Research Article

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On Bohr's inequality for special subclasses of stable starlike harmonic mappings

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Abstract: The focus of this article is to explore the Bohr inequality for a specific subset of harmonic starlike mappings introduced by Ghosh and Vasudevarao (*Some basic properties of certain subclass of harmonic univalent functions*, Complex Var. Elliptic Equ. **63** (2018), no. 12, 1687–1703.). This set is denoted as $\mathcal{B}_H^0(M) := \{f = h + \overline{g} \in \mathcal{H}_0 : |zh''(z)| \le M - |zg''(z)|\}$ for $z \in \mathbb{D}$, where $0 < M \le 1$. It is worth mentioning that the functions belonging to the class $\mathcal{B}_H^0(M)$ are recognized for their stability as starlike harmonic mappings. With this in mind, this research has a twofold goal: first, to determine the optimal Bohr radius for this specific subclass of harmonic mappings, and second, to extend the Bohr-Rogosinski phenomenon to the same subclass.

Keywords: stable starlike harmonic mappings, Bohr radius, Bohr's inequality, Bohr-Rogosinski's inequalities

MSC 2020: 30C35, 30C45, 35Q30

1 Introduction

In recent years, Bohr's inequality and its various improved versions have once again sparked extensive research among many scholars. The classical statement of Bohr's inequality asserts that when the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfies $|f(z)| \le 1$ for all $z \in \mathbb{D}$, the classical form of the Bohr inequality is expressed as follows:

$$\sum_{n=1}^{\infty} |a_n| r^n \le 1, \tag{1.1}$$

for $|z| = r \le 1/3$, and the constant 1/3 cannot be improved. This inequality was defined by Bohr [1] in 1914. In fact, Bohr obtained the conclusion for $r \le 1/6$; later, Weiner, Riesz, and Schur have independently proved it to 1/3. Then, more and more theories about the Bohr radius and Bohr inequality have been extensively studied. For example, the Bohr phenomenon for certain close-to-convex analytic functions was established in [2], and Ahamed [3] extended this to the case of harmonic mappings. Kayumov and Ponnusamy [4] established a more precise version of the Bohr-type inequalities for bounded analytic functions. In their work, they introduced the concept of p-Bohr radius for the class of odd analytic functions and harmonic functions. Later, more and more scholars extended the analytical Bohr's inequality to the harmonic case [5–7]. Bohr's phenomenon in subordination and bounded harmonic classes has been widely researched [8,9]. In 2021, Ahamed and Allu [10] introduced the improved Bohr radius for the class of starlike log-harmonic mappings. For more interesting aspects of the Bohr phenomenon, we refer to [8,11–14] and references therein.

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Similar to the Bohr radius, there is also the notion of Rogosinski radius [15], which is defined as follows: if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function on $\mathbb{D}$ such that $|f(z)| < 1$ in $\mathbb{D}$, then for every $N \geq 1$, we have $|S_N(z)| = |\sum_{n=0}^{N} a_n z^n| < 1$ for $r < 1/2$, where 1/2 is the best possible quantity [15]. The number $r = 1/2$ is called the Rogosinski radius. And the Bohr-Rogosinski inequality, which is considered by Kayumov et al. [16], is given by:

$$R_N^f(z) = |f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq d(f(0), \partial f(\mathbb{D})),$$

for $|z| = r \leq R_N$; here, $d(f(0), \partial f(\mathbb{D}))$ denotes the Euclidean distance between $f(0)$ and the boundary of $f(\mathbb{D})$, where R_N is the positive root of the equation $2(1+r)r^N - (1-r)^2 = 0$. The aforementioned area of research is crucial for exploring the Bohr phenomenon. In recent years, there has been a notable surge of interest in extending the Bohr-Rogosinski inequality into the realm of harmonic mappings, as demonstrated by recent studies such as those referenced in [17]. Consequently, our objective is to delve deeper into this field, with a focus on the investigation of the specific Bohr-Rogosinski radius and the development of refined inequalities.

A complex-valued function $f = u + iv$ is said to be harmonic in a domain $\Omega \subset \mathbb{C}$ if u and v are real-valued harmonic functions in Ω. Let $\mathcal{H}$ denote the class of complex-valued harmonic functions f in the unit disk $\mathbb{D}$ normalized by $f(0) = 0 = f_z(0) - 1$. Each function $f \in \mathcal{H}$ can be expressed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ with the following power series representations:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

We call h and g the analytic and co-analytic parts of f, respectively. The Jacobian of $f = h + \overline{g}$ is defined as $J_f = |h'|^2 - |g'|^2$. Lewy's theorem [18] implies that a harmonic function f is locally univalent and sense-preserving if and only if $J_f > 0$ in $\mathbb{D}$. Note that $J_f(z) > 0$ in $\mathbb{D}$ is equivalent to the existence of an analytic function ω, called the (second complex) dilatation of f, given by $\omega(z) = g'(z)/h'(z)$, with $|\omega(z)| < 1$ for all $z \in \mathbb{D}$, where $h'(z) \neq 0$ in $\mathbb{D}$.

Let $\mathcal{S}_H$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving harmonic mappings on $\mathbb{D}$. This class was introduced and investigated by Clunie and Sheil-Small [19], the class $\mathcal{S}_H$ contains the standard class $\mathcal{S}$ of analytic univalent functions, and they showed that $\mathcal{S}_H$ is normal, but not compact. They also investigated the subclass $\mathcal{S}_H^0$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_H$ with $g'(0) = 0$. The series expansions of h and g for the subclass are as follows:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1.2}$$

We define the majorant series associated with $f = h + \overline{g}$ as:

$$\mathcal{M}_f(r) = \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n = r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n. \tag{1.3}$$

In [20], the authors introduced the following subclass of harmonic univalent mappings.

