Research Article

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The structure fault tolerance of burnt pancake networks

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Abstract: One of the symbolic parameters to measure the fault tolerance of a network is its connectivity. The H-structure connectivity and H-substructure connectivity extend the classical connectivity and are more practical. For a graph G and its connected subgraph H, the H-structure connectivity $\kappa(G;\ H)$ (resp. H-substructure connectivity $\kappa^s(G;\ H)$) of G is the cardinality of a minimum subgraph set such that every element of the set is isomorphic to H (resp. every element of the set is isomorphic to a connected subgraph of H) in G, whose vertices removal disconnects G. In this article, we investigate the H-structure connectivity and H-substructure connectivity of the n-dimensional burnt pancake network BP_n for each $H \in \{K_1, K_{1,1}, \ldots, K_{1,n-1}, P_4, \ldots, P_7, C_8\}$.

Keywords: interconnection networks, H-structure connectivity, H-substructure connectivity, burnt pancake networks

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1 Introduction

An interconnection network is often represented as a graph, in which a vertex corresponds to a processor, and an edge corresponds to a communication link. One fundamental consideration in the design of networks is fault tolerance, which can be measured by the connectivity of graphs. In general, if the connectivity is larger, then its fault tolerance is higher. For a connected graph G, its *connectivity* $\kappa(G)$ is defined as the minimum cardinality of a vertex subset whose removal makes the remaining graph disconnected. In recent years, the conditional connectivity [1] and the restricted connectivity [2,3] were introduced in succession for more accurate assessment of the fault tolerance of an interconnection network.

However, the connectivity parameters mentioned above are still disadvantageous because they just take into account the influence of a private vertex failure on the networks rather than the influence of a vertex or the vertices around it. In reality, one failing vertex is bound to have some adverse effects on the surrounding vertices. Furthermore, stimulated by the current situation that networks and subnetworks of large scale are increasingly made into chips, people think that it is becoming more and more feasible to consider the fault situation of a structure. Lin et al. [4] came up with the structure connectivity and substructure connectivity, whose proposition perfectly accommodates this disadvantageous realistic environment. Instead of focusing on the effects of a single vertex failure, they started to pay attention to the influence caused by some structure failures.

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Table 1: The H-(sub)structure connectivity of BP_n

H	K_1	$K_{1,r}$	P_4	P_5	P_6	P_7	C_8
$\kappa(BP_n; H)$	n	n	n	n	$n-1$	$\lceil \frac{n}{2} \rceil$	/
$\kappa^s(BP_n; H)$	n	n	n	n	$n-1$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$

Due to its advantages, there have been a number of results about the structure connectivity and substructure connectivity on some well-known networks, such as hypercube Q_n [4,5], folded hypercube FQ_n [6], balanced hypercube BH_n [7], k-ary n-cube network $Q_{n,k}$ [8,9], twisted hypercube H_n [10], crossed cube CQ_n [11], bubble-sort star graph BS_n [12], star graph S_n [13], (n,k)-star graph $S_{n,k}$ [14], alternating group graph AG_n [15,16], wheel network CW_n [17], circulant graph $\mathrm{Cir}(n, \Omega)$ [18], and divide-and-swap cube DSC_n [19].

This article determines the H-structure connectivity and H-substructure connectivity of the n-dimensional burnt pancake network BP_n, where $H \in \{K_1, K_{1,r}, P_4, \ldots, P_7, C_8\}$ and $1 \le r \le n-1$. For detailed results, see Table 1.

The remaining of the article is organized as follows: Section 2 presents the basic notations and definitions and introduces burnt pancake networks and their relevant structural properties; Section 3–5 dedicate the H-structure connectivity and H-substructure connectivity of BP_n such that H are $K_{1,r}$ $(1 \le r \le n-1)$, P_ℓ $(4 \le \ell \le 7)$, and C_8, respectively; Section 6 concludes the article.

2 Preliminaries

We simply describe some terminologies and notations of graph theory, give the definitions of the structure connectivity and substructure connectivity, and provide the topological structure and properties of BP_n in this section.

2.1 Terminologies and notations

For notation and terminology not mentioned here, the reader can refer to the study by Bondy and Murty [20]. Given two graphs G and H, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, denoted by $H \subseteq G$; if they have identical structure, then H is *isomorphic* to G, denoted by $H \cong G$; if H is isomorphic to a connected subgraph of G, denoted by $H \preccurlyeq G$. For $S \subseteq V(G)$, $G[S]$ is the *induced subgraph* by S; $N_G(S) = \cup_{v \in S} N_G(v) \backslash S$ and $N_G[S] = N_G(S) \cup S$, where $N_G(v)$ denotes the neighbors of v in G (the subscript G can be omitted without ambiguity); $G - S$ is the graph, which is obtained by removing all vertices of S and the edges incident with vertices of S in G. Note that $K_{1,r}$ is a *star* with r pendant vertices and P_ℓ is a *path* with ℓ vertices. For any two subgraphs G_1 and G_2 in G, $E(G_1, G_2)$ is the set of edges joining one vertex in G_1 to another vertex in G_2.

Here are the definitions of the H-structure connectivity and H-substructure connectivity of G, where G and H are two connected graphs and $V(\mathcal{H}) = \cup_{H \in \mathcal{H}} V(H)$.

Definition 1. [4] Let $H \subseteq G$ and $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a set of connected subgraph of G such that every $H_i \cong H$. If $G - V(\mathcal{H})$ is disconnected, then $\mathcal{H}$ is an *H-structure cut*. The *H-structure connectivity* of G is defined as follows:

$$\kappa(G; H) = \min\{|\mathcal{H}| : \mathcal{H} \text{ is an } H\text{-structure cut}\}.$$

Definition 2. [4] Let $H \subseteq G$ and $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a set of connected subgraph of G such that every $H_i \preccurlyeq H$. If $G - V(\mathcal{H})$ is disconnected, then $\mathcal{H}$ is an *H-substructure cut*. The *H-substructure connectivity* of G is defined as follows:

$$\kappa^s(G;\ H) = \min\{|\mathcal{H}| : \mathcal{H} \text{ is an } H\text{-substructure cut}\}.$$

As a matter of fact, the structure connectivity and substructure connectivity are the natural generalizations of the classical connectivity with $\kappa(G;\ K_1) = \kappa^s(G;\ K_1) = \kappa(G)$ in this sense. It is obvious to obtain the following results from the definitions above.

Observation 2.1. $\kappa^s(G;\ H) \leq \kappa(G;\ H)$ and $\kappa^s(G;\ H) \leq \kappa^s(G;\ H')$ for $H' \preccurlyeq H$.

2.2 Burnt pancake networks

Burnt pancake networks originate from the Burnt Pancake Problem discussed in [21–24]. Incidentally, one of the authors of [22] is Microsoft co-founder Bill Gates. In the Burnt Pancake Problem, one is tasked with sorting a stack of burnt pancakes in the proper order and orientation. Similarly to pancake networks, burnt pancake networks are also Cayley graphs [25,26].

Given a positive integer n, let $[n]$ and $[\pm n]$ be the set $\{1, 2, \ldots, n\}$ and $\{-n, -(n-1), \ldots, -1\} \cup [n]$, respectively. To simplify notation, it is usual to use $\bar{i}$ instead of $-i$, and we also replace $\bar{i}\bar{j}$ with $\overline{ij}$. A *signed permutation* on $[\pm n]$ is an n-permutation $x_1x_2\cdots x_n$ of $[\pm n]$ such that after taking the absolute value of each element x_i can make up a permutation of $[n]$, i.e., $|x_1||x_2|\cdots|x_n|$ forms a permutation of $[n]$. For example, all of the signed permutations on $[\pm 2]$ are $\{12, 1\bar{2}, \bar{1}2, \overline{12}, 21, 2\bar{1}, \bar{2}1, \overline{21}\}$.

Definition 3. [27] Let BP_n be an *n-dimensional burnt pancake network*, whose vertex set and edge set are defined as follows:

- $V(\mathrm{BP}_n) = \{$all the signed permutations of $[\pm n]\}$.
- $E(\mathrm{BP}_n) = \{(u, u^i) : u = x_1x_2\cdots x_i\cdots x_n,\ u^i = \overline{x_ix_{i-1}\cdots x_2x_1}x_{i+1}\cdots x_n$ and $i \in [n]\}$.

