Research Article

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Properties of locally semi-compact Ir-topological groups

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Abstract: This study investigates some topological properties of locally semi-compact Ir-topological groups and establishes the relationship between Ir-topological groups and semi-compact spaces. The proved theorems generalize the corresponding results of Ir-topological group. Finally, we define a quotient topology on the Ir-topological group and study some topological properties of the space.

Keywords: semi-open sets, locally semi-compact spaces, Ir-topological groups

MSC 2020: 22B05, 22B10, 22D05, 54A05

1 Introduction

Recently, topological groups have garnered significant attention from topologists due to their applications in graph algebras and the study of hyperbolic groups [1–6]. Many fundamental concepts and constructions related to topological groups have been introduced. One of the generic questions in topological algebra is how the relationship between topological properties depends on the underlying algebraic structure [7]. Locally compact groups are essential because many examples of groups that arise throughout mathematics are locally compact [8–11]. The rules that describe the relationship between a locally compact group and an algebraic operation are almost always continuous. It is natural to explore the properties of topological groups by relaxing the continuity conditions. Consequently, Levine [12] introduced the concepts of semi-open sets and semi-continuity within general topological spaces. These concepts have now become research topics for topologists worldwide, including the study of semi-separation axioms and binary topological spaces [13–15].

One of the critical applications of semi-open sets is compactness. In 1984, the definition of locally semi-compact spaces [16] was introduced. Furthermore, many scholars use semi-open sets to study topological groups, and the study of the topological group is wide open. In 2014, Bosan et al. [17] studied the class of s-topological groups and a wider class of S-topological groups defined using semi-open sets and semi-continuity. In 2015, two types of topological groups, which are called irresolute-topological and Ir-topological [18], were introduced and studied.

These groups form a generalization of topological groups, and some preliminary results and applications of these groups are presented. However, the relationships between these topological groups and other topological spaces have yet to be obtained. One of the main operations on topological groups is taking quotient groups. The quotient spaces of some topological groups have yet to be proposed either.

To solve these problems mentioned above, the primary purpose of this study is to establish relationships between Ir-topological groups and semi-compact spaces and define a quotient topology on the Ir-topological group. We will introduce some basic properties of locally semi-compact spaces and Ir-topological groups in Section 2. In Section 3, by introducing the concept of countable semi-stars, the connection between

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Ir-topological groups and semi-compact spaces is constructed, and the concept of quotient spaces is also introduced. Also, we generalize a series of results on Ir-topological groups.

Throughout this study, X and Y are always topological spaces on which no separation axioms are assumed. The set of positive integers is denoted as $\mathbb{N}$ and the real line is denoted as $\mathbb{R}$. The family of all semi-open sets in X is denoted by $\mathrm{SO}(X)$. The interior and semi-interior of A in X are denoted as A^0 and $\mathrm{sop}(A)$. The closure and semi-closure of A in X are denoted as $\overline{A}$ and $\mathrm{scl}A$. In this study, G denotes a group endowed with a topology. We do not require the group operations to be continuous but to satisfy some weaker forms of continuity. If G is a group, then e denotes its identity element and τ denotes its topology. For definitions not defined here, we refer the reader to [7].

2 Preliminaries

Ir-topological groups are one of the essential classes of topological groups, and many results have been obtained. We study in this section the most elementary general properties of locally semi-compact spaces and Ir-topological groups and obtain some new properties.

We need to recall some basic notations. A subset A of a topological space X is said to be *semi-open* [12] if there exists an open set B in X such that $B \subset A \subset \overline{B}$. Obviously, any open set is semi-open, but the converse need not be true. The complement of a semi-open set is said to be *semi-closed*. A collection $\mathscr{B}$ of subsets of a topological space X is said to be *s-locally finite* [19] if for each $x \in X$, there exists a semi-open set U in X containing x and U intersects at most finitely many members of $\mathscr{B}$. Clearly, if a collection is locally finite, it is s-locally finite, but the converse need not be true.

A collection $\mathscr{C}$ of subsets of a topological space X is a *net* [20] if for each point $x \in X$ and any open set V containing x, there exists a set $C \in \mathscr{C}$ such that $x \in C \subset V$. A topological space X is said to be *s-regular* [21] if for any closed set B and $x \notin B$, there exist disjoint semi-open sets U and V such that $x \in U$ and $B \subset V$.

A topological space X is said to be *semi-compact* [22] if every cover of X by semi-open sets has a finite subcover. A topological space X is said to be *locally semi-compact* [16] if every point of X has an open semi-compact neighborhood. Obviously, each semi-compact space is a locally semi-compact space. But the converse need not be true. For example, let X be an infinite discrete topological space. Thus, X is not semi-compact. Since the singletons can serve as semi-compact neighborhoods, it follows that X is locally semi-compact.

Proposition 2.1. *Suppose X is a locally semi-compact s-regular space. Then, the collection of semi-compact semi-closed sets in X is a net.*

Proof. Suppose x in X and A is an open set containing x. Since X is locally semi-compact, it follows that there is an open neighborhood B which is semi-compact. Then, $A \cap B$ is a neighborhood of x. Let $C = X - A \cap B$. Then, x is not in C, and C is closed. Since X is s-regular, it follows that there exist disjoint semi-open sets E and F such that x in E and $C \subset F$. Let $D = X - F$. Then, D is semi-closed and $X - F \subset X - C$. Thus, $D \subset A \cap B$ and $x \in D \subset A$.