Definition 1. For $M > 0$, let

$$\mathcal{B}_H^0(M) := \{f = h + \overline{g} \in \mathcal{S}_H^0 : |zh''(z)| \leq M - |zg''(z)| \text{ for } z \in \mathbb{D}\}. \tag{1.4}$$

We consider Bohr's radius for the class of stable starlike harmonic mappings in $\mathcal{S}_H^0$, which were introduced in [21].

Definition 2. A (sense-preserving) harmonic mapping $f = h + \overline{g}$ is *stable starlike* (resp. *stable convex*) *harmonic* in the unit disk $\mathbb{D}$ if all the mappings $f_\lambda = h + \lambda g$ with $|\lambda| = 1$ are starlike (resp. *convex*) in $\mathbb{D}$.

The remainder of this article is organized as follows: Section 2 presents essential lemmas and conclusions required to establish our main conclusions. These primarily encompass the coefficient estimation theorem, the growth theorem, and the distortion theorem for the subclass $\mathcal{B}_H^0(M)$. Section 3 delves into the investigation of Bohr's phenomenon and the refinement of Bohr's inequalities within the specific subclass of harmonic mappings, while considering their stability conditions, as discussed in [22,23]. Additionally, this article explores Bohr-Rogosinski inequalities. The research results of this article provide a theoretical basis and new perspectives for further studying the properties of harmonic mappings.

2 Lemmas and preliminaries

It is well known that functions in the subclass $\mathcal{B}_H^0(M)$ are stable harmonic starlike with respect to origin for $M \le 1$ [4]. Furthermore, the forthcoming results concerning the coefficient estimation and the growth distortion theorem play a vital role in proving our main results.

Lemma 2.1. [20, Theorem 2.2] *Let $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ and be given by* (1.2). *Then, for $n \ge 2$,*

(1) $|a_n| \le \frac{M}{n(n-1)}$;

(2) $|b_n| \le \frac{M}{n(n-1)}$;

(3) $|a_n| + |b_n| \le \frac{M}{n(n-1)}$;

(4) $|a_n| - |b_n| \le \frac{M}{n(n-1)}$.

All inequalities are sharp.

Lemma 2.2. [20, Theorem 2.3] *Let $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ and be given by* (1.2). *Then,*

$$|z| - \frac{M}{2}|z|^2 \le |f(z)| \le |z| + \frac{M}{2}|z|^2. \tag{2.1}$$

Both inequalities are sharp for the function f_M given by $f_M(z) = z + \frac{M}{n(n-1)}z^n$.

The following result gives the sharp upper bound of J_f for functions f in the class $B_H^0(M)$.

Theorem A. [20, Theorem 2.5] *Let $f = h + \overline{g} \in \mathcal{B}_H^0(M)$, then $J_f(z) \le (1 + M|z|)^2$, with equality for the function $f(z) = z + \frac{M}{2}z^2$.*

In [23], the authors obtained the Bohr radius for stable convex harmonic mappings.

Theorem B. [23, Theorem 2.1.1(i)] *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be a stable convex harmonic mapping on the unit disk $\mathbb{D}$. Then,*

$$\mathcal{M}_f(r) \le d(f(0), \partial f(\mathbb{D})), \tag{2.2}$$

if $|z| \le r_0 = 1/3$, where r_0 is the unique root in $(0, 1)$ of

$$\frac{r}{1-r} = \frac{1}{2}.$$

Theorem C. [23, Theorem 2.1.2(i)] *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ be a stable convex harmonic mapping on the unit disk $\mathbb{D}$, and let S_r be the area of the image $f(D_r)$, with $D_r = |z| = r$. Then,*

$$\mathcal{M}_f(r) + \left(\frac{S_r}{\pi}\right)^k \le d(f(0), \partial f(\mathbb{D})), \tag{2.3}$$

if $|z| \leq r_0$, *where* r_0 *is the unique root in* $(0,1)$ *of*

$$\frac{r}{1-r} + \frac{r^{2k}}{(1-r^2)^{2k}} = \frac{1}{2}. \tag{2.4}$$

3 Main results

Before proving the main results of this article, initially, we recall the definition of dilogarithm [24]. The polylogarithm function $Li_k(z)$ is defined for $|z| < 1$ by:

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

and by analytic continuation to the entire complex plane. In particular, the Euler dilogarithm function $\mathrm{Li}_2(z)$ is defined for $|z| < 1$ by:

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-t)}{t} \mathrm{d}t.$$

Next, we will find the minimum Euclidean distance between $f(0)$ and the boundary of $f(\mathbb{D})$.

For $f = h + \overline{g} \in \mathcal{B}_H^0(M)$, by Lemma 2.2, we have

$$|f(z)| \geq |z| - \frac{M}{2}|z|^2.$$

Then, after simple calculation, we can obtain

$$d(f(0), \partial f(\mathbb{D})) = \left| \lim_{|z|\to 1} |f(z)| - |f(0)| \right| \geq \left| \lim_{|z|\to 1} |z| - \frac{M}{2}|z|^2 \right| = 1 - \frac{M}{2}. \tag{3.1}$$

In this section, we obtain the Bohr-type inequality and a series of improved Bohr inequalities for the functions in the subclass $\mathcal{B}_H^0(M)$. First, we obtain the Bohr radius for stable starlike harmonic mappings.