Moreover, u^i is the unique i-neighbor of u for $i \in [n]$, u^i is an *in-neighbor* of u if $i \in [n-1]$ and u^i is an *out-neighbor* of u if $i = n$. We label the edge (u, u^i) (for short uu^i) by i, and the edge uu^i is called an *i-edge*.

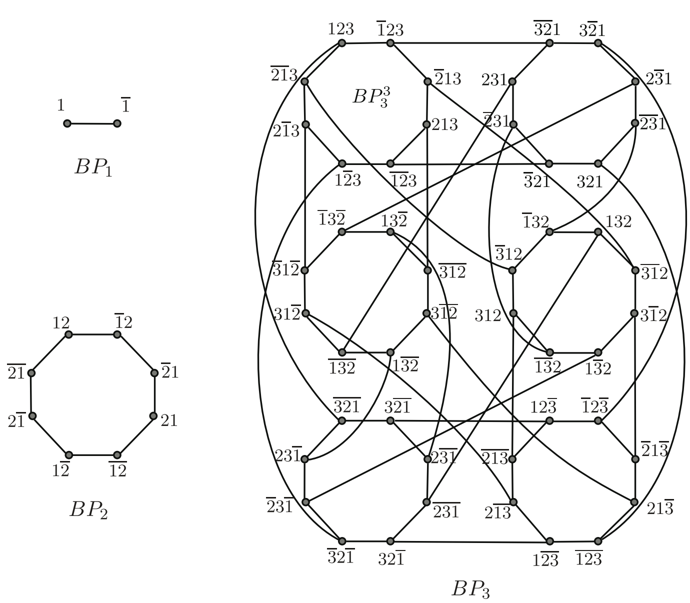

Figure 1: Burnt pancake networks BP_1, BP_2, and BP_3.

The n-dimensional burnt pancake networks for $n = 1, 2$, and 3 are shown in Figure 1. It is obvious that $|V(\mathrm{BP}_n)| = 2^n \times n!$ and $|E(\mathrm{BP}_n)| = n \times 2^{n-1} \times n!$. We denote a cycle or path by the edges it traverses. For example, an (a, b, c, d, e, f, g, h)-cycle traverses an a-edge, followed by a b-edge, and successively until the last edge traversed is an h-edge. In fact, BP_2 is a $(1, 2, 1, 2, 1, 2, 1, 2)$-cycle.

Lemma 2.2. *Let BP_n be an n-dimensional burnt pancake network, $i, j \in [\pm n]$, and $i \neq j$. Then, the following results hold*:

(1) [27, Theorem 3] BP_n *is n-regular and $\kappa(\mathrm{BP}_n) = \lambda(\mathrm{BP}_n) = n$.*

(2) [28, Lemma 2] BP_n *can be decomposed into $2n$ vertex-disjoint subgraphs BP_n^i, which are induced by all of those signed permutations whose last position is i, where $i \in [\pm n]$. Clearly, $\mathrm{BP}_n^i \cong \mathrm{BP}_{n-1}$, and every edge (if exists) between BP_n^i and BP_n^j is labeled by n.*

(3) [27, Lemma 1] *For any two distinct subgraphs BP_n^i and BP_n^j, the number of edges between them are*

$$|E(\mathrm{BP}_n^i, \mathrm{BP}_n^j)| = \begin{cases} (n-2)! \times 2^{n-2}, & \text{if } i \neq \bar{j}, \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 2.3. *Let $n \geq 2$ be an integer, $i, j \in [\pm n]$, and $i \neq j$.*

(1) [29, Theorem 10] *The girth* (*the length of a shortest cycle*) *of* BP_n *equals to 8 for* $n \geq 2$.

(2) [25, Theorem 4.1] *An eight-cycle in* BP_n *is one of the following forms*:
- $(k, j, i, j, k, k-j+i, i, k-j+i)$-*cycle for* $1 \leq i < j \leq k-1$ *and* $3 \leq k \leq n$;
- (k, j, k, i, k, j, k, i)-*cycle for* $2 \leq i, j \leq k-2$, $i + j \leq k$, *and* $4 \leq k \leq n$;
- $(k, i, k, 1, k, i, k, 1)$-*cycle for* $2 \leq i \leq k-1$ *and* $3 \leq k \leq n$;
- $(k, 1, k, 1, k, 1, k, 1)$-*cycle for* $2 \leq k \leq n$.

Proposition 2.4. *Let $i, j \in [\pm n]$, $i \neq j$, and $u \in V(\mathrm{BP}_n^i)$.*

(1) *Let $U = N_{\mathrm{BP}_n^i}[u]$. Then, the induced subgraph by U in BP_n is isomorphic to $K_{1,n-1}$. Furthermore, if the out-neighbors of vertices in U are denoted by U^n, then they lie in different copies.*

(2) *Each subgraph that is isomorphic to one of P_4, P_5, P_6, and C_8 in BP_n^i is incident with at most two edges between BP_n^i and BP_n^j in BP_n for all $j \neq i$.*

Proof. (1) Let $u = x_1x_2\cdots x_n$ be a vertex of BP_n^i and $U = N_{\mathrm{BP}_n^i}[u]$. Then, the induced subgraph by U in BP_n is isomorphic to $K_{1,n-1}$ because of

$$\begin{aligned} U &= \{u, u^1, u^2, \ldots, u^{n-1}\} \\ &= \{x_1x_2\cdots x_n,\ \overline{x_1}x_2\cdots x_n,\ \overline{x_2x_1}\cdots x_n,\ \ldots,\ \overline{x_{n-1}\cdots x_2x_1}x_n\}. \\ U^n &= \{\overline{x_nx_{n-1}\cdots x_1},\ \overline{x_nx_{n-1}\cdots x_2}x_1,\ \overline{x_nx_{n-1}\cdots x_3}x_1x_2,\ \ldots,\ \overline{x_n}x_1x_2\cdots x_{n-1}\}. \end{aligned} \tag{1}$$

These vertices of U^n belong to $\mathrm{BP}_n^{\overline{x_1}}, \mathrm{BP}_n^{x_1}, \mathrm{BP}_n^{x_2}, \ldots, \mathrm{BP}_n^{x_{n-1}}$, respectively. This implies that a subgraph isomorphic to $K_{1,n-1}$ in BP_n^i is incident with at most one edge between BP_n^i and BP_n^j in BP_n.

(2) Let F be a subgraph in BP_n^i. If F is isomorphic to one of P_4, P_5, and P_6, there are two subgraphs F_1 and F_2 such that $F_1 \preccurlyeq K_{1,2}$, $F_2 \preccurlyeq K_{1,2}$, and $V(F_1) \cup V(F_2) = V(F)$. By equation (1), every F_i for $i \in \{1, 2\}$ is incident with at most one edge between BP_n^i and BP_n^j. Therefore, F is incident with at most two edges between BP_n^i and BP_n^j. Suppose that $F \cong C_8$. If F is incident with three edges between BP_n^i and BP_n^j, then there exists a subgraph isomorphic to $K_{1,2}$ in F, and it is incident with two edges between BP_n^i and BP_n^j, a contradiction. □

Lemma 2.5. [26, Theorems 3.5, 3.7 and 3.9] $\kappa_0(\mathrm{BP}_n) = n$, $\kappa_1(\mathrm{BP}_n) = 2n-2$, *and* $\kappa_2(\mathrm{BP}_n) = 3n-4$ *for* $n \geq 4$, *where $\kappa_h(H)$, the h-extra connectivity of H, is defined the minimum number of vertices whose deletion yields the resulting graph disconnected and each remaining component has more than h vertices.*

3 $\kappa(\mathrm{BP}_n;\, K_{1,r})$ and $\kappa^s(\mathrm{BP}_n;\, K_{1,r})$

In light of the definition of structure connectivity, substructure connectivity, and $\kappa(\mathrm{BP}_n) = n$, the following result is immediate.

Theorem 3.1. $\kappa(\mathrm{BP}_n;\, K_1) = \kappa^s(\mathrm{BP}_n;\, K_1) = n$.

Next, we determine $K_{1,r}$-structure connectivity and $K_{1,r}$-substructure connectivity of BP_n for $1 \le r \le n-1$ by establishing the upper and lower bounds, respectively.