It remains to show that D is semi-compact. Suppose $\{H_\alpha : \alpha \in I\}$ is a semi-open cover of D. For each H_α, there exists an open set $L_\alpha \subset D$ such that $L_\alpha \subset H_\alpha \subset \overline{L_\alpha}$. Then, there exists an open set M_α in X such that $L_\alpha = M_\alpha \cap D$. Thus, $\overline{H_\alpha} \subset \overline{L_\alpha} \subset \overline{M_\alpha}$. Let $O_\alpha = M_\alpha \cup H_\alpha$. Then, $O_\alpha \cap D = H_\alpha$. Since $M_\alpha \subset O_\alpha \subset \overline{O_\alpha}$ and $\overline{O_\alpha} = \overline{M_\alpha \cup H_\alpha} = \overline{M_\alpha}$, it follows that O_α is semi-open in B. Then, $\{O_\alpha : \alpha \in I\} \cup \{B - D\}$ is a semi-open cover of B. There exists a semi-open cover $\{O_1, O_2, \cdots, O_r\} \cup \{B - D\}$ in B. Thus, there exists a semi-open cover $\{H_1, H_2, \cdots, H_r\}$ of D, and D is semi-compact. □

Definition 2.2. [23] A subset A of a topological space X is said to be *pre-open* if $A \subset \overline{A}^0$.

Obviously, each open set is pre-open, but the converse need not be true. Also, we note that the notions of pre-open and semi-open are independent of each other. For example, let $X = \{x_1, x_2, x_3, x_4\}$ and $\tau = \{X, \varnothing, \{x_1, x_4\}, \{x_3\}, \{x_1, x_3, x_4\}\}$. Then, (X, τ) is a topological space, and $\{x_1, x_2, x_3\}$ is a pre-open set, but it is not semi-open. Since $\{x_3\} \subset \{x_2, x_3\} \subset \overline{\{x_3\}}$ and $\overline{\{x_2, x_3\}}^0 = \{x_3\}$, it follows that $\{x_2, x_3\}$ is semi-open but not pre-open. From the definition of pre-open, we have the following remark.

Remark 2.3. If A is pre-open in topological space X, then there exists an open set B in X such that $A \subset B \subset \overline{A}$.

Definition 2.4. [24] A mapping $f : X \to Y$ is said to be *irresolute* (respectively, *pre-semi-open*) if $f^{-1}(V)$ (respectively, $f(V)$) is semi-open in X (respectively, Y) for each semi-open set V in Y (respectively, X).

Definition 2.5. A collection $\mathscr{C}$ of subsets of a topological space X is a *semi-net* if for each point $x \in X$ and any semi-open set V containing x, there exists a set $C \in \mathscr{C}$ such that $x \in C \subset V$.

Obviously, each semi-net in a topological space is a net, but the following example shows that the converse need not be true.

Example 2.6. Suppose $X = \{x_1, x_2, x_3\}$ and $\mathscr{B}_1 = \{\varnothing, X, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$ is a topology of X. Thus, $\mathscr{B}_2 = \{\varnothing, X, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$ is the family of all semi-open sets of X. Suppose $\mathscr{B}_3 = \{X, \{x_1\}, \{x_2\}\}$. Then, $\mathscr{B}_3$ is a net of X. Note that $\{x_1, x_3\}$ is a semi-open set and $x_3 \in \{x_1, x_1\}$. For each set, A in $\mathscr{B}_3$ does not satisfy the condition that $x_3 \in A \subset \{x_1, x_3\}$. Therefore, $\mathscr{B}_3$ is not a semi-net of X.

Proposition 2.7. *Suppose X is a countably semi-compact space and $f : X \to Y$ is an irresolute injection. If X has a countable semi-net, then Y is locally semi-compact.*

Proof. Assume that $\mathscr{B}$ is a countable semi-net of X. Suppose $\mathscr{A}$ is a semi-open cover of X. For each point x in A and $A \in \mathscr{A}$, there exists a set B in $\mathscr{B}$ such that x in B and $B \subset A$. Then, there exists a family $\mathscr{B}_A \subset \mathscr{B}$ such that $A = \bigcup_{B \in \mathscr{B}_A} B$. Let $\bigcup_{A \in \mathscr{A}} \mathscr{B}_A = \mathscr{B}_0$. Then, $\mathscr{B}_0$ is a countable cover of X. Let $\mathscr{B}_0 = \{B_n : n \in \mathbb{N}\}$. If $B_n \in \mathscr{B}_0$, and then there exists a set $A_n \in \mathscr{A}$ such that $B_n \subset A_n$. Let

$$\mathscr{A}_1 = \{A_n : A_n \in \mathscr{A}, B_n \in \mathscr{B}_0, B_n \subset A_n\}.$$

Then, $\mathscr{A}_1 \subset \mathscr{A}$ is a countable cover of X, and X is semi-Lindelöf. Since X is countably semi-compact, it follows that X is semi-compact. Suppose $\mathscr{C} = \{C_\alpha : \alpha \in I\}$ is a semi-open cover of Y. Hence, $\{f^{-1}(C_\alpha) : \alpha \in I\}$ is a semi-open cover of X. Thus, there exists a finite cover $\{f^{-1}(C_{\alpha_1}), f^{-1}(C_{\alpha_2}), \cdots, f^{-1}(C_{\alpha_r})\}$. Then, $\{C_{\alpha_1}, C_{\alpha_2}, \cdots, C_{\alpha_r}\}$ is a finite cover of Y. Thus, Y is semi-compact, and Y is locally semi-compact. □

Definition 2.8. [18] A topologized group is said to be an *Ir-topological* group if both the multiplication mapping and the inverse mapping are irresolute.