Theorem 3.1. *Let* $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ *be a stable starlike harmonic mapping with* $0 < M \leq 1$ *on the unit disk* $\mathbb{D}$. *Then,*

$$\mathcal{M}_f(r) \leq d(f(0), \partial f(\mathbb{D})),$$

for $|z| = r \leq r_M$, *where* r_M *is the unique root of the equation*:

$$r + M(r + (1-r)\log(1-r)) - 1 + \frac{M}{2} = 0 \tag{3.2}$$

in $(0,1)$. *The radius* r_M *is the best possible.*

Proof. Let $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ be given by (1.4). For $|z| = r$, by Lemma 2.1 (3) and (3.1), we obtain

$$\begin{aligned}
\mathcal{M}_f(r) &= r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \\
&\leq r + \sum_{n=2}^{\infty} \frac{M}{n(n-1)} r^n \\
&= r + M(r + (1-r)\log(1-r)) \\
&\leq d(f(0), \partial f(\mathbb{D})).
\end{aligned}$$

Hence, we have

$$\mathcal{M}_f(r) \leq d(f(0), \partial f(\mathbb{D})),$$

if $|z| \leq r_M$, where r_M is the unique root of the equation:

$$r + M(r + (1-r)\log(1-r)) - 1 + \frac{M}{2} = 0.$$

In order to show that r_M is the best possible radius, we consider the following function $f = f_M$, which is defined by:

$$f_M(z) = z + \frac{M}{n(n-1)}z^n. \tag{3.3}$$

For $|z| = r_M$ and the function $f = f_M$, a simple computation shows that

$$\begin{aligned}\mathcal{M}_f(r) &= |z| + \sum_{n=2}^{\infty}(|a_n| + |b_n|)|z|^n \\ &= r_M + M(r_M + (1-r_M)\log(1-r_M)) \\ &= 1 - \frac{M}{2} \\ &= d(f(0), \partial f(\mathbb{D})).\end{aligned}$$

Hence, the radius r_M is the best possible. This completes the proof. □

Remark 1. The values in Table 1 correspond to the Bohr radius r_M for various values of M in Theorem 3.1. Based on the information presented in Figure 1, we can infer that r_M falls in the range $0.404289 \leq r_M < 1$ for $0 < M \leq 1$.

We will now derive the refined Bohr-Rogosinski inequality for functions belonging to the subclass $\mathcal{B}_H^0(M)$.

Theorem 3.2. *Let $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ be a stable starlike harmonic mapping on the unit disk $\mathbb{D}$, with $0 < M \leq 1$. Then,*

$$|f(z^m)| + \sum_{n=N}^{\infty}(|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(\mathbb{D})),$$

Table 1: Roots r_M of (3.2) for different values of M

M	0.01	0.1	0.2	0.3	0.5	0.8	1.0
r_M	0.985748	0.886128	0.803307	0.735056	0.622586	0.48526	0.404289

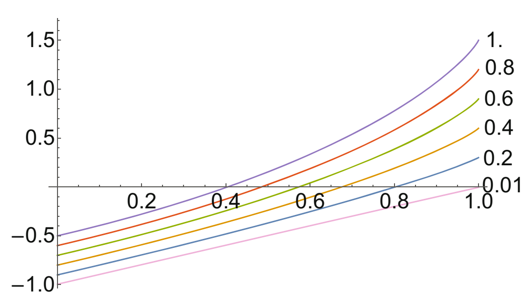

Figure 1: Graph of r_M when M = 0.01, 0.2, 0.4, 0.6, 0.8, and 1.0.

for $|z| = r \le r^N_{m,M}$, *where* $r^N_{m,M}$ *is the unique root of the equation*:

$$r^m + \frac{M}{2}r^{2m} + \sum_{n=N}^{\infty}\frac{M}{n(n-1)}r^n - 1 + \frac{M}{2} = 0 \tag{3.4}$$

in $(0,1)$. *The radius* $r^N_{m,M}$ *is the best possible.*

Proof. Since $f \in \mathcal{B}^0_H(M)$, then by Lemmas 2.1 and 2.2, we can obtain

$$|f(z^m)| + \sum_{n=N}^{\infty}(|a_n| + |b_n|)|z|^n \le r^m\left(1 + \frac{M}{2}r^m\right) + \sum_{n=N}^{\infty}\frac{M}{n(n-1)}r^n.$$

Let $H_1(r) : [0,1] \to \mathbb{R}$ be defined by

$$H_1(r) \coloneqq r^m + \frac{M}{2}r^{2m} + \sum_{n=N}^{\infty}\frac{M}{n(n-1)}r^n - 1 + \frac{M}{2}.$$

It is easy to see that $H_1(r)$ is continuous on $[0,1]$ and differentiable on $(0,1)$. A straightforward calculation reveals that

$$H_1(1) = 1 + \frac{M}{2} + \frac{M}{N-1} - 1 + \frac{M}{2} = \frac{MN}{N-1} > 0, \quad (N \ge 2).$$

Furthermore, we observe that

$$H_1(0) = -1 + \frac{M}{2} < 0.$$

It is evident that $H_1(1)H_1(0) < 0$, and thus, according to the intermediate value theorem, $H_1(r)$ has at least one root in $(0,1)$. We now show that $H_1(r)$ has exactly one root in $(0,1)$. Performing a direct computation yields

$$\frac{\mathrm{d}}{\mathrm{d}r}(H_1(r)) = mr^{m-1} + mMr^{2m-1} + \sum_{n=N}^{\infty}\frac{M}{n-1}r^{n-1} > 0,$$

for all $r \in (0,1)$. Consequently, $H_1(r)$ strictly increases over $(0,1)$. Therefore, $H_1(r)$ has the unique root in $(0,1)$.