Lemma 3.2. *$\kappa(\mathrm{BP}_n;\, K_{1,r}) \le n$ for $n \ge 2$ and $1 \le r \le n-1$.*

Proof. Let $u \in V(\mathrm{BP}_n)$ and U_i be a set of r vertices in $N(u^i)\backslash\{u\}$ for $i = 1, 2, \ldots, n$. Using H_i to represent the induced subgraph by $U_i \cup \{u^i\}$ in BP_n, we have $H_i \cong K_{1,r}$. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Then, $\mathrm{BP}_n - V(\mathcal{H})$ is disconnected, and u becomes an isolated vertex. Therefore, $\kappa(\mathrm{BP}_n, K_{1,r}) \le |\mathcal{H}| = n$. □

For convenience, we define some notations throughout the article.

(1) $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ is a set of connected subgraphs of BP_n;

(2) $\mathcal{H}^i = \mathcal{H} \cap \mathrm{BP}_n^i = \{H_1 \cap \mathrm{BP}_n^i, H_2 \cap \mathrm{BP}_n^i, \ldots, H_t \cap \mathrm{BP}_n^i\}$ for $i \in [\pm n]$;

(3) $\mathrm{BP}_n^{a_1}, \mathrm{BP}_n^{a_2}, \ldots, \mathrm{BP}_n^{a_{2n}}$ are $2n$ vertex-disjoint sub-burnt pancake networks of BP_n such that $|\mathcal{H}^{a_1}| \ge |\mathcal{H}^{a_2}| \ge \cdots \ge |\mathcal{H}^{a_{2n}}|$;

(4) $I = \{i : \mathcal{H} \cap \mathrm{BP}_n^{a_i} \ne \emptyset\}$ and $I_0 = \{i : \mathcal{H} \cap \mathrm{BP}_n^{a_i} = \emptyset\}$;

(5) $\mathrm{BP}_n^I = \mathrm{BP}_n[\cup_{i\in I} V(\mathrm{BP}_n^{a_i})]$ and $\mathrm{BP}_n^{I_0} = \mathrm{BP}_n[\cup_{i\in I_0} V(\mathrm{BP}_n^{a_i})]$;

(6) $|E(i,j)| = |E(\mathrm{BP}_n^{a_i}, \mathrm{BP}_n^{a_j})|$.

Lemma 3.3. *$\kappa^s(\mathrm{BP}_n;\, K_{1,n-1}) \ge n$ for $n \ge 2$.*

Proof. Let $\mathcal{H}$ and $\mathcal{H}^{a_i}$ be defined as above. We will show that $\mathrm{BP}_n - V(\mathcal{H})$ is connected if $|\mathcal{H}| \le n-1$ and every element of $\mathcal{H}$ is isomorphic to a subgraph of $K_{1,n-1}$ by induction on n. For $n = 2$, as $\mathrm{BP}_2 \cong C_8$, it is clear that $\mathrm{BP}_2 - V(K_{1,1})$ and $\mathrm{BP}_2 - V(K_1)$ are still connected. For $2 \le \ell \le n-1$, suppose that the statement holds for BP_ℓ.

Since $|\mathcal{H}| \le n-1$ and every element of $\mathcal{H}$ distributes in at most two sub-burnt pancake networks, we have $|\mathcal{H}^i| \le n-1$, $\sum_{j=1}^{2n} |\mathcal{H}^{a_j}| \le 2n-2$, $|\mathcal{H}^{a_{2n-1}}| = |\mathcal{H}^{a_{2n}}| = 0$, and $|\mathcal{H}^{a_{2n-2}}| \le 1$.

Case 1 $|\mathcal{H}^{a_{2n-2}}| = 1$.

In this case, $|\mathcal{H}^{a_1}| = |\mathcal{H}^{a_2}| = \cdots = |\mathcal{H}^{a_{2n-3}}| = 1$ and every element of $\mathcal{H}$ distributes in exactly two distinct sub-burnt pancake networks. It implies that every H_i of $\mathcal{H}$ contains exactly one n-edge. Notice that $|\mathcal{H}^{a_k}| \le 1$, it is easy to see that $\mathrm{BP}_n^{a_k} - V(\mathcal{H}^{a_k})$ is connected for $k \in [2n]$ by the induction hypothesis for $n \ge 3$. Without loss of generality, let $\mathcal{H}^{a_j} \preccurlyeq K_{1,n-2}$, $\mathcal{H}^{a_{j+1}} \preccurlyeq K_1$, and $V(H_i) = V(\mathcal{H}^{a_j}) \cup V(\mathcal{H}^{a_{j+1}})$ for $j = 2i-1$ and $i \in [n-1]$.

On the one hand, let $G_i = \mathrm{BP}_n[V(\mathrm{BP}_n^{a_j} \cup \mathrm{BP}_n^{a_{j+1}})] - V(H_i)$ for $i \in [n-1]$. Then, G_i is connected because every H_i contains just one edge between $\mathrm{BP}_n^{a_j}$ and $\mathrm{BP}_n^{a_{j+1}}$ and $|E(j, j+1)| = (n-2)! \times 2^{n-2} > 1$ for $n \ge 3$. On the other hand, since every H_i is incident with at most one edge between G_i and $\mathrm{BP}_n^{a_{2n-1}}$ (resp. $\mathrm{BP}_n^{a_{2n}}$), there is at least one edge connecting G_i and $\mathrm{BP}_n^{a_{2n-1}}$ (resp. $\mathrm{BP}_n^{a_{2n}}$). Therefore, $\mathrm{BP}_n - V(\mathcal{H})$ is connected.

Case 2 $|\mathcal{H}^{a_{2n-2}}| = 0$.

In this case, $\{2n-2, 2n-1, 2n\} \subseteq I_0$, $|I_0| \ge 3$, and $\mathrm{BP}_n^{I_0}$ is connected. We consider the following cases.

Case 2.1 $|\mathcal{H}^{a_1}| \le n-2$.

By the induction hypothesis, $\mathrm{BP}_n^{a_j} - V(\mathcal{H}^{a_j})$ is connected for $j \in [2n]$.

In light of Lemma 2.2(3) and Proposition 2.4(1), $|E(i, i_0)| = (n-2)! \times 2^{n-2}$ for $a_i \ne \overline{a_{i_0}}$, and every element of $\mathcal{H}^{a_i} \cup \mathcal{H}^{a_{i_0}}$ is incident with at most one edge between $\mathrm{BP}_n^{a_i}$ and $\mathrm{BP}_n^{a_{i_0}}$ for $i \in [2n-3]$, $i_0 \in I_0$. Clearly, $(n-2)! \times 2^{n-2} > n-2$ for $n \ge 3$, and thus, there is at least one edge connecting $\mathrm{BP}_n^{a_i}$ and $\mathrm{BP}_n^{I_0}$. Therefore, $\mathrm{BP}_n - V(\mathcal{H})$ is connected.

Case 2.2 $|\mathcal{H}^{a_1}| = n-1$ and $|\mathcal{H}^{a_j}| \le n-2$ for $2 \le j \le 2n$.

By the pigeonhole principle, we have $|\mathcal{H}^{a_n}| \le 1$ and $|\mathcal{H}^{a_{n+1}}| = 0$.

Case 2.2.1 $|\mathcal{H}^{a_n}| = 1$.

It is clear that $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$ is connected by Case 2.1. Since $|\mathcal{H}^{a_n}| = 1$, $|\mathcal{H}^{a_2}| = |\mathcal{H}^{a_3}| = \cdots = |\mathcal{H}^{a_{n-1}}| = 1$ and every element H_i of $\mathcal{H}$ distributes in exactly two distinct sub-burnt pancakes $BP_n^{a_1}$ and $BP_n^{a_i}$ for $i = 2, 3, \ldots, n$.

Let D be a connected component of $BP_n^{a_1} - V(\mathcal{H}^{a_1})$, and so $N_{BP_n^{a_1}}(D) \subseteq V(\mathcal{H}^{a_1})$.

Suppose that $|V(D)| = 1$. Let $D = \{w\}$, as $|N_{BP_n}(w)| = n$ and $|N_{BP_n}(w) \cap V(H_i)| \le 1$ for $H_i \in \mathcal{H}$, then $w^n \notin V(\mathcal{H})$. Now, we set $|V(D)| \ge 2$. Let $uv \in E(D)$ such that $N_{BP_n^{a_1}}(v) \cap V(\mathcal{H}^{a_1}) \ne \emptyset$. Then, $v^n \notin V(\mathcal{H})$. Otherwise, there exists an $i \in \{2, 3, \ldots, n\}$ such that $\mathcal{H}^{a_i} \preccurlyeq K_{1,n-2}$ and $\mathcal{H}^{a_i}$ is incident with two n-edges between $BP_n^{a_1}$ and $BP_n^{a_i}$, contradicts Proposition 2.4(1). Therefore, $BP_n - V(\mathcal{H})$ is connected.