Definition 2.9. [24] A mapping $f : X \to Y$ is said to be a *semi-homeomorphism* if f is bijective, irresolute, and pre-semi-open.

Proposition 2.10. *Suppose G is an Ir-topological group and each $a \in X$. Then, the right multiplication r_a is a semi-homeomorphism.*

Proof. Suppose x and y are in G with $x \neq y$. Assume that $r_a(x) = r_a(y)$. Then, $xa = ya$ and $x = y$. It is contradictory, and r_a is an injection. For each b in G, there exists a point $c = ba^{-1}$ in G such that $r_a(c) = b$. Thus, r_a is a surjection and r_a is bijection.

Define a mapping $f_1 : G \to G \times G, x \to (x, a)$. Let $f : G \times G \to G$ be the multiplication mapping. Thus, $r_a(x) = f(f_1(x))$ for each x in G. For each x in G and each semi-open neighborhood A of $f_1(x)$, there exist semi-open sets B and C such that $f_1(x) \in B \times C \subset A$. Then, x in B and a in C. Thus, $f_1(B) \subset A$ and f_1 is irresolute. Since G is an Ir-topological group, it follows that f is irresolute. Hence, r_a is irresolute. Since $r_a^{-1} = r_{a^{-1}}$, it follows that r_a is pre-semi-open. Therefore, r_a is a semi-homeomorphism. □

According to Proposition 2.10, we obtain the following remarks directly.

Remark 2.11. Suppose G is an Ir-topological group and each $a \in X$. Then, the left multiplication l_a is a semi-homeomorphism.

Remark 2.12. Suppose G is an Ir-topological group and A is semi-open in G. Then, Ax and xA are semi-open for each x in G.

3 Properties of Ir-topological group

This section contains theorems on topological properties of locally semi-compact Ir-topological groups. We will use some mappings to investigate the relationships between Ir-topological groups and other topological spaces. Also, we obtained some properties of the Ir-topological group.

It is well known that the intersection of two semi-open sets need not be semi-open. Thus, every family of semi-open sets in a topological space need not be a topology. Example 3.1 [18] shows that even if the family of semi-open sets is a topology on G, the Ir-topological group need not be a topological group.

Example 3.1. The set $G = \{1, 3, 5, 7\}$ is a group under multiplication $\odot_8$, the multiplication modulo 8. Let $\tau = \{\emptyset, G, \{1\}, \{1, 3, 5\}\}$ be the topology on G. Then, $\mathrm{SO}(G) = \{\emptyset, G, \{1\}, \{1, 3, 5\}, \{1, 3\}, \{1, 5\}, \{1, 7\}, \{1, 3, 7\}, \{1, 5, 7\}\}$.

Thus, the family $\mathrm{SO}(G)$ is a topology on G, and different from τ. Obviously, each semi-open set in G is pre-open.

For any $V \in \mathrm{SO}(G)$, the preimage $f^{-1}(V)$ contains the open set $\{(1, 1)\} \subset G \times G$, which is dense in $G \times G$, that is, for each $V \in \mathrm{SO}(G)$, we have $\{(1, 1)\} \subset f^{-1}(V) \subset \overline{\{(1, 1)\}}$. It means that $f^{-1}(V)$ is semi-open in $G \times G$. Hence, f is irresolute. On the other hand, for each $V \in \mathrm{SO}(G)$, it holds $g^{-1}(V) = V$, which means that g is an irresolute mapping. So, $(G, \odot, \tau)$ is an Ir-topological group which is not a topological group (f is not continuous, for instance, at $(3, 3) \in G \times G$).

According to Example 3.1, we obtain the following proposition directly.

Proposition 3.2. *G is a locally semi-compact Ir-topological group, and each semi-open set in G is pre-open, but G does not need to be a topological group.*

In order to prove the following result, we need to define some new definitions. A topological space is said to be *σ-compact* [20] if it is the union of countably many compact subspaces. A topological space is said to be *σ semi-compact* if it is the union of countably many semi-compact subspaces. Obviously, each σ semi-compact space is σ-compact. The following example shows that each σ-compact space need not be σ semi-compact.

Example 3.3. Let $X = \mathbb{R}$ have the standard topology, and $Y_n = [-1, 1]$ have the subspace topology τ for each $n \in \mathbb{N}$. Then, $\cup_{n\in\mathbb{N}} Y_n$ is σ-compact. Suppose $\mathscr{A}_n = [-1, 0] \cup \{(1/(n + 1), 1/n) : n \in \mathbb{N}\}$. Then, $\mathscr{A}_n$ is a semi-open cover of Y_n and does not have a finite subcover. Hence, $\cup_{n\in\mathbb{N}} Y_n$ is not σ semi-compact.

Obviously, semi-compact space is σ semi-compact, but the converse need not be true. For example, let $X = \mathbb{R}$ have the standard topology. Then, X is σ semi-compact, but not semi-compact. A space X is *locally σ semi-compact* if for every point x of X, there exists an open neighborhood V such that V is σ semi-compact. A collection is said to be *star-finite* [20] if every member of the collection intersects only finitely many members. A space X is said to have the *countable semi-star* property if for each countable star-finite semi-open cover $\mathscr{A}$ of X, there exists a finite set B in X such that $St(B, \mathscr{A}) = \cup\{A : A \in \mathscr{A}, A \cap B \neq \emptyset\} = X$.