To show that $r^N_{m,M}$ is the best possible, we consider the function $f = f_M$ defined by (3.3). In view of (3.1), for $f = f_M$ and $|z| = r^N_{m,M}$, we obtain

$$\begin{aligned}|f_M(z^m)| + \sum_{n=N}^{\infty}(|a_n| + |b_n|)|z|^n &= (r^N_{m,M})^m + \frac{M}{2}(r^N_{m,M})^{2m} + \sum_{n=N}^{\infty}\frac{M}{n(n-1)}(r^N_{m,M})^n \\ &= 1 - \frac{M}{2} \\ &= d(f(0), \partial f(\mathbb{D})).\end{aligned}$$

This implies that $r^N_{m,M}$ is the best possible. Thus, the proof is complete. □

Let $m = 1$ in Theorem 3.2, we have the following result.

Corollary 3.3. *Let* $f = h + \overline{g} \in \mathcal{B}^0_H(M)$ *be a stable starlike harmonic mapping on the unit disk* $\mathbb{D}$, *with* $0 < M \le 1$. *Then,*

$$|f(z)| + \sum_{n=N}^{\infty}(|a_n| + |b_n|)|z|^n \le d(f(0), \partial f(\mathbb{D})),$$

for $|z| = r \le r_{1,M}$, *where* $r_{1,M}$ *is the unique root of the equation*:

$$r + \frac{M}{2}r^2 + \sum_{n=N}^{\infty}\frac{M}{n(n-1)}r^n - 1 + \frac{M}{2} = 0. \tag{3.5}$$

Remark 2. The aforementioned equation is the sharp Bohr-Rogosinski inequality and sharp Bohr-Rogosinski radius as well known. For certain specific values of M and m, a straightforward computation yields the Bohr-Rogosinski radius $r^N_{m,M}$ as presented in Table 2. Figure 2 illustrates that the roots $r^2_{m,M}$ of (3.4) exceed 0.360456 when $N = 2$ and $m = 1$.

Theorem 3.4. *Let $f = h + \overline{g} \in \mathcal{B}^0_H(M)$ be a stable starlike harmonic mapping with $0 < M \leq 1$ on the unit disk $\mathbb{D}$, and let S_r be the area of the image $f(\mathbb{D}_r)$, with $\mathbb{D}_r = |z| = r$. Then,*

$$\mathcal{M}_f(r) + \left(\frac{S_r}{\pi}\right)^k \leq d(f(0), \partial f(\mathbb{D})),$$

for $|z| = r \leq r_{k,M}$, where $r_{k,M}$ is the unique root of the equation:

$$r + M(r + (1-r)\log(1-r))(r^2 + M^2(r^2 \operatorname{Li}_2(r^2) - r^2 + r^2\log(1-r^2) - \log(1-r^2)))^k - 1 + \frac{M}{2} = 0 \tag{3.6}$$

in $(0,1)$. The radius $r_{k,M}$ is the best possible.

Proof. First, we need to find the bound on S_r in the case where f is stable starlike harmonic. For $f = h + \overline{g}$ given by (1.4), it is well known that

$$\frac{S_r}{\pi} = \frac{1}{\pi}\iint_D J_f \, dA = \sum_{n=1}^{\infty} n(|a_n| - |b_n|)(|a_n| + |b_n|)r^{2n} = r^2 + \sum_{n=2}^{\infty} n(|a_n| - |b_n|)(|a_n| + |b_n|)r^{2n}.$$

Since $f \in \mathcal{B}^0_H(M)$, then by Lemma 2.1, we obtain that

$$\frac{S_r}{\pi} \leq r^2 + \sum_{n=2}^{\infty} \frac{M^2}{n(n-1)^2} r^{2n}.$$

Therefore, it follows that

$$\begin{aligned}
\mathcal{M}_f(r) + \left(\frac{S_r}{\pi}\right)^k &= r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n + \left(r^2 + \sum_{n=2}^{\infty} n(|a_n| - |b_n|)(|a_n| + |b_n|)r^{2n}\right)^k \\
&\leq r + \sum_{n=2}^{\infty} \frac{M}{n(n-1)} r^n + \left(r^2 + \sum_{n=2}^{\infty} \frac{M^2}{n(n-1)^2} r^{2n}\right)^k \\
&= r + M(r + (1-r)\log(1-r)) + (r^2 + M^2(r^2 Li_2(r^2) - r^2 - (1-r^2)\log(1-r^2)))^k \\
&\leq d(f(0), \partial f(\mathbb{D})).
\end{aligned}$$

Now, we consider the function $H_2(r) : [0,1] \to \mathbb{R}$, where $H_2(r)$ is defined by:

$$H_2(r) := r + \sum_{n=2}^{\infty} \frac{M}{n(n-1)} r^n + \left(r^2 + \sum_{n=2}^{\infty} \frac{M^2}{n(n-1)^2} r^{2n}\right)^k - 1 + \frac{M}{2}.$$

Table 2: Roots $r^N_{m,M}$ of (3.4) for different values of N and (m, M)

(m, M)	(1,0.2)	(1,0.4)	(1,0.6)	(1,0.8)	(1,1)
$r^2_{m,M}$	0.759124	0.621340	0.519008	0.434651	0.360456
(m, M)	(1,0.2)	(1,0.4)	(1,0.6)	(1,0.8)	(1,1)
$r^5_{m,M}$	0.822941	0.696518	0.591388	0.498657	0.413617
(m, M)	(2,0.2)	(2,0.4)	(2,0.6)	(2,0.8)	(2,1)
$r^2_{m,M}$	0.855228	0.755342	0.673436	0.600602	0.532128
(m, M)	(2,0.2)	(2,0.4)	(2,0.6)	(2,0.8)	(2,1)
$r^5_{m,M}$	0.902969	0.828416	0.762589	0.700316	0.638296