Case 2.2.2 $|\mathcal{H}^{a_n}| = 0$.

In light of the discussion of Case 2.1, $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$ is connected. It suffices to show that an arbitrary vertex u of $BP_n^{a_1} - V(\mathcal{H}^{a_1})$ can connect to $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$.

To the contrary, suppose that there exists a vertex $u = x_1x_2\cdots x_{n-1}a_1$ in $BP_n^{a_1} - V(\mathcal{H}^{a_1})$, which cannot connect to $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$. If $x_1 \in \{\overline{a_n}, \overline{a_{n+1}}, \ldots, \overline{a_{2n}}\}$, then $u^n \in \{\overline{a_1x_{n-1}\cdots x_2}a_n, \overline{a_1x_{n-1}\cdots x_2}a_{n+1}, \ldots, \overline{a_1x_{n-1}\cdots x_2}a_{2n}\}$, and these vertices belong to $BP_n^{a_n}, BP_n^{a_{n+1}}, \ldots, BP_n^{a_{2n}}$, respectively. Thus, uu^n is a path connecting u and $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$, a contradiction. It implies that $x_1 \notin \{\overline{a_n}, \overline{a_{n+1}}, \ldots, \overline{a_{2n}}\}$, and there exists at least a pair of a_i and a_j such that $a_i = \overline{a_j}$ for $n \le i, j \le 2n$. Without loss of generality, let $a_n = \overline{a_{n+1}} = 2$. It means that $x_1 \ne 2$ and $x_1 \ne \overline{2}$, but 2 or $\overline{2}$ must occur at the vertex u, assuming $x_2 = 2$.

If $u^n \notin V(\mathcal{H})$, then uu^n is a path connecting u and $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$; if $u^1 \notin V(\mathcal{H})$ and $(u^1)^2 \notin V(\mathcal{H})$, then $uu^1(u^1)^2((u^1)^2)^n$ is a path connecting u and $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$; if $u^2 \notin V(\mathcal{H})$, then $uu^2(u^2)^n$ is a path connecting u and $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$; if there exists an integer $i \in \{3, 4, \ldots, n-1\}$ such that $u^i \notin V(\mathcal{H})$ and $(u^i)^{i-1} \notin V(\mathcal{H})$, then $uu^i(u^i)^{i-1}((u^i)^{i-1})^n$ is a path connecting u and $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$ (Figure 2). All of these cases get contradictions. Summing up above, we know that $u^n \in V(\mathcal{H})$, at least one of u^1 and $(u^1)^2$ is contained in $V(\mathcal{H})$, $u^2 \in V(\mathcal{H})$, and at least one of u^i, $(u^i)^{i-1}$ is contained in $V(\mathcal{H})$ for $i = 3, 4, \ldots, n-1$.

Choose $v_1 \in \{u^1, (u^1)^2\} \cap V(\mathcal{H})$ and $v_i \in \{u^i, (u^i)^{i-1}\} \cap V(\mathcal{H})$ for $i = 3, 4, \ldots, n-1$ and denote $v_n = u^n$ and $v_2 = u^2$. Let $U = \{v_1, v_2, v_3, \ldots, v_n\}$. Then, $U \subseteq V(\mathcal{H})$.

If we can show that every element of $\mathcal{H}$ contains at most one vertex in U, then $|\mathcal{H}| \ge |U| = n$, which contradicts $|\mathcal{H}| \le n-1$. If not, assume that there are two distinct vertices $v_i, v_j \in U$, which lie in an element of $\mathcal{H}$. By the definition of v_i, v_j, there exist a v_i-u path R_i and a v_j-u path R_j whose lengths are at most 2. Furthermore, since $u \notin V(\mathcal{H})$ and v_i, v_j lie in an element of $\mathcal{H}$, there is a v_i-v_j path R whose length is at most 2. Consequently, R_iR_jR forms a cycle whose length is at most 6, which contradicts the girth of BP_n.

Hence, any vertex u of $BP_n^{a_1} - V(\mathcal{H}^{a_1})$ connects to $BP_n - V(BP_n^{a_1}) - V(\mathcal{H})$, i.e., $BP_n - V(\mathcal{H})$ is connected.

Case 2.3 $|\mathcal{H}^{a_1}| = |\mathcal{H}^{a_2}| = n-1$ and $|\mathcal{H}^{a_j}| = 0$ for $3 \le j \le 2n$.

Assume that $u = x_1x_2\cdots x_n$ is a vertex in $BP_n^{a_1} - V(\mathcal{H}^{a_1})$; if $u^n \in BP^{a_3} \cup BP^{a_4} \cup \cdots \cup BP^{a_{2n}}$, then the result is true. Next, let $u^n \in BP^{a_2}$, and it is obvious that $(u^1)^n, (u^2)^n, \ldots, (u^{n-1})^n \notin BP^{a_2}$. If one of $u^1, u^2, \ldots, u^{n-1}$ is not in $V(\mathcal{H})$, then it is true. Assume that $u^1, u^2, \ldots, u^{n-1}$ belong to $V(\mathcal{H})$. If $u^n \in V(\mathcal{H})$, then $|\mathcal{H}| \ge n$, a contradiction.

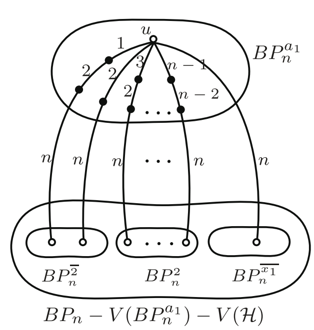

Figure 2: Illustration of Case 2.2.2 of Lemma 3.3.

Assume that $u^n \notin V(\mathcal{H})$, similarly, if $((u^n)^1)^n, ((u^n)^2)^n, \ldots, ((u^n)^{n-1})^n \notin BP^{a_1}$ and one of them is not in $V(\mathcal{H})$, then it is true. Now, $(u^n)^1, (u^n)^2, \ldots, (u^n)^{n-1} \in V(\mathcal{H})$ and $|\mathcal{H}| = 2n-2$, a contradiction.

Therefore, any vertex u of $\mathrm{BP}_n^{a_1} - V(\mathcal{H}^{a_1})$ connects to $\mathrm{BP}_n - V(\mathrm{BP}_n^{a_1}) - V(\mathcal{H})$, i.e., $\mathrm{BP}_n - V(\mathcal{H})$ is connected. □

Combining Lemmas 3.2 and 3.3, we can obtain the following result.

Theorem 3.4. $\kappa(\mathrm{BP}_n; K_{1,r}) = \kappa^s(\mathrm{BP}_n; K_{1,r}) = n$ *for* $n \geq 2$ *and* $1 \leq r \leq n-1$.

4 $\kappa(\mathrm{BP}_n; P_\ell)$ and $\kappa^s(\mathrm{BP}_n; P_\ell)$

In this section, we investigate the P_ℓ-structure connectivity and P_ℓ-substructure connectivity of BP_n for $4 \leq \ell \leq 7$.

Lemma 4.1. $\kappa(\mathrm{BP}_n; P_4) \leq n$ *and* $\kappa(\mathrm{BP}_n; P_5) \leq n$ *for* $n \geq 3$.

Proof. Let $u \in V(\mathrm{BP}_n)$ and H_i (resp. F_i) be the (i, j, i, j)-path (resp. (i, j, i, j, i)-path) that starts at u, where $i = 1, 2, \ldots, n$ and $j \equiv (i+1) \bmod n$. Clearly, $H_i \cong P_4$ (resp. $F_i \cong P_5$). Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ (resp. $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$). Then, $\mathrm{BP}_n - V(\mathcal{H})$ (resp. $\mathrm{BP}_n - V(\mathcal{F})$) is disconnected and u is an isolated vertex. Therefore, $\kappa(\mathrm{BP}_n; P_4) \leq |\mathcal{H}| = n$ (resp. $\kappa(\mathrm{BP}_n; P_5) \leq |\mathcal{F}| = n$). □

Lemma 4.2. $\kappa^s(\mathrm{BP}_3; P_5) \geq 3$.