Theorem 3.4. *Suppose G is a locally σ semi-compact Ir-topological group and each semi-open set is pre-open. If G has the countable semi-star property, then G is semi-compact.*

Proof. Since G is a locally σ semi-compact Ir-topological group, it follows that there exists an open neighborhood B of the identity e in G such that B is σ semi-compact. Let $B = \cup_{n\in\mathbb{N}}B_n$, and each B_n is semi-compact. Then, B is a subgroup of G, and B is semi-Lindelöf. For each a in $G - B$, according to Proposition 2.10, the set $r_a(B) = Ba$ is semi-open in G. Thus, $G - B = \cup_{a\notin B}Ba$ is semi-open and B is semi-closed. Since G is the disjoint union of the right cosets of B, it follows that G is semi-homeomorphism to B and G is semi-Lindelöf.

Suppose $C \subset G$ is a semi-closed set and e is not in C. Then, $D = G - C$ is an open neighborhood of e. Since the multiplication mapping $f : G \times G \to G$, $(x, y) \to xy$ is irresolute, it follows that $f^{-1}(D)$ is a semi-open set containing (e, e). Thus, there exist semi-open sets E and F such that e in E, e in F, and $E \times F \subset f^{-1}(D)$. Then, there exist open sets E_1 such that $E_1 \subset E \subset \overline{E_1}$ and $F \subset \overline{F}^0 \subset \overline{F}$. Hence, $E \cap F \subset \overline{E_1} \cap \overline{F}^0$. Suppose x_0 in $\overline{E_1} \cap \overline{F}^0$ and U_{x_0} is a neighborhood of x_0. Thus, $U_{x_0} \cap \overline{F}^0$ is a neighborhood of x_0. Then, $U_{x_0} \cap \overline{F}^0 \cap E_1 \neq \varnothing$ and $x_0 \in \overline{E_1 \cap \overline{F}^0}$. Hence, $E \cap F \subset \overline{E_1 \cap \overline{F}^0}$. Suppose y_0 in $E_1 \cap \overline{F}^0$. Then, y_0 is in E, and y_0 is not in $X - \overline{F}^0$. Thus, there exists a neighborhood V_{y_0} of y_0 such that $V_{y_0} \cap (X - \overline{F}^0) = \varnothing$. Assume that y_0 is not in F. Then, $y_0 \in \overline{F} - F = Fr\{F\}$. Hence, y_0 in $Fr\{\overline{F}^0\}$ and $V_{y_0} \cap (X - \overline{F}^0) \neq \varnothing$. It is contradictory. Then, $y_0 \in F$ and $E_1 \cap \overline{F}^0 \subset E \cap F$. Thus,

$$E_1 \cap \overline{F}^0 \subset E \cap F \subset \overline{E_1 \cap \overline{F}^0}.$$

Hence, $E \cap F$ is semi-open.

Let $H = E \cap F$. Then, H is a semi-open set containing e and $H \times H \subset f^{-1}(D)$. Then, $H^2 \subset D$. Suppose b in $\mathrm{scl}H$. Since e in H^{-1}, it follows that b in bH^{-1} and bH^{-1} is semi-open. Hence, $bH^{-1} \cap H \neq \varnothing$. Suppose c in $bH^{-1} \cap H$. Then, $c = bh^{-1}$, and $b = ch \in H^2$. Thus, $\mathrm{scl}H \subset H^2 \subset D$. Suppose $L = G - \mathrm{scl}H$. Then, L is semi-open and $C \subset L$. Since $H \subset \mathrm{scl}H$ and e in H, it follows that $H \cap L = \varnothing$.

Suppose O and P are disjoint semi-closed sets, and x in O. Then, x in $X - P$ and $X - P$ is semi-open. Then, there exists a semi-open set Q_x such that x in Q_x and $Q_x \subset \mathrm{scl}Q_x \subset X - P$ and $\mathrm{scl}Q_x \cap P = \varnothing$. Hence, $\mathscr{Q} = \{Q_x : x \in O\}$ is a semi-open cover of O. Then, there exists a semi-open set R_y such that $\mathrm{scl}R_y \cap O = \varnothing$ for each y in P. Let $\mathscr{R} = \{R_y : y \in P\}$. Then, $\mathscr{R}$ is a semi-open cover of P. Thus, $\mathscr{Q} \cup \mathscr{R} \cup \{X - P \cup O\}$ is a semi-open cover of G. Then, there exists a countable subcover of G. Thus, there exists a countable semi-open cover $\{Q_n : n \in \mathbb{N}\}$ of O and a countable semi-open cover $\{R_n : n \in \mathbb{N}\}$ of P. Let

$$S_n = Q_n - \cup\{\mathrm{scl}R_k : k \leq n\},$$

$$T_n = R_n - \cup\{\mathrm{scl}Q_k : k \leq n\}.$$

Then, S_n and T_n are semi-open sets. Hence, $S_n \cap T_m = \varnothing$, n in $\mathbb{N}$ and m in $\mathbb{N}$. Let $S = \cup_{n\in\mathbb{N}}S_n$, $T = \cup_{n\in\mathbb{N}}T_n$. Then, $O \subset S$, $P \subset T$, and $S \cap T = \varnothing$. Thus, G is semi-normal.