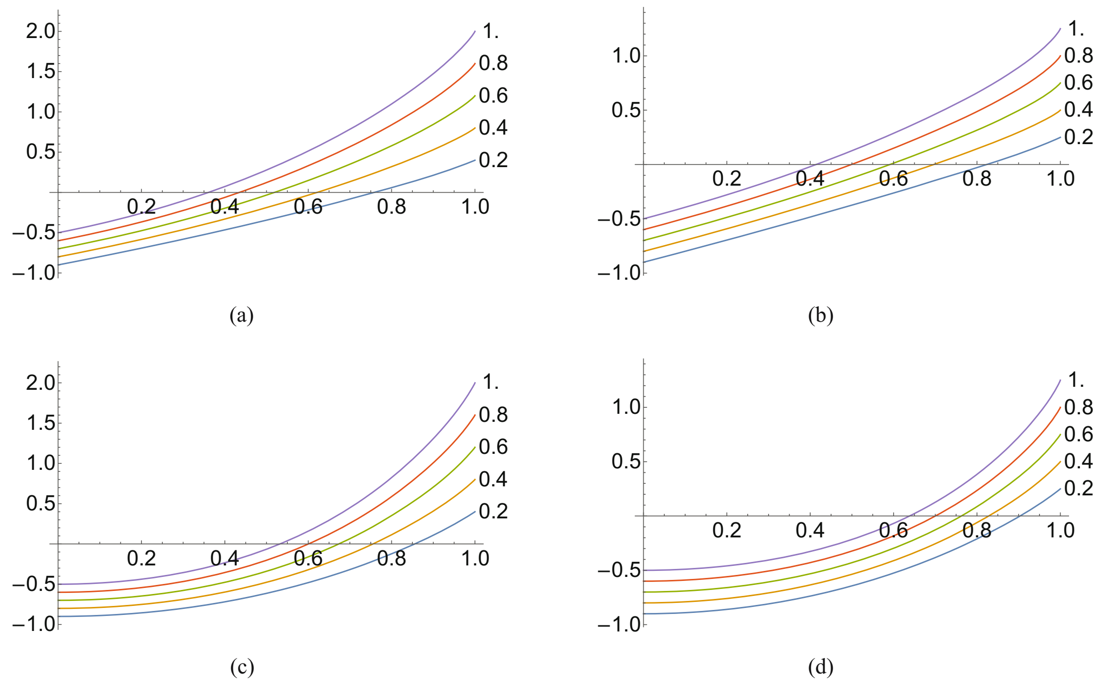

Figure 2: Graph of particular $r_{m,M}^N$ in (3.4) when $M = 0.2, 0.4, 0.6, 0.8$, and 1.0. (a) $m = 1, N = 2$. (b) $m = 1, N = 5$. (c) $m = 2, N = 2$. (d) $m = 1, N = 5$.

Using the method similar to Theorem 3.2, we obtain that $H_2(0)H_2(1) < 0$ and $\frac{\mathrm{d}}{\mathrm{d}r}(H_2(r)) > 0$ in $(0, 1)$. As a result, the function $H_2(r)$ contains exactly one root in $(0, 1)$, denote it as $r_{k,M}$. Therefore, $H_2(r_{k,M}) = 0$, and for all $0 < r < r_{k,M}$, $H_2(r) < 0$.

To prove that $r_{k,M}$ is the best possible, we consider the function $f = f_M$ defined by (3.3). Using (3.1) for $f = f_M$ and $r > r_{k,M}$, it is easy to see that

$$\begin{aligned} & r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n + \left(r^2 + \sum_{n=2}^{\infty} n(|a_n| - |b_n|)(|a_n| + |b_n|)r^{2n}\right)^k \\ > \; & r_{k,M} + \sum_{n=2}^{\infty}\frac{M}{n(n-1)}(r_{k,M})^n + \left((r_{k,M})^2 + \sum_{n=2}^{\infty}\frac{M^2}{n(n-1)^2}(r_{k,M})^{2n}\right)^k \\ = \; & 1 - \frac{M}{2} = d(f(0), \partial f(\mathbb{D})), \end{aligned}$$

which shows that $r_{k,M}$ is best possible. This completes the proof of Theorem 3.4. □

Remark 3. Table 3 and Figure 3 present the values of the Bohr radius $r_{k,M}$ associated with various values of k and M in Theorem 3.4. If $k \to \infty$ and $M \to 0$, clearly, $r_{k,M} \to 1$. Consequently, we can discern that the radius lies in the range $0.326888 \le r_{k,M} < 1$ for varying values of $0 < M \le 1$ and non-negative integer k.

Theorem 3.5. *Let* $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ *be given by* (1.4) *with* $0 < M \le 1$. *Then,*

$$\mathcal{M}_f(r) + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)r^n \le d(f(0), \partial f(\mathbb{D})),$$

for $|z| = r \le r_M$, *where* r_M *is the unique root of the equation:*

$$2r + M(r + (1-r)\log(1-r)) + 2M^2(r\,\mathrm{Li}_2(r) + \mathrm{Li}_2(r) - 3r + 2r\log(1-r) - 2\log(1-r)) = 1 - \frac{M}{2} \tag{3.7}$$

in $(0, 1)$. *The radius* r_M *is the best possible.*

Table 3: Roots $r_{k,M}$ of (3.6) for different values of k and M

M	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
$r_{1,M}$	0.553262	0.492872	0.435728	0.380841	0.326888
$r_{2,M}$	0.654969	0.586322	0.519333	0.453293	0.386992
$r_{5,M}$	0.754437	0.662210	0.570533	0.484580	0.404174
$r_{10,M}$	0.794532	0.675369	0.573788	0.485259	0.404289