Proof. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ and $H_i \preccurlyeq P_5$ for $i \in [t]$. It is easy to check that $\mathrm{BP}_3 - V(\mathcal{H})$ is connected when $t = 0$ or 1. Suppose that $t = 2$. If $H_1, H_2 \preccurlyeq P_3$, then $\mathrm{BP}_3 - V(\mathcal{H})$ is connected by Theorem 3.4. Suppose that at least one of H_1 and H_2 is isomorphic to P_4 or P_5. Since there are at most four n-edges in $\mathcal{H}$, we have the following cases.

Suppose that $\mathcal{H}$ contains no n-edges. When $H_1 \cup H_2 \subseteq \mathrm{BP}_3^{a_1}$, clearly, $\mathrm{BP}_3 - V(\mathcal{H})$ is connected. When $H_1 \subseteq \mathrm{BP}_3^{a_1}$ and $H_2 \subseteq \mathrm{BP}_3^{a_2}$, $\mathrm{BP}_3^{a_1} - V(H_1)$ is connected and $|V(\mathrm{BP}_3^{a_1} - V(H_1))| \geq 3$, and there is at least one n-edge connecting $\mathrm{BP}_3^{a_1} - V(H_1)$ and $\mathrm{BP}_3^{I_0}$. Similarly, $\mathrm{BP}_3^{a_2} - V(H_2)$ can connect to $\mathrm{BP}_3^{I_0}$, where $I_0 = \{3, 4, 5, 6\}$. Therefore, $\mathrm{BP}_3 - V(\mathcal{H})$ is connected.

First, assume that $\mathcal{H}$ contains one n-edge. When $|\mathcal{H}^{a_1}| = 2$ and $|\mathcal{H}^{a_2}| = 1$, clearly, $\mathrm{BP}_3 - V(\mathrm{BP}_3^{a_1}) - V(\mathcal{H})$ is connected and $\mathrm{BP}_3^{a_1} - V(\mathcal{H}^{a_1})$ contains at most two components D_1 and D_2. It is easy to check that $N_{\mathrm{BP}_3}(D_j) \cap V(\mathrm{BP}_3 - V(\mathrm{BP}_3^{a_1}) - V(\mathcal{H})) \neq \emptyset$ for $j = 1, 2$. Therefore, $\mathrm{BP}_3 - V(\mathcal{H})$ is connected. When $|\mathcal{H}^{a_1}| = |\mathcal{H}^{a_2}| = |\mathcal{H}^{a_3}| = 1$, $\mathrm{BP}_3^{a_i} - V(\mathcal{H}^{a_i})$ is connected and $|V(\mathrm{BP}_3^{a_i} - V(\mathcal{H}^{a_i}))| \geq 3$ for $i = 1, 2, 3$, and there is at least one n-edge connecting $\mathrm{BP}_3^{a_i} - V(\mathcal{H}^{a_i})$ and $\mathrm{BP}_3^{I_0}$ for $I_0 = \{4, 5, 6\}$. Therefore, $\mathrm{BP}_3 - V(\mathcal{H})$ is connected.

Second, assume that $\mathcal{H}$ contains two n-edges. Then, H_1 and H_2, respectively, contain one n-edge, or H_1 has exactly two n-edges. When $|\mathcal{H}^{a_1}| = |\mathcal{H}^{a_2}| = |\mathcal{H}^{a_3}| = |\mathcal{H}^{a_4}| = 1$, every $\mathrm{BP}_n^{a_i} - V(\mathcal{H}^{a_i})$ is connected for $1 \leq i \leq 4$, and there is at least one n-edge connecting $\mathrm{BP}_3^{a_i} - V(\mathcal{H}^{a_i})$ and $\cup_{i=5}^{6} \mathrm{BP}_3^{a_i}$. When $|\mathcal{H}^{a_1}| = 2$, and $|\mathcal{H}^{a_2}| = |\mathcal{H}^{a_3}| = 1$, $\mathrm{BP}_3 - V(\mathrm{BP}_3^{a_1}) - V(\mathcal{H})$ is connected and every component D of $\mathrm{BP}_n^{a_1} - V(\mathrm{BP}_n^{a_1})$ satisfies that $N_{\mathrm{BP}_3}(D) \cap V(\mathrm{BP}_3 - V(\mathrm{BP}_3^{a_1}) - V(\mathcal{H})) \neq \emptyset$. When $|\mathcal{H}^{a_1}| = |\mathcal{H}^{a_2}| = 2$, $\mathrm{BP}_3 - V(\mathrm{BP}_3^{a_1}) - V(\mathrm{BP}_3^{a_2})$ is connected and every component D_i of $\mathrm{BP}_n^{a_i} - V(\mathrm{BP}_n^{a_i})$ satisfies that $N_{\mathrm{BP}_3}(D) \cap V(\mathrm{BP}_3 - V(\mathrm{BP}_3^{a_1}) - V(\mathrm{BP}_3^{a_2})) \neq \emptyset$ for $i = 1, 2$. Summing up above, $\mathrm{BP}_3 - V(\mathcal{H})$ is connected.

Third, assume that $\mathcal{H}$ contains three n-edges. Then, H_1 contains exactly two n-edges and H_2 contains one n-edge. It easy to check that $\mathrm{BP}_3 - V(\mathcal{H})$ is connected from the above discussion.

Fourth, assume that $\mathcal{H}$ contains four n-edges. Both H_1 and H_2 contain two n-edges. When $|\mathcal{H}^{a_1}| = \cdots = |\mathcal{H}^{a_6}| = 1$, every $\mathcal{H}^{a_i} \preccurlyeq K_{1,1}$ or K_1 and $\mathrm{BP}_n^{a_i} - V(\mathcal{H}^{a_i})$ is connected for $i = 1, \ldots, 6$. Without loss of generality, let $H_1 \subseteq \mathrm{BP}_3[\cup_{i=1}^{3} V(\mathrm{BP}_3^{a_i})] = G_1$ and $H_2 \subseteq \mathrm{BP}_3[\cup_{i=4}^{6} V(\mathrm{BP}_3^{a_i})] = G_2$, and we have that $G_1 - V(H_1)$ is connected since there are two n-edges between any pair of $\mathrm{BP}_3^{a_i}$ and $\mathrm{BP}_3^{a_j}$ for $a_i \neq \overline{a_j}$ and that every element of $\mathcal{H}^{a_i} \cup \mathcal{H}^{a_j}$ is incident with exactly one edge between $BP_3^{a_i}$ and $BP_3^{a_j}$ for $i, j \in \{1, 2, 3\}$. Similarly, $G_2 - V(H_2)$ is connected. Since the number of deleted n-edges between G_1 and G_2 are at most 4 and $|E(G_1, G_2)| \geq 12$, there exist edges connecting $G_1 - V(H_1)$ and $G_2 - V(H_2)$. Hence, $BP_3 - V(\mathcal{H})$ is connected. Similarly, when $|\mathcal{H}^{a_1}| = |\mathcal{H}^{a_2}| = |\mathcal{H}^{a_3}| = 2$ or $|\mathcal{H}^{a_1}| = |\mathcal{H}^{a_2}| = 2, |\mathcal{H}^{a_3}| = |\mathcal{H}^{a_4}| = 1$ or $|\mathcal{H}^{a_1}| = 2, |\mathcal{H}^{a_2}| = \cdots = |\mathcal{H}^{a_5}| = 1$, we also can obtain that $BP_3 - V(\mathcal{H})$ is connected. □

Lemma 4.3. *$\kappa^s(BP_n; P_5) \geq n$ for $n \geq 3$.*

Proof. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a set of subgraph of P_5 in BP_n such that $t \leq n-1$, and we show that $BP_n - V(\mathcal{H})$ is connected by induction on n. By Lemma 4.2, this statement holds for BP_3. Now, assume that the statement holds for BP_ℓ $(3 \leq \ell \leq n-1)$.

Since every element of $\mathcal{H}$ lies in at most three different copies of $BP_n^{a_i}$ for $i \in [2n]$ and contains at most two n-edges, we have $|\mathcal{H}^{a_i}| \leq n-1$ and $\sum_{i=1}^{2n} |\mathcal{H}^{a_i}| \leq 3n-3$.

Case 1 $|\mathcal{H}^{a_1}| \leq n-2$.