Now, we will show that G is semi-compact. Suppose $\mathscr{U}$ is a semi-open cover of G. Then, there exists a countable cover $\mathscr{M} = \{M_n : n \in \mathbb{N}\}$ and $\mathscr{M} \subset \mathscr{U}$. Suppose x in M_n. Then, there exists a semi-open set N_x such that x in N_x and $N_x \subset \mathrm{scl}N_x \subset M_n$. Thus, $\{N_x : x \in G\}$ is a semi-open cover of G. Then, there exists a countable subcover $\mathscr{N} = \{N_{x_n} : x \in G\}$ of G. For each N_{x_n} in $\mathscr{N}$, pick M_n in $\mathscr{M}$ such that $N_{x_n} \subset M_n$. Thus, $\mathrm{scl}N_{x_n} \subset M_n$ and $\{\mathrm{scl}N_{x_n} : n \in \mathbb{N}\}$ is a semi-closed cover of G. Since G is semi-normal, it follows that there exists a semi-open set $U(n, 1)$ such that $\mathrm{scl}N_{x_n} \subset U(n, 1) \subset \mathrm{scl}U(n, 1) \subset M_n$. Arguing by induction, there exists a family $\{U(n, k) : k \in \mathbb{N}\}$ such that

$$\mathrm{scl}N_{x_n} \subset U(n, 1) \subset \mathrm{scl}U(n, 1) \subset U(n, 2) \subset \cdots \subset U(n, k) \subset \mathrm{scl}U(n, k) \subset \cdots \subset M_n.$$

Thus, $\{U(n, k) : n \in \mathbb{N}, k \in \mathbb{N}\}$ is a semi-open cover of G. For each $k \in \mathbb{N}$, let $V_k = U(1, k) \cup U(2, k) \cup \cdots \cup U(n, k)$. Thus, V_k is a semi-open set and $V_k \subset V_{k+1}$ for each $k \in \mathbb{N}$.

Let $W(1, 1) = V_2 \cap U(1, 2)$, $W(n, k) = (V_{k+1} - \mathrm{scl}V_{k-1}) \cap U(n, k + 1)$, which holds for $1 \leq n \leq k$ and $k > 1$. Thus, $W(n, k)$ is semi-open and $W(n, k) \subset U(n, k + 1) \subset M_n$. Suppose that d is a point of G. Let

$$n_0 = \min\{n : d \in U(n, k), n \in \mathbb{N}, k \in \mathbb{N}\}, \quad k_0 = \min\{k : d \in U(n_0, k), k \in \mathbb{N}\}.$$

Then, $d \in U(n_0, k_0)$ and d is not in $U(n_0, k_0 - 1)$. Thus, d is not in $\mathrm{scl}U(n_0, k_0 - 2)$. Hence, $d \in U(n_0, k_0) - \mathrm{scl}U(n_0, k_0 - 2)$. Then, $d \in U(n_0, k_0) \cap (V_{k_0+1} - \mathrm{scl}V_{k_0-2})$ and $d \in W(n_0, k_0 - 1)$. Thus, $\mathscr{W} = \{W(n, k) : n \in \mathbb{N}, k \in \mathbb{N}\}$

is a semi-open cover of G and we will show that $\mathscr{W}$ is star-finite. Suppose $W(n,k) \in \mathscr{W}$. Then, $W(n,k) \subset U(n,k+1) \subset V_{k+1}$. Thus, $W(n,k) \cap (V_{k+3} - \mathrm{scl} V_{k+1}) = \varnothing$. Hence, $W(n,k) \cap (V_{k+3} - \mathrm{scl} V_{k+1}) \cap U(n,k+3) = \varnothing$ and $W(n,k) \cap W(n,k+2) = \varnothing$. Then, $W(n,k) \cap W(n,i) = \varnothing$ and $i \geq k+2$ for each $k \in \mathbb{N}$. Thus, $\mathscr{W}$ is a star-finite cover.

Since each star-finite semi-open cover has the semi-star property, it follows that there exists a finite set $W_0 = \{x_1, x_2, \cdots, x_t\}$ such that

$$X = St(W_0, \mathscr{W}) = St(x_1, \mathscr{W}) \cup \cdots \cup St(x_t, \mathscr{W}).$$

Since $\mathscr{W}$ is a star-finite cover, it follows that $\mathscr{W}$ is point-finite. Thus, $St(x_i, \mathscr{W})$ and $i = 1, \cdots, t$ is the union of finitely many members of $\mathscr{W}$. Let it be $\{W_{i_1}, \cdots, W_{i_t}\}$. Then, $\{W_{1_1}, \cdots, W_{1_t}, W_{2_1}, \cdots, W_{t_t}\}$ is a finite subcover of G. Therefore, G is semi-compact. □

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.5. *Suppose G is a locally σ semi-compact Ir-topological group and each semi-open set in G is pre-open. If $f : G \to X$ is an irresolute surjection, then X is semi-compact.*

Corollary 3.6. *Suppose G is a locally σ semi-compact Ir-topological group. If each semi-open set in G is pre-open, then each semi-closed set in G is semi-compact.*

Corollary 3.7. *Suppose G is a locally σ semi-compact Ir-topological group. If each semi-open set in G is pre-open, then G is semi-normal.*

Since each T_2 locally compact space is a Tychonoff space [20], we obtain the following theorem directly by Corollary 3.7.