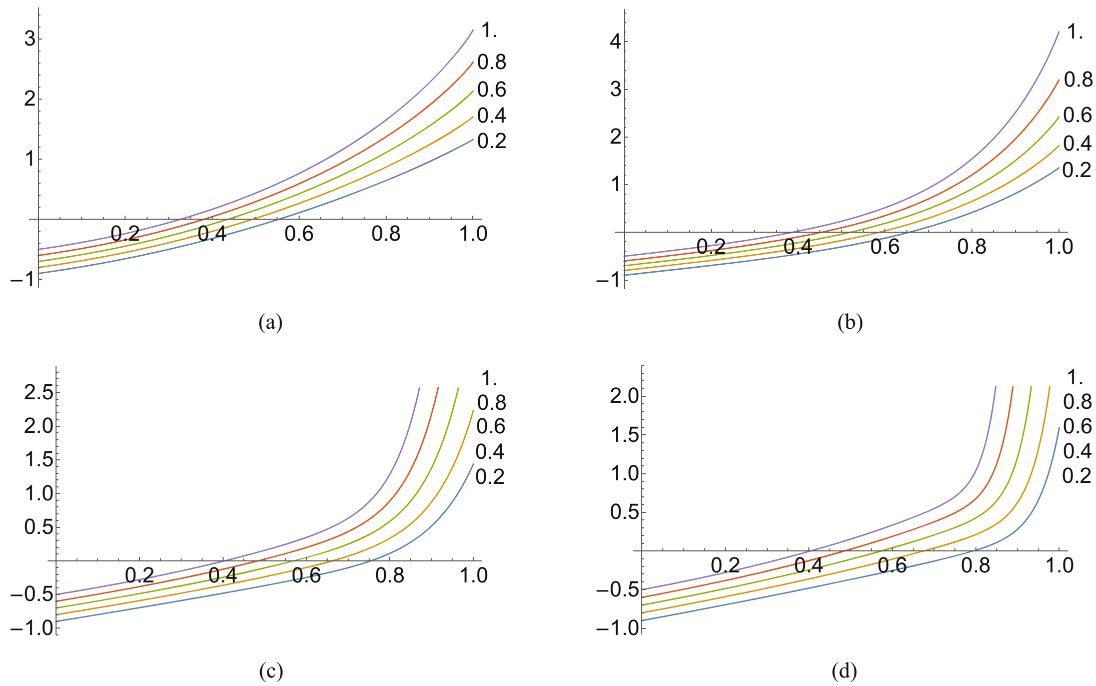

Figure 3: Graph of particular $r_{k,M}$ of (3.6) when $M = 0.2, 0.4, 0.6, 0.8$, and 1.0. (a) $k = 1$, (b) $k = 2$, (c) $k = 5$, and (d) $k = 10$.

Proof. Let $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ be given by (1.4). In view of Lemmas 2.1 and 2.2, we have

$$|a_n| \le \frac{M}{n(n-1)}, \quad |b_n| \le \frac{M}{n(n-1)}$$

and

$$|f(z)| \le |z| + \frac{M}{2}|z|^2.$$

Now, we have

$$\begin{aligned} \mathcal{M}_f(r) + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)r^n &= 2r + \sum_{n=2}^{\infty}(|a_n| + |b_n| + |a_n|^2 + |b_n|^2)r^n \\ &\le 2r + \sum_{n=2}^{\infty}\left(\frac{M}{n(n-1)} + \frac{2M^2}{n^2(n-1)^2}\right)r^n \\ &= 2r + M(r + (1-r)\log(1-r)) + 2M^2(r\operatorname{Li}_2(r) + \operatorname{Li}_2(r) - 3r \\ &\quad + 2r\log(1-r) - 2\log(1-r)) \\ &\le d(f(0), \partial f(\mathbb{D})). \end{aligned} \tag{3.8}$$

Hence, it is easy to see that

$$\mathcal{M}_f(r) + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)r^n \le d(f(0), \partial f(\mathbb{D})),$$

for $r \le r_M$, where r_M is the unique root of (3.7) in $(0, 1)$.

To show the sharpness of the radius r_M, we consider the function $f = f_M$ defined by (3.3). For $|z| = r_M$, we write that

$$\begin{aligned}\mathcal{M}_f(r) + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)r^n &= 2r_M + \sum_{n=2}^{\infty}\left(\frac{M}{n(n-1)} + \frac{2M^2}{n^2(n-1)^2}\right)r_M^n\\ &= 2r_M + M(r_M + (1 - r_M)\log(1 - r_M)) + 2M^2(r_M \operatorname{Li}_2(r_M) + \operatorname{Li}_2(r_M) - 3r_M\\ &\quad + 2r_M\log(1 - r_M) - 2\log(1 - r_M))\\ &= 1 - \frac{M}{2} = d(f(0), \partial f(\mathbb{D})),\end{aligned}$$

and hence, the radius r_M is the best possible. This completes the proof. □

Remark 4. Table 4 provides the values of the Bohr radius r_M corresponding to varying values of M as outlined in Theorem 3.5. As depicted in Figure 4, we discern that the radius lies in the interval $0.223616 \le r_M < \frac{1}{2}$ for different values of $0 < M \le 1$.

Theorem 3.6. *Let $f = h + \overline{g} \in \mathcal{B}^0_{\mathcal{H}}(M)$ be given by (1.4) for $0 < M \le 1$. Then, for any integer $p \ge 1$,*

$$|f(z)|^p + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n + \sum_{n=2}^{\infty}(|a_n| + |b_n|)^2 r^{2n} \le d(f(0), \partial f(\mathbb{D})),$$

for $|z| = r \le r_{p,M}$, where $r_{p,M}$ is the unique root of

$$\left(r + \frac{M}{2}r^2\right)^p + M(r - (r-1)\log(1-r)) + M^2(-3r^2 + 2(r^2 - 1)\log(1 - r^2) + (1 + r^2)\operatorname{Li}_2(r^2)) = 1 - \frac{M}{2} \tag{3.9}$$

in $(0, 1)$. Here, $r_{p,M}$ is best possible.