By the inductive hypothesis, each $BP_n^{a_i} - V(\mathcal{H}^{a_i})$ is connected for $i \in [2n]$. In light of Lemma 2.2(3) and Proposition 2.4(2), when $a_i \neq \overline{a_j}$, we know that $|E(i,j)| = (n-2)! \times 2^{n-2}$ and the number of deleted n-edges between $BP_n^{a_i}$ and $BP_n^{a_j}$ is at most $4(n-2)$. Since $(n-2)! \times 2^{n-2} > 4(n-2)$ for $n \geq 5$, there exists at least one edge connecting $BP_n^{a_i} - V(\mathcal{H}^{a_i})$ and $BP_n^{a_j} - V(\mathcal{H}^{a_j})$. For $n = 4$, $(n-2)! \times 2^{n-2} = 8$. Let $J = \{j : |\mathcal{H}^{a_j}| = 2\}$. Then, $2 \leq |J| \leq 3$ and $\mathcal{H}^{a_i} \cup \mathcal{H}^{a_j}$ is incident with at most four n-edges for $i, j \in J$. Thus, there exists at least one edge connecting $BP_4^{a_i} - V(\mathcal{H}^{a_i})$ and $BP_4^{a_j} - V(\mathcal{H}^{a_j})$. Therefore, $BP_n - V(\mathcal{H})$ is connected.

Case 2 $|\mathcal{H}^{a_1}| = n-1$.

By the pigeonhole principle, $|\mathcal{H}^{a_2}| \leq n-1$, $|\mathcal{H}^{a_3}| \leq n-2$, $|\mathcal{H}^{a_{2n-1}}| \leq 1$, and $|\mathcal{H}^{a_{2n}}| = 0$.

By the inductive hypothesis, each $BP_n^{a_i} - V(\mathcal{H}^{a_i})$ is connected for $|\mathcal{H}^{a_i}| \leq n-2$. By Lemma 2.2(3) and Proposition 2.4(2), $|E(i, 2n)| = (n-2)! \times 2^{n-2}$ and every element of $\mathcal{H}^{a_i} \cup \mathcal{H}^{a_{2n}}$ is incident with at most two n-edges between $BP_n^{a_i}$ and $BP_n^{a_{2n}}$ for $|\mathcal{H}^{a_i}| \leq n-2$ and $a_i \neq \overline{a_{2n}}$. Since $(n-2)! \times 2^{n-2} > 2(n-2)$ for $n \geq 4$, there is at least one edge connecting $BP_n^{a_i} - V(\mathcal{H}^{a_i})$ and $BP_n^{a_{2n}}$.

Let $J = \{j : |\mathcal{H}^{a_j}| = n-1\}$. Then, every element of $\mathcal{H}$ is intersecting with $BP_n^{a_j}$.

Case 2.1 There exists $j \in J$ such that $a_j = \overline{a_{2n}}$.

Clearly, $BP_n - V(BP_n^J) - V(\mathcal{H})$ is connected. By symmetry, assume that $a_1 = \overline{a_{2n}}$. Next, we will show that an arbitrary vertex u of $BP_n^{a_1} - V(\mathcal{H}^{a_1})$ can connect to $BP_n - V(BP_n^J) - V(\mathcal{H})$. Note that $BP_n^{a_{2n}} \subseteq BP_n - V(BP_n^J) - V(\mathcal{H})$.

On the contrary, assume that u is a vertex that belongs to $BP_n^{a_1} - V(\mathcal{H}^{a_1})$ and disconnects $BP_n - V(BP_n^J) - V(\mathcal{H})$. Without loss of generality, let $u = x_1x_2 \ldots x_{n-1}1$. Then, $a_{2n} = \bar{1}$. If there exists an integer $i \in \{1, 2, \ldots, n-1\}$ such that $u^i, (u^i)^n, ((u^i)^n)^1, (((u^i)^n)^1)^n \notin V(\mathcal{H})$, then $uu^i(u^i)^n((u^i)^n)^1(((u^i)^n)^1)^n$ is a path connecting u and $BP_n^{a_{2n}}$; if $u^n, (u^n)^1, ((u^n)^1)^n \notin V(\mathcal{H})$, then $uu^n(u^n)^1((u^n)^1)^n$ is a path connecting u and $BP_n^{a_{2n}}$ (see the left of Figure 3). All of these cases get contradictions.

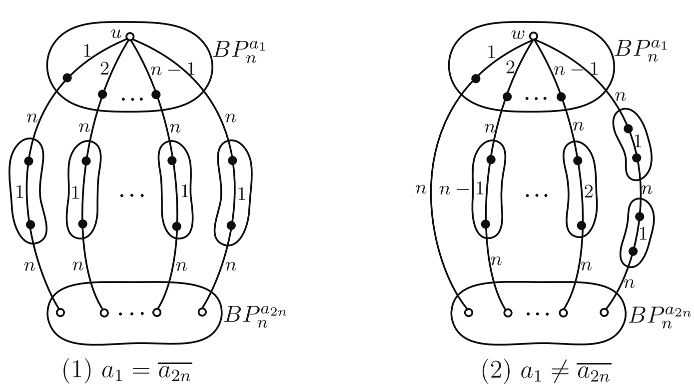

Figure 3: Illustration of Case 2 of Lemma 4.3.

Summing up above, at least one of u^i, $(u^i)^n$, $((u^i)^n)^1$, and $(((u^i)^n)^1)^n$ is contained in $V(\mathcal{H})$ for $i = 1, 2, \ldots, n-1$ and at least one of u^n, $(u^n)^1$, and $((u^n)^1)^n$ is contained in $V(\mathcal{H})$. Choose $v_i \in \{u^i, (u^i)^n, ((u^i)^n)^1, (((u^i)^n)^1)^n\} \cap V(\mathcal{H})$ for $i = 1, 2, \ldots, n-1$ and $v_n \in \{u^n, (u^n)^1, ((u^n)^1)^n\} \cap V(\mathcal{H})$. Let $U = \{v_1, v_2, \ldots, v_n\}$. Then, $U \subseteq V(\mathcal{H})$. If we can show that every element of $\mathcal{H}$ contains at most one vertex in U, then $|\mathcal{H}| \geq |U| = n$, which contradicts $|\mathcal{H}| \leq n-1$.

If not, assume that there are two distinct vertices $v_i, v_j \in U$, which lie in an element of $\mathcal{H}$. By the definition of v_i and v_j, there exist a v_i-u path R_i and a v_j-u path R_j whose lengths are at most 3. Recall that $u \notin V(\mathcal{H})$ and $V(\mathrm{BP}_n^{a_{2n}}) \nsubseteq V(\mathcal{H})$. Since v_i and v_j lie in an element of $\mathcal{H}$, there is a v_i-v_j path R whose length is at most 4. Consequently, R_iR_jR forms a cycle C whose length is at most 10. When C is a C_8, we can check that either the edge of R_iR_j does not match with a C_8 of BP_n (Figure 4 shows four kinds of C_8 containing n-edges) or $R \cap V(\mathrm{BP}_n^{a_1}) = \emptyset$, a contradiction. When C is a C_9 or C_{10}, R is an element of $\mathcal{H}$ but $R \cap V(\mathrm{BP}_n^{a_1}) = \emptyset$, also a contradiction.

Therefore, any vertex u of $\mathrm{BP}_n^{a_1} - V(\mathcal{H}^{a_1})$ connects to $\mathrm{BP}_n - V(\mathrm{BP}_n^J) - V(\mathcal{H})$.

Case 2.2 $a_j \neq \overline{a_{2n}}$ for every $j \in J$.

On the one hand, $\mathrm{BP}_n^{\overline{a_{2n}}} - V(\mathcal{H}^{\overline{a_{2n}}})$ can connect to any $\mathrm{BP}_n^{a_i} - V(\mathcal{H}^{a_i})$ for $|\mathcal{H}^{a_i}| \leq n-2$ when $n \geq 5$ because of $(n-2)! \times 2^{n-2} > 4(n-2)$. On the other hand, there exist $i \notin J$ and $|\mathcal{H}^{a_i}| \leq n-3$ such that $\mathrm{BP}_n^{a_i} - V(\mathcal{H}^{a_i})$ can connect to $\mathrm{BP}_n^{\overline{a_{2n}}} - V(\mathcal{H}^{\overline{a_{2n}}})$ when $n = 4$. Hence, $\mathrm{BP}_n - V(\mathrm{BP}_n^J) - V(\mathcal{H})$ is connected.