Theorem 3.8. *Suppose G is an Ir-topological group, and $f : G \to Y$ is a semi-open irresolute bijection. If G is locally semi-compact and each semi-open set in G is pre-open, then Y is a Tychonoff space.*

Definition 3.9. [23] A subset A in a topological space X is said to be a *regular-open* set if and only if $A = \overline{A}^0$.

It is well known that, a set is regular open if and only if it is semi-closed and pre-open.

Theorem 3.10. *Suppose G is an Ir-topological group and each semi-open set in G is pre-open. If A is a locally semi-compact pre-open subgroup, then A is a regular-open Ir-topological subgroup.*

Proof. Let us show that A is Ir-topological group. Let $f : G \times G \to G$ and $f_1 : A \times A \to A$ be the multiplication mapping. Suppose (a, b) in $A \times A$ and U is semi-open containing $f_1(a \times b) = ab$ in $A \times A$. Since A is pre-open in G, Then, there exists a semi-open set A_0 in G such that $U = A \cap A_0$. Thus, $f^{-1}(A_0)$ is semi-open in $G \times G$. Hence, there exist two semi-open sets U_a and V_b in G such that $(a, b) \in U_a \times V_b \subset f^{-1}(A_0) \subset G \times G$. Hence, $U_a \cap A$ and $V_b \cap A$ are semi-open in A. Then, $f((U_a \cap A) \times (V_b \cap A)) = f_1((U_a \cap A) \times (V_b \cap A)) = (U_a \cap A)(V_b \cap A) \subset A \cap A_0 \subset U$. Thus, f_1 is irresolute.

Let $g : G \to G$ and $g_1 : A \to A$ be the inverse mapping. Suppose W is semi-open in A. Since A is pre-open in G, it follows that there exists a semi-open set W_0 in G. Then, $g^{-1}(W_0)$ is semi-open in G. Hence, $g^{-1}(W_0) \cap A = g_1^{-1}(W)$ is semi-open in A and g_1 is irresolute. Therefore, A is an Ir-topological group.

Now, we will show that A is semi-open in G. Suppose l in A. Since A is locally semi-compact, it follows that there exists an open semi-compact neighborhood L in A. According to Corollary 3.7, there exists a semi-open set M in A such that $l \in M \subset \mathrm{scl}_A M \subset L \subset A$, and $\mathrm{scl}_A M$ is semi-compact in A, where the set $\mathrm{scl}_A M$ is defined as the semi-closed set of M in A. Then, $\mathrm{scl}_A M$ is semi-compact of B. Since G is semi-normal, it follows that $\mathrm{scl}_A M$ is a semi-closed set in B. Since $M \subset \mathrm{scl}_A M$, it follows that $\mathrm{scl}_G M \subset \mathrm{scl}_A M$, where the set $\mathrm{scl}_G M$ is defined as the semi-closed set of M in G. Thus, $\mathrm{scl}_G M = \mathrm{scl}_A M$. Since A is pre-open, it follows that there exists a semi-open set

N in G such that $M = N \cap A$ and l in N. Hence, $\mathrm{scl}_G M \subset \mathrm{scl}_G N$. Suppose $m \in \mathrm{scl}_G N$ and O is a semi-open neighborhood of m in G. Then, $O \cap N \neq \varnothing$. Since each semi-open set in G is pre-open, it follows that $O \cap N$ is semi-open in G. Then, $O \cap N \cap A = O \cap M \neq \varnothing$ and m in $\mathrm{scl}_G M$. Hence, $\mathrm{scl}_G N \subset \mathrm{scl}_G M$. Thus,

$$\mathrm{scl}_G M = \mathrm{scl}_G N = \mathrm{scl}_A M.$$

Thus, $N \subset \mathrm{scl}_G N = \mathrm{scl}_A M \subset A$, and A is semi-open in G.

Since the left multiplication $l_x : G \to G$ is a pre-semi-open mapping, it follows that each left coset $l_x(A) = xA$ of A is semi-open. Let us show that $G - A = \cup_{x \in G-A} xA$. Obviously, $G - A \subset \cup_{x \in G-A} xA$. Suppose y in $\cup_{x \in G-A} xA$. Then, there exists c in A such that $y = xc$. Thus, c^{-1} in A. Assume that y is not in $G - A$. Then, $yc^{-1} = x$ in A, which contradicts the assumption, and y in $G - A$. Hence, $G - A = \cup_{x \in G-A} xA$ is semi-open and A is semi-closed. Thus, $A^{-0} \subset A$. Since A is pre-open, it follows that $A \subset A^{-0}$. Therefore, $A = A^{-0}$, and A is regular-open. □

Theorem 3.11. *Suppose G is a locally semi-compact Ir-topological group and each semi-open set is pre-open. If Y is an open subgroup of G, then Y is a locally semi-compact Ir-topological group.*

Proof. Suppose x is in G. According to Proposition 2.10, Yx is semi-open. Hence, $Y_0 = \cup_{x \in G-Y} Yx$ is semi-open. Then, $Y = G - Y_0$ is semi-closed.