Table 4: Roots r_M of (3.7) for different values of M

M	**0.01**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
r_M	0.496738	0.436652	0.377543	0.322634	0.271524	0.223616

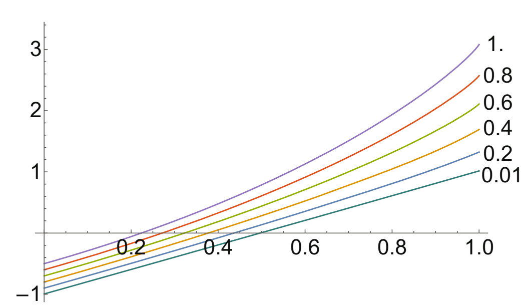

Figure 4: Graph of r_M when M = 0.01, 0.2, 0.4, 0.6, 0.8, and 1.0.

Proof. According to Lemma 2.1, we have that

$$(|a_n| + |b_n|)^2 \le \frac{M^2}{n^2(n-1)^2} \quad \text{and} \quad |f(z)| \le |z| + \frac{M}{2}|z^2|.$$

It is obvious that

$$\begin{aligned} &|f(z)|^p + \sum_{n=2}^{\infty}(|a_n| + |b_n|)|z|^n + \sum_{n=2}^{\infty}(|a_n| + |b_n|)^2 r^{2n} \\ &\le \left(r + \frac{M}{2}r^2\right)^p + \sum_{n=2}^{\infty}\frac{M}{n(n-1)}r^n + \sum_{n=2}^{\infty}\frac{M^2}{n^2(n-1)^2}r^{2n} \\ &= \left(r + \frac{M}{2}r^2\right)^p + M(r-(r-1)\log(1-r)) + M^2(-3r^2 + 2(r^2-1)\log(1-r^2) + (1+r^2)\operatorname{Li}_2(r^2)) \\ &\le d(f(0), \partial f(\mathbb{D})). \end{aligned}$$

Hence, we have

$$|f(z)|^p + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n + \sum_{n=2}^{\infty}(|a_n| + |b_n|)^2 r^{2n} \le d(f(0), \partial f(\mathbb{D})),$$

if $|z| \le r_{p,M}$, where $r_{p,M}$ is the unique root of equation (3.9). Moreover, let $H_3(r) : [0,1] \to \mathbb{R}$, where $H_3(r)$ is defined by:

$$H_3(r) := \left(r + \frac{M}{2}r^2\right)^p + \sum_{n=2}^{\infty}\frac{M}{n(n-1)}r^n + \sum_{n=2}^{\infty}\frac{M^2}{n^2(n-1)^2}r^{2n}.$$

By the similar argument being used in the proof of the previous theorems, it is easy to show that $H_3(0)H_3(1) < 0$ and $\frac{d}{dr}(H_3(r)) > 0$ in $(0,1)$. Hence, $H_3(r)$ being continuous and monotone increasing, $r_{p,M}$ is the unique root of the equation $H_3(r) = 0$ in $(0,1)$.

Thus, we have

$$\begin{aligned} &\left(r_{p,M} + \frac{M}{2}(r_{p,M})^2\right)^p + \sum_{n=2}^{\infty}\frac{M}{n(n-1)}(r_{p,M})^n + \sum_{n=2}^{\infty}\frac{M^2}{n^2(n-1)^2}(r_{p,M})^{2n} \\ &= \left(r_{p,M} + \frac{M}{2}(r_{p,M})^2\right)^p + M(r_{p,M} - (r_{p,M}-1)\log(1-r_{p,M})) \\ &\quad + M^2(-3(r_{p,M})^2 + 2((r_{p,M})^2-1)\log(1-(r_{p,M})^2) + (1+(r_{p,M})^2)\operatorname{Li}_2((r_{p,M})^2)) = 1 - \frac{M}{2}. \end{aligned} \tag{3.10}$$

Consider the function $f = f_M$ defined by (3.3). Using (3.1) and (3.10) for $f = f_M$ and $r > r_{p,M}$, it can be shown that

$$|f(r_{p,M})|^p + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n + \sum_{n=2}^{\infty}(|a_n| + |b_n|)^2 r^{2n} > d(f(0), \partial f(\mathbb{D})),$$

which shows that $r_{p,M}$ is best possible. This completes the proof. □

Remark 5. While examining the roots $r_{p,M}$ of equation (3.9), several intriguing observations come to light.
(1) Table 5 shows radius data for some specific values of M and p in Theorem (3.6).
(2) Figure 5 clearly illustrates different situations of $r_{p,M}$.
(3) When $p = 1$ and $M = 1$, we obtain that $r_{min} = 0.358143$. Moreover, when $p \to \infty$ and $M \to 0$, we obtain that $r_{p,M} \to 1$.

Theorem 3.7. *Let* $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ *be given by* (1.4) *with* $0 < M \le 1$. *Then,*

$$\mathcal{M}_f(r) + \sqrt{J_f(z)}\,|z| \le d(f(0), \partial f(\mathbb{D})),$$

Table 5: Roots $r_{p,M}$ of (3.6) for different values of k and M

M	0.2	0.4	0.6	0.8	1.0
$r_{1,M}$	0.756680	0.617611	0.515262	0.431503	0.358143
$r_{5,M}$	0.873153	0.781823	0.708167	0.644363	0.585636
$r_{10,M}$	0.893369	0.814365	0.749851	0.693220	0.639704
$r_{20,M}$	0.904360	0.833072	0.755118	0.724470	0.675943