It suffices to show that an arbitrary vertex w of $\mathrm{BP}_n^{a_1} - V(\mathcal{H}^{a_1})$ can connect to $\mathrm{BP}_n - V(\mathrm{BP}_n^J) - V(\mathcal{H})$ for $a_1 \neq \overline{a_{2n}}$. Recall that $\mathrm{BP}_n^{a_{2n}} \subseteq \mathrm{BP}_n - V(\mathrm{BP}_n^J) - V(\mathcal{H})$.

On the contrary, assume that $w \in \mathrm{BP}_n^{a_1} - V(\mathcal{H}^{a_1})$ and w disconnects to $\mathrm{BP}_n - V(\mathrm{BP}_n^J) - V(\mathcal{H})$. Without loss of generality, let $a_1 = 1$ and $w = x_1x_2 \ldots x_{n-1}1$. Then, $a_{2n} \neq \bar{1}$. Let $a_{2n} = \bar{2}$. If $x_1 = 2$, then ww^n is an edge connecting w and $\mathrm{BP}_n^{a_{2n}}$, a contradiction. It means that $x_1 \neq 2$, assuming $x_1 = \bar{2}$.

If $w^1 \notin V(\mathcal{H})$, then $ww^1(w^1)^n$ is a path connecting w and $\mathrm{BP}_n^{a_{2n}}$; if there exists an integer $i \in \{2, 3, \ldots, n-1\}$ such that $w^i, (w^i)^n, ((w^i)^n)^{n+1-i}, (((w^i)^n)^{n+1-i})^n \notin V(\mathcal{H})$, then $ww^i(w^i)^n((w^i)^n)^{n+1-i}(((w^i)^n)^{n+1-i})^n$ is a path connecting w and $\mathrm{BP}_n^{a_{2n}}$; if $w^n, (w^n)^1, ((w^n)^1)^n, (((w^n)^1)^n)^1 \notin V(\mathcal{H})$, then $ww^n(w^n)^1((w^n)^1)^n(((w^n)^1)^n)^1((((w^n)^1)^n)^1)^n$ is a path connecting w and $\mathrm{BP}_n^{a_{2n}}$ (see the right of Figure 3). All of these cases get contradictions. Thus, $w^1 \in V(\mathcal{H})$, at least one of $w^i, (w^i)^n, ((w^i)^n)^{n+1-i}, (((w^i)^n)^{n+1-i})^n$ belongs to $V(\mathcal{H})$ for $i = 2, 3, \ldots, n-1$, and at least one of $w^n, (w^n)^1, ((w^n)^1)^n, (((w^n)^1)^n)^1$ belongs to $V(\mathcal{H})$.

Let $v_1 = w^1$, and choose $v_i \in \{w^i, (w^i)^n, ((w^i)^n)^{n+1-i}, (((w^i)^n)^{n+1-i})^n\} \cap V(\mathcal{H})$ for $i = 2, 3, \ldots, n-1$ and $v_n \in \{w^n, (w^n)^1, ((w^n)^1)^n, (((w^n)^1)^n)^1\} \cap V(\mathcal{H})$. Let $W = \{v_1, v_2, \ldots, v_n\}$. Then, $W \subseteq V(\mathcal{H})$. We will show that every element of $\mathcal{H}$ contains at most one vertex in W.

If not, assume that there are two distinct vertices $v_i, v_j \in U (i > j)$, which lie in an element of $\mathcal{H}$. By the definition of v_i and v_j, there is a v_i-w path Q_i whose length is at most 4 and a v_j-w path Q_j whose length is at most 3. Recall that $w \notin V(\mathcal{H})$ and $V(\mathrm{BP}_n^{a_{2n}}) \nsubseteq V(\mathcal{H})$. Since v_i and v_j lie in an element of $\mathcal{H}$, there is a v_i-v_j path Q whose length is at most 4. Consequently, Q_iQ_jQ forms a cycle C whose length is at most 11. When C is a C_8, we can check that either the edge of Q_iQ_j does not match with a C_8 of BP_n or $Q \cap V(\mathrm{BP}_n^{a_1}) = \emptyset$, a contradiction. When C is a C_9, C_{10}, or C_{11}, Q is an element of $\mathcal{H}$ but $Q \cap V(\mathrm{BP}_n^{a_1}) = \emptyset$, also a contradiction.

Thus, every element of $\mathcal{H}$ contains at most one vertex in U. Hence, $|\mathcal{H}| \geq |U| = n$, a contradiction. Therefore, any vertex w of $\mathrm{BP}_n^{a_1} - V(\mathcal{H}^{a_1})$ connects to $\mathrm{BP}_n - V(\mathrm{BP}_n^J) - V(\mathcal{H})$.

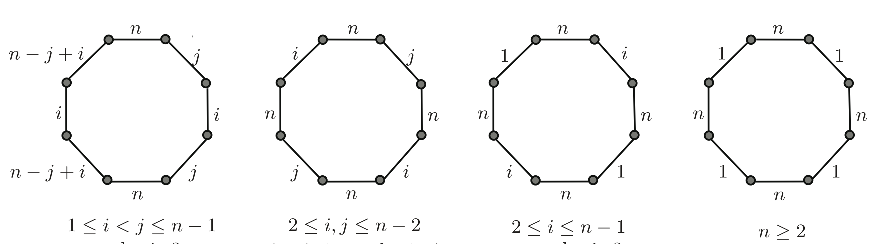

Figure 4: Four kinds C_8 with n-edge in BP_n.

By symmetry, any vertex of $BP_n^{a_j} - V(\mathcal{H}^{a_j})$ for $j \in J$ can connect to $BP_n - V(BP_n^J) - V(\mathcal{H})$. Therefore, $BP_n - V(\mathcal{H})$ is connected. □

Theorem 4.4. $\kappa(BP_n; P_5) = \kappa^s(BP_n; P_5) = \kappa^s(BP_n; P_4) = \kappa(BP_n; P_4) = n$ *for* $n \geq 3$.

Proof. By Observation 2.1 and Lemmas 4.1 and 4.3, $\kappa(BP_n; P_5) = n = \kappa^s(BP_n; P_5) \leq \kappa^s(BP_n; P_4) \leq \kappa(BP_n; P_4) = n$. □

Lemma 4.5. $\kappa(BP_n; P_6) \leq n - 1$ *for* $n \geq 3$.

Proof. For any vertex $u \in V(BP_n)$, let C_i be $(i, 1, i, 1, i, 1, i, 1)$-cycle containing u and $H_i = C_i - \{u, u^1\}$ for $i \in \{2, 3, \dots, n\}$. Denote $\mathcal{H} = \{H_2, H_3, \dots, H_n\}$. It is easy to see that $H_i \cong P_6$, $N(\{u, u^1\}) = \cup_{i=2}^{n}\{u^i, (u^1)^i\} \subseteq V(\mathcal{H})$ and $u, u^1 \notin V(\mathcal{H})$. Hence, $BP_n - V(\mathcal{H})$ is disconnected, uu^1 is an isolated edge, and $\kappa(BP_n, P_6) \leq |\mathcal{H}| = n - 1$. □

Lemma 4.6. $\kappa^s(BP_n; P_6) \geq n - 1$ *for* $n \geq 3$.

Proof. Let $\mathcal{H} = \{H_1, \dots, H_t\}$ and $H_i \lessdot P_6$ for $i \in [t]$. For $n = 3$, it suffices to show that $BP_3 - V(\mathcal{H})$ is connected with $|\mathcal{H}| \leq 1$. It is clear that $BP_3 - V(\mathcal{H})$ is connected if $|\mathcal{H}| = 0$. Assume that $|\mathcal{H}| = 1$ and $\mathcal{H} = \{H_1\}$, either H_1 is isomorphic to a connected subgraph of P_5 or there are $H_{1,1}$ and $H_{1,2}$ such that $H_{1,1}$ and $H_{1,2}$ are isomorphic to a connected subgraph of P_5 and $V(H_{1,1}) \cup V(H_{1,2}) = V(H_1)$. By Theorem 4.4, $\kappa^s(BP_3;P_5) = 3$, then $BP_3 - V(\mathcal{H})$ is connected.

For $n \geq 4$, let $\mathcal{H}$ be a minimum P_6-substructure cut set of BP_n. It suffices to show that $|\mathcal{H}| \geq n - 1$.

Case 1 There exists an isolated vertex u in $BP_n - V(\mathcal{H})$.

Since $|N(u)| = n$ and $|N(u) \cap H_i| \leq 1$, we have $|\mathcal{H}| \geq n$. Hence, $|\mathcal{H}| > n - 1$.

