\frac{1}{(d-c)^2}\left(\frac{(d^3-c^3)(d-c)}{6r^2}+\frac{(d^3-c^3)(d-c)}{6s^2}\right)\\&=\frac{1}{(d-c)^2}\cdot\frac{(d^3-c^3)(d-c)}{6}\left(\frac{1}{r^2}+\frac{1}{s^2}\right)\\&\geq G(M_m(c,d),M_m(c,d))=\frac{(M_m(c,d))^2}{2}\left(\frac{1}{r^2}+\frac{1}{s^2}\right),\end{aligned}$$

namely

$$H_e(c^2,d^2)=\frac{c^2+cd+d^2}{3}=\frac{d^3-c^3}{3(d-c)}\geq(M_m(c,d))^2.\qquad\square$$

**Theorem 3.** *Let $c>0$, $d>0$, and $m\geq 2$. Then*

$$S_{m+1}(c,d)\geq M_m(c,d),\tag{14}$$

*where $S_{m+1}(c,d)=\left(\frac{d^{m+1}-c^{m+1}}{(m+1)(d-c)}\right)^{\frac{1}{m}}$ and $M_m(c,d)=\sqrt[m]{\frac{c^m+d^m}{2}}$ are $m+1$-order Stolarsky mean and $m$-order power mean of positive numbers $c$ and $d$, respectively.*

**Proof.** For $m\geq 2$, it is obvious that the function $\psi(x,y)=x^my^m$ is increasing and convex on the coordinates on $(0,+\infty)\times(0,+\infty)$, by Theorem 1, $G(c,d)$ is Schur-$m$-power convex function. For $c>0$, $d>0$, and $c\neq d$, from

$$\left(\frac{(M_m(c,d))^m-1}{m},\frac{(M_m(c,d))^m-1}{m}\right)\prec\left(\frac{c^m-1}{m},\frac{d^m-1}{m}\right),$$

it follows that

$$\begin{aligned}G(c,d)&=\frac{1}{(d-c)^2}\int_c^d\int_c^d x^my^m\,\mathrm{d}x\mathrm{d}y=\frac{1}{(d-c)^2}\int_c^d x^m\mathrm{d}x\int_c^d y^m\,\mathrm{d}y\\&=\left(\frac{d^{m+1}-c^{m+1}}{(m+1)(d-c)}\right)^2\geq G(M_m(c,d),M_m(c,d))=(M_m(c,d))^{2m},\end{aligned}$$

extracting the $2m$-th root on both sides, we obtain

$$S_{m+1}(c,d) = \left(\frac{d^{m+1} - c^{m+1}}{(m+1)(d-c)}\right)^{\frac{1}{m}} \geq M_m(c,d). \qquad \square$$

## 5 Conclusion

We explore the ideas of monotonicity, Schur-geometric convexity, Schur-harmonic convexity, and Schur-power convexity in relation to the lower and upper bounds of integral means. Our focus is on convex functions on coordinate axes.

Through the utilization of two coordinate convex functions and the application of Theorem 1, we have established two binary mean inequalities. We firmly believe that by seeking out other interesting coordinate convex functions and utilizing Theorem 1, we can generate a plethora of innovative binary mean inequalities. We encourage interested readers to explore and experiment with these possibilities.

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