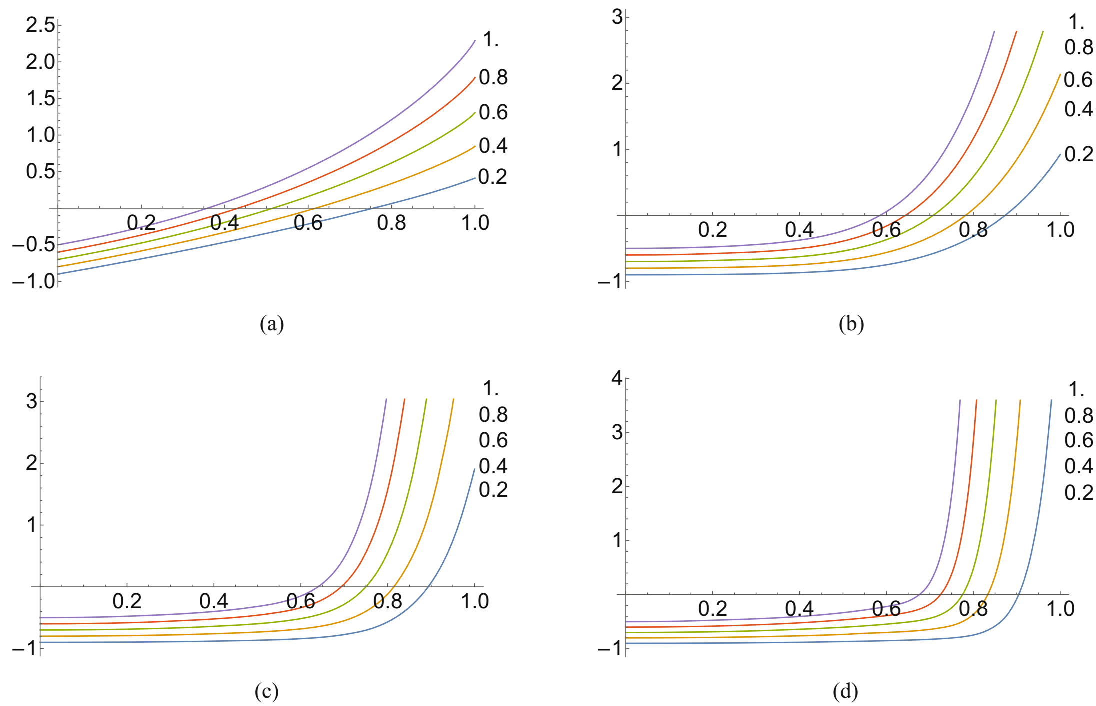

Figure 5: Graph of particular $r_{p,M}$ in Remark 5: (a) $p = 1$, (b) $p = 5$, (c) $p = 10$, and (d) $p = 20$.

for $|z| = r \leq r_M$, *where* r_M *is the unique root of the equation*:

$$r + M(r + (1 - r)\log(1 - r)) + r(1 + Mr) = 1 - \frac{M}{2} \tag{3.11}$$

in $(0, 1)$. *The radius* r_M *is the best possible.*

Proof. According to Theorem A, if $f = h + \overline{g} \in \mathcal{B}_H^0(M)$ with equality for the function $f(z) = z + \frac{M}{2}z^2$, then

$$\sqrt{J_f(z)} \leq 1 + M|z|. \tag{3.12}$$

Therefore, considering Lemmas 2.1 and 2.2, along with equation (3.12), this leads to

$$\begin{aligned} \mathcal{M}_f(r) + \sqrt{J_f(z)}\,|z| &\leq r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n + |z|(1 + M|z|) \\ &\leq r + \sum_{n=2}^{\infty}\frac{M}{n(n-1)}r^n + r(1 + Mr) \\ &= r + M(r + (1 - r)\log(1 - r)) + r(1 + Mr). \end{aligned} \tag{3.13}$$

Table 6: Roots r of (3.11) for different values of M

M	0.01	0.2	0.4	0.6	0.8	1.0
r_M	0.495521	0.421716	0.359355	0.306125	0.258559	0.214551

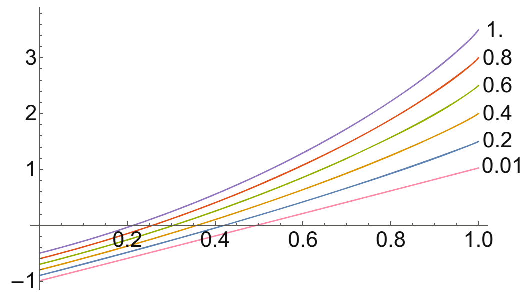

Figure 6: Graph of r_M when $M = 0.01, 0.2, 0.4, 0.6, 0.8$, and 1.0.

It is easy to see that $M_f(r) + \sqrt{J_f(z)}\,|z| \le d(f(0), \partial f(\mathbb{D}))$ holds for $|z| = r \le r_M$, where r_M is the unique root of the equation in (3.11).

For the function $f = f_M$ and $|z| = r_M$, a simple computation using (3.1) shows that

$$\begin{aligned} r_M + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r_M^n + |z|(1 + M|z|) &= r_M + \sum_{n=2}^{\infty}\frac{M}{n(n-1)}r_M^n + r_M(1 + Mr_M) \\ &= r_M + M(r_M + (1 - r_M)\log(1 - r_M)) + r_M(1 + Mr_M) \\ &= 1 - \frac{M}{2} = d(f(0), \partial f(\mathbb{D})). \end{aligned}$$

Therefore, the radius r_M is the best possible. This completes the proof. □

Remark 6. Table 6 presents the values of the Bohr radius r_M for various values of M as specified in Theorem 3.7. As depicted in Figure 6, it is evident that the radius falls in the range $0.214551 \le r_M < \frac{1}{2}$ for different values of $0 < M \le 1$.

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