Suppose a in Y. Then, there exists an open semi-compact neighborhood A containing a. Then, $A \cap Y$ is open in Y. Let us show that $A \cap Y$ is semi-closed in Y. Fix b in $G - A$. Then, there exist two semi-open sets, C_a and D_a, such that $a \in C_a$, $b \in D_a$, and $C_a \cap D_a = \varnothing$. Then, $\{C_a : a \in A\}$ is a semi-open cover of A. Thus, there exists a finite subcover $\{C_{a_1}, C_{a_2}, \cdots, C_{a_r}\}$ of A such that $A \subset \cup_{i=1}^{r} C_{a_i}$. Let $C = \cup_{i=1}^{r} C_{a_i}$ and $D = \cap_{i=1}^{r} D_{a_i}$. Since each semi-open set in G is pre-open, it follows that D is semi-open and $C \cap D = \varnothing$. Then, $A \cap D = \varnothing$. Thus, A is semi-closed in G. Since Y is semi-closed, it follows that $A \cap Y$ is semi-closed in G. Thus, $A \cap Y$ is semi-closed in Y.

Now we will show that $A \cap Y$ is semi-compact in A. Suppose $\mathscr{E} = \{E_\alpha : \alpha \in I\}$ is a semi-open cover of $A \cap Y$ in A. Then, there exists an open set F_α in $A \cap Y$ such that $F_\alpha \subset E_\alpha \subset \overline{F_\alpha}$. Then, there exists an open set G_α in A such that $F_\alpha = G_\alpha \cap A \cap Y$. Hence, $\overline{E_\alpha} \subset \overline{F_\alpha} \subset \overline{G_\alpha}$. Let $H_\alpha = E_\alpha \cup G_\alpha$. Then, $H_\alpha \cap A \cap Y = E_\alpha$ and $\overline{H_\alpha} = \overline{G_\alpha}$. Hence, H_α is semi-open in A. Then, $\{H_\alpha : \alpha \in I\} \cup \{A - A \cap Y\}$ is a semi-open cover of A. Thus, there exists a finite semi-open cover

$$\{H_{\alpha_1}, H_{\alpha_2}, \cdots, H_{\alpha_s}\} \cup \{A - A \cap Y\}.$$

Then, there exists a finite semi-open cover $\{H_{\alpha_1}, H_{\alpha_2}, \cdots, H_{\alpha_s}\}$ of $A \cap Y$, and $A \cap Y$ is semi-compact in A.

Suppose $\{U_\beta : \beta \in j\}$ is a semi-open cover of $A \cap Y$ in Y. Then, $\{U_\beta \cap A : \beta \in j\}$ is a semi-open cover of $A \cap Y$ in A. Thus, a finite semi-open cover $\{U_{\beta_1} \cap A, U_{\beta_2} \cap A, \cdots, U_{\beta_t} \cap A\}$ of $A \cap Y$ exists such that $A \cap Y \subset \cup_{i=1}^{t} U_{\beta_i} \cap A \subset \cup_{i=1}^{t} U_{\beta_i}$. Then, $A \cap Y$ is semi-compact in Y. Thus, Y is a locally semi-compact space.

Let $f_1 : Y \times Y \to Y$ and $f : G \times G \to G$ be the multiplication mapping. Let $g_1 : Y \to Y$ and $g : G \to G$ be the inverse mapping. Then, f_1 and g_1 are irresolute. The proof is similar to Theorem 3.10 and is omitted. Therefore, Y is a locally semi-compact Ir-topological group. □

We introduce some additional notations for brevity in the following theorems. In [25], the semi-quotient topology was introduced on s-topological groups and irresolute topological groups. This kind of construction will be applied here to topologized groups: Ir-topological groups. Suppose A is an invariant subgroup of an Ir-topological group G and each semi-open is pre-open. Let $G/A = \{gA : g \in G\}$ and $f : G \to G/A$, $g \to gA$ be the natural projection. Let

$$\mathscr{B} = \{B : B \subset G/A, f^{-1}(B) \text{ is semi open in } G\}.$$

Let $\mathscr{A} = \{\varnothing, G/A\} \cup \mathscr{B}$.

Now, we will show that $\mathscr{A}$ is a topology. It means that we only need to verify that the intersection of any two sets in $\mathscr{B}$ belongs to $\mathscr{B}$. Suppose B_1 and B_2 are subsets in $\mathscr{B}$. Then, $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are semi-open. Thus, there exists an open set C_1 such that $C_1 \subset f^{-1}(B_1) \subset \overline{C_1}$. Hence, $f^{-1}(B_2)$ is pre-open and $f^{-1}(B_2) \subset \overline{f^{-1}(B_2)}^0 \subset \overline{f^{-1}(B_2)}$. Then, $f^{-1}(B_1) \cap f^{-1}(B_2) \subset \overline{C_1} \cap \overline{f^{-1}(B_2)}^0$. Take an element x in $\overline{C_1} \cap \overline{f^{-1}(B_2)}^0$ and any neighborhood U_x of x. Thus, $U_x \cap \overline{f^{-1}(B_2)}^0$ is a neighborhood of x and $U_x \cap \overline{f^{-1}(B_2)}^0 \cap$