Case 2 There is no isolated vertex in $BP_n - V(\mathcal{H})$.

Let D be a minimum connected component of $BP_n - V(\mathcal{H})$. Then, $|V(D)| \geq 2$.

By Lemma 2.5, if $n \geq 4$ and $|V(D)| = 2$, then $|N(D)| = 2n - 2$. Since $|N(D) \cap H_i| \leq 2$, we have $|\mathcal{H}| \geq n - 1$.

By Lemma 2.5, if $n \geq 4$ and $|V(D)| \geq 3$, then $|N(D)| \geq 3n - 4$, and $N(D)$ contains at least $3n - 4$ mutually nonadjacent vertices. However, there are at most $3n - 6$ mutually nonadjacent vertices in $V(\mathcal{H})$ if $|\mathcal{H}| \leq n - 2$. Therefore, $|\mathcal{H}| \geq n - 1$. □

Applying Observation 2.1 and Lemmas 4.5 and 4.6, the following results are obtained immediately.

Theorem 4.7. $\kappa(BP_n; P_6) = \kappa^s(BP_n; P_6) = n - 1$ *for* $n \geq 3$.

Lemma 4.8. $\kappa(BP_n; P_7) \leq \left\lceil \frac{n}{2} \right\rceil$ *for* $n \geq 3$.

Proof. For any $u \in V(BP_n)$, let C_k be $(k-2, k-1, k, k-1, k-2, k-1, k, k-1)$-cycle starting at u for $k = 2i + 1$ and $i = \{1, 2, \dots, \left\lceil \frac{n-1}{2} \right\rceil\}$.

If n is odd, let C_n be $(n, 1, n, 2, n, 1, n)$-path starting at u.

If n is even, let C_n be $(n-1, 1, n-1, n, 2, 1, 2, n)$-cycle starting at u.

Let $H_k = C_k - \{u\}$ for $k = 2i + 1$ and $i = \{1, 2, \dots, \left\lceil \frac{n}{2} \right\rceil\}$. Denote $\mathcal{H} = \{H_1, H_2, \dots, H_{\lceil \frac{n}{2} \rceil}\}$. It is not hard to see that $H_k \cong P_7$, $N(u) = \{u^1, u^2, \dots, u^n\} \subseteq V(\mathcal{H})$, and $u \notin V(\mathcal{H})$. Hence, $BP_n - V(\mathcal{H})$ is disconnected, u is an isolated vertex, and $\kappa(BP_n, P_7) \leq |\mathcal{H}| = \left\lceil \frac{n}{2} \right\rceil$. □

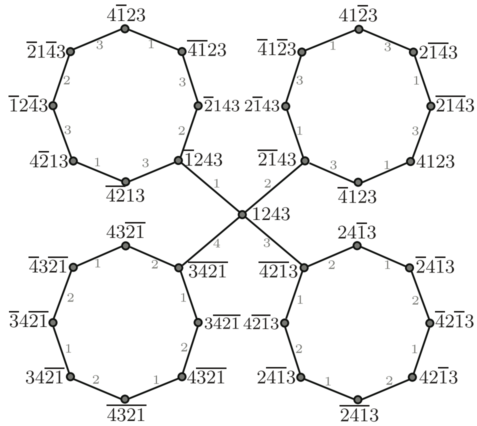

Figure 5: A C_8-structure cut in BP_4.

5 $\kappa(\mathrm{BP}_n; C_8)$ and $\kappa^s(\mathrm{BP}_n; C_8)$

We study the C_8-substructure connectivity and C_8-structure connectivity of BP_n as follows.

Lemma 5.1. $\kappa^s(\mathrm{BP}_n; C_8) \geq \lceil \frac{n}{2} \rceil$ *for* $n \geq 3$.

Proof. It suffices to show that $\mathrm{BP}_n - V(\mathcal{H})$ is connected for $\mathcal{H} = \{H_1, \dots, H_s, H_{s+1}, \dots, H_t\}$ with $t \leq \lceil \frac{n}{2} \rceil - 1$ and every $H_i \preccurlyeq C_8$. Suppose that every $H_i (s + 1 \leq i \leq t)$ is just isomorphic to a subgraph of P_4, then every $H_i (1 \leq i \leq s)$ can be divided into two subgraphs of P_4, set $V(H_i) = V(H_{i,1}) \cup V(H_{i,2})$. Let $\mathcal{H}' = \{H_{1,1}, H_{1,2}, \dots, H_{s,1}, H_{s,2}, H_{s+1}, \dots, H_t\}$. Then, $V(\mathcal{H}) = V(\mathcal{H}')$. Moreover, every element of $\mathcal{H}'$ is isomorphic to a subgraph of P_4 and $|\mathcal{H}'| = 2s + (t - s) = t + s \leq 2t \leq 2(\lceil \frac{n}{2} \rceil - 1) \leq n - 1$. But $\kappa^s(\mathrm{BP}_n; P_4) = n$ by Theorem 4.4, hence, $\mathrm{BP}_n - V(\mathcal{H})$ is connected. □

By Observation 2.1 and Lemmas 4.8 and 5.1, we obtain $\left\lceil \frac{n}{2} \right\rceil \leq \kappa^s(\mathrm{BP}_n; C_8) \leq \kappa^s(\mathrm{BP}_n; P_7) \leq \kappa(\mathrm{BP}_n; P_7) \leq \left\lceil \frac{n}{2} \right\rceil$. Then, the following theorem is obvious.

Theorem 5.2. $\kappa^s(\mathrm{BP}_n; C_8) = \kappa^s(\mathrm{BP}_n; P_7) = \kappa(\mathrm{BP}_n; P_7) = \left\lceil \frac{n}{2} \right\rceil$ *for* $n \geq 3$.

Lemma 5.3. $\kappa(\mathrm{BP}_n; C_8) \leq n$ *for* $n \geq 3$.

Proof. For $u \in V(\mathrm{BP}_n)$, let H_i be $(1, 2, 1, 2, 1, 2, 1, 2)$-cycle going through u^i for $i = 3, \dots, n$. In particular, let H_1 be the $(2, 3, 1, 3, 2, 3, 1, 3)$-cycle going through u^1 and H_2 be the $(1, 3, 1, 3, 1, 3, 1, 3)$-cycle going through u^2. Denote $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$. Clearly, $H_i \cong C_8$, $N(u) = \{u^1, u^2, \dots, u^n\} \subseteq V(\mathcal{H})$, and $u \notin V(\mathcal{H})$. Hence, $\mathrm{BP}_n - V(\mathcal{H})$ is disconnected, u is an isolated vertex, and $\kappa(\mathrm{BP}_n, C_8) \leq |\mathcal{H}| = n$. (Figure 5 shows a C_8-structure cut in BP_4, where $u = 1243$.) □

Combining Observation 2.1, Theorem 5.2, and Lemma 5.3, we have the following result.

Theorem 5.4. $\left\lceil \frac{n}{2} \right\rceil \leq \kappa(\mathrm{BP}_n; C_8) \leq n$ *for* $n \geq 3$.

6 Conclusion

In this article, we investigate the H-structure and H-substructure connectivity of the n-dimensional burnt pancake network BP_n when H is isomorphic to $K_{1,t}$ $(1 \le t \le n-1)$, P_ℓ $(4 \le \ell \le 7)$, and C_8. More details,

(1) for $H = K_{1,t}(1 \le t \le n-1)$, $\kappa^s(BP_n; K_{1,t}) = \kappa(BP_n; K_{1,t}) = n$;

(2) for $H = P_\ell(4 \le \ell \le 7)$, $\kappa^s(BP_n; P_\ell) = \kappa(BP_n; P_\ell) = n$ when $\ell = 4, 5$, $\kappa^s(BP_n; P_6) = \kappa(BP_n; P_6) = n - 1$ and $\kappa^s(BP_n; P_7) = \kappa(BP_n; P_7) = \left\lceil \frac{n}{2} \right\rceil$;

(3) for $H = C_8$, $\kappa^s(BP_n; C_8) = \left\lceil \frac{n}{2} \right\rceil$ and $\left\lceil \frac{n}{2} \right\rceil \le \kappa(BP_n; C_8) \le n$.

This work provides constructive ideas for other networks in the process of showing the structure and substructure connectivity. In addition, the exact value of $\kappa(BP_n; C_8)$ remains open, and we have the following conjecture.

Conjecture 6.1. $\kappa(BP_n; C_8) = n$ *for* $n \ge 3$.

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