$C_1 \neq \emptyset$. Hence, x in $\overline{C_1 \cap \overline{f^{-1}(B_2)}^0}$ and $\overline{C_1} \cap \overline{f^{-1}(B_2)}^0 \subset \overline{C_1 \cap \overline{f^{-1}(B_2)}^0}$. Suppose $y \in C_1 \cap \overline{f^{-1}(B_2)}^0$. Then, y is not in $X - \overline{f^{-1}(B_2)}^0$ and there exists an open neighborhood V_y of y such that $V_y \cap (X - \overline{f^{-1}(B_2)}^0) = \emptyset$. Assume that y is not in $f^{-1}(B_2)$. Then, $y \in \overline{f^{-1}(B_2)} - f^{-1}(B_2) = Fr\{f^{-1}(B_2)\} \subset Fr\{\overline{f^{-1}(B_2)}^0\}$. Thus, $V_y \cap (X - \overline{f^{-1}(B_2)}^0) \neq \emptyset$. It is contradictory. Then, $y \in f^{-1}(B_2)$. Since $y \in C_1$, it follows that $y \in f^{-1}(B_1)$. Thus,

$$C_1 \cap \overline{f^{-1}(B_2)}^0 \subset f^{-1}(B_1) \cap f^{-1}(B_2) \subset \overline{\overline{f^{-1}(B_2)}^0 \cap C_1}.$$

Hence, $f^{-1}(B_1) \cap f^{-1}(B_2)$ is semi-open in G/A. Then, $B_1 \cap B_2 \in \mathscr{B}$ and $\mathscr{A}$ is a topology on G/A. We call it the Ir-s-quotient topology, and $(G/A, \mathscr{A})$ is called the Ir-s-quotient topological group.

The following example shows that $\mathscr{B}$ is different from the topology $\mathscr{C} = \{C : C \subset G/A, f^{-1}(B) \textit{ is open in } G\}$.

Example 3.12. Suppose $(X, \mathscr{T})$ is a topological space, $X = \{1, 4, 7\}$ and

$$\mathscr{T} = \{\emptyset, X, \{1\}, \{4\}, \{1, 4\}\}.$$

Then, the semi-open sets of X are $\mathscr{T}_1 = \{\emptyset, X, \{1\}, \{4\}, \{1, 4\}, \{1, 7\}, \{4, 7\}\}$. Let R be a relation from X to X defined by xRy if and only if $x \equiv y (\text{mod } 3)$ when x and y are in X. Then, R is an equivalence relation of X. Let $R(4) = \{4, 7\}$. Thus,$f^{-1}(R(4))$ is semi-open and is not open.

It is well known that if $f : G \to G/A, g \to gA$ be a natural projection mapping, then $f^{-1}(f(C)) = CA$ for each subset C in G [7], and f is irresolute. The next results are elementary, but important.

Proposition 3.13. *Suppose $(G, \mathscr{T})$ is an Ir-topological group and each semi-open set is pre-open. If A is an invariant subgroup, then $f : (G, \mathscr{T}) \to (G/A, \mathscr{A})$ is pre-semi-open.*

Proposition 3.14. *Suppose $(G, \mathscr{T})$ is an Ir-topological group and A is a subgroup. If $(G/A, \mathscr{A})$ is semi-T_1, then A is semi-closed in G.*

Theorem 3.15. *Suppose G is a locally semi-compact Ir-topological group and each semi-open set is pre-open. If A is an invariant subgroup, then $(G/A, \mathscr{A})$ is a locally semi-compact Ir-topological group.*

Proof. Let $f : G \to G/A, g \to gA$ and g be the multiplication mapping $(x, y) \to xy$ from $G \times G$ to G, and g_1 be the inverse mapping $g \to g^{-1}$ from G to G. Then, g and g_1 are irresolute. Let h be the multiplication mapping $(x, y) \to xy$ from $G/A \times G/A$ to G/A, and h_1 be the inverse mapping $g \to g^{-1}$ from G/A to G/A.

Since $f(g(x, y)) = h((f \times f)(x, y))$, it follows that $h((f \times f)(x, y))$ is irresolute. Assume that a semi-open set D in G/A such that $h^{-1}(D)$ is not semi-open. Then, $(h(f \times f))^{-1}(D) = (f \times f)^{-1}h^{-1}(D)$ is semi-open. According to Proposition 3.13, $f \times f$ is pre-semi-open. Then,

$$h^{-1}(D) = (f \times f)(f \times f)^{-1}h^{-1}(D)$$

is semi-open, which contradicts the assumption, and hence h is irresolute. In the same way, h_1 is irresolute. Thus, $(G/A, \mathscr{A})$ is an Ir-topological group.

Suppose aA in G/A. Then, there exists an open semi-compact neighborhood E of a such that aA in $f(E)$. Thus, $f(E)$ is open. Suppose $\mathscr{B}$ is a semi-open cover of $f(E)$. Hence, $f^{-1}(\mathscr{B})$ is a semi-open cover of E. Then, there exists a finite subcover $\{B_1, B_2, \cdots, B_r\}$, and $\{f(B_1), f(B_2), \cdots, f(B_r)\}$ is a semi-open cover of $f(E)$ by Proposition 3.13. Therefore, $(G/A, \mathscr{A})$ is a locally semi-compact Ir-topological group. □

4 Conclusion and future direction

In this article, we continued the study of the properties of Ir-topological groups, and some new properties have been obtained. At the same time, we establish the connections between locally semi-compact spaces and Ir-topological groups.

Several directions for future research are discussed below. For example, to obtain different types of topological groups in further research, we suggest adopting an Ir-paratopological group, which has a topology such that multiplication mapping is jointly irresolute instead of an Ir-topological group. The work initiated here is the starting point for continuing work towards that direction and motivating others to do so.

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