Research Article

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On Graham partitions twisted by the Legendre symbol

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Abstract: We investigate when there is a partition of a positive integer n,

$$n = f(\lambda_1) + f(\lambda_2) + \cdots + f(\lambda_\ell),$$

satisfying that

$$1 = \frac{\chi_p(\lambda_1)}{\lambda_1} + \frac{\chi_p(\lambda_2)}{\lambda_2} + \cdots + \frac{\chi_p(\lambda_\ell)}{\lambda_\ell},$$

where χ_p is the Legendre symbol modulo prime p and $f(k) = k$ or the kth m-gonal number with $m = 3, 4$, or 5.

Keywords: Graham partition, sum of reciprocals, quadratic twist, Legendre symbol, polygonal number

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1 Introduction

In a seminal article [1], Graham initiated a quest to search for a partition of a positive integer n,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell,$$

satisfying that

$$1 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_\ell}.$$

In a recent article by Kim et al. [2] in honor of Graham, we call such a partition a *Graham partition*. For example, (12, 6, 4, 2) is a Graham partition of 24 as

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$$24 = 12 + 6 + 4 + 2 \quad \text{and} \quad 1 = \frac{1}{12} + \frac{1}{6} + \frac{1}{4} + \frac{1}{2}.$$

In 1963, Graham [1] proved that there is a Graham partition of n into distinct parts, provided that $n \geq 78$. Though there is no written proof of the ordinary partition version, by adopting Graham's idea, one can easily prove that there is a Graham partition of a positive integer n for $n \geq 24$. The number of ordinary Graham partitions is listed at OEIS [3].

A common theme in the theory of q-series and partitions considers a weight by a quadratic character. For example, due to a connection with the Rogers-Ramanujan continued fraction, Berndt and Sohn [4] studied the asymptotic expansion of

$$\prod_{n=1}^{\infty}(1 - x^n)^{\chi_5(n)},$$

where $\chi_5(n)$ is the Legendre symbol modulo 5. A quadratic twist is also common in the theory of elliptic curves and modular forms [5–7].

This article deals with Graham partitions twisted by the Legendre symbol modulo odd primes. Let $\chi_p(n)$ be the Legendre symbol modulo p, and let $\mathcal{G}_{\chi_p}(n)$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

$$n = \sum_{i=1}^{\ell}\lambda_i \quad \text{and} \quad 1 = \sum_{i=1}^{\ell}\frac{\chi_p(\lambda_i)}{\lambda_i}.$$

We call a partition $\lambda \in \mathcal{G}_{\chi_p}(n)$ a *Graham partition twisted by* χ_p. For example, $(6, 4, 4, 3, 1) \in \mathcal{G}_{\chi_7}(18)$ as

$$18 = 6 + 4 + 4 + 3 + 1 \quad \text{and} \quad 1 = -\frac{1}{6} + 2 \cdot \frac{1}{4} - \frac{1}{3} + 1.$$

To show the nonemptiness of the set $\mathcal{G}_{\chi_p}(n)$ for a fixed prime p, one can adopt Graham's idea [1], which uses recurrence relations. For example, when $p = 13$, we can observe that if $\lambda \in \mathcal{G}_{\chi_{13}}(n)$, then

$$3\lambda \cup (3, 3) \in \mathcal{G}_{\chi_{13}}(3n + 6),$$
$$3\lambda \cup (3, 3, 2, 2, 1) \in \mathcal{G}_{\chi_{13}}(3n + 11),$$

and

$$3\lambda \cup (6, 6, 1) \in \mathcal{G}_{\chi_{13}}(3n + 13),$$

where 3λ is the partition consisting of three times of parts in λ and the partition $\lambda \cup \mu$ is one whose parts are a union of parts from λ and μ. While the idea of using recurrence relations is powerful for a fixed prime p, the nature of the proof is ad hoc on p, so we need a different idea to show the existence of Graham partitions twisted by χ_p for varying primes p. Indeed, the set $\mathcal{G}_{\chi_p}(n)$ is a nonempty set for odd primes $p \neq 5$ and for a sufficiently large integer n. On the other hand, the set $\mathcal{G}_{\chi_5}(n)$ is empty unless n is in a certain arithmetic progression.

Theorem 1. *The set $\mathcal{G}_{\chi_5}(n)$ is not empty if and only if n is congruent to* 1 *modulo* 5. *For an odd prime $p \neq 5$ and an integer $n \geq n_{\chi_p}$, the set $\mathcal{G}_{\chi_p}(n)$ is not empty, where*

$$n_{\chi_p} = 18.8p^{3/2}(\log p)^2 .$$

Table 1: Lower bounds in Theorem 1

p	N_{χ_p}	n_{χ_p}
3	9	118
7	14	1,319
13	19	5,798

Remark. Our proof is constructive so that we can derive a sharper n_{χ_p} for a fixed prime p. For example, we can obtain $n_{\chi_3} = 9$ (resp. $n_{\chi_7} = 55$) by taking $\alpha = 2$ (resp. $\alpha = 3$) in the proof of Theorem 1.

We remark that the lower bound n_{χ_p} is far from the tight bound for the nonemptiness of $\mathcal{G}_{\chi_p}(n)$. Let N_{χ_p} be the smallest integer such that $\mathcal{G}_{\chi_p}(n)$ is nonempty for integers $n \geq N_{\chi_p}$. Table 1 provides some numerics on N_{χ_p} and n_{χ_p}.

One can verify N_{χ_p} by checking $\mathcal{G}_{\chi_p}(n)$ for $n < n_{\chi_p}$. However, the computational cost for the exhaustive search could be expensive for large primes p as the number of partitions of n grows exponentially.

Motivated by Euler's partition identity, i.e., the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts, and Kim et al. [2] investigated the existence of a Graham partition of n into odd parts. Here, we show that there is also a twisted Graham partition of n into odd parts.

Corollary 2. *For an odd prime $p \geq 7$ and an integer $n \geq p(\sqrt{p} + 6)^2$, there is a partition $\lambda \in \mathcal{G}_{\chi_p}(n)$ such that all parts of λ are odd.*

Graham [1] proposed a general conjecture, which claims that for a sufficiently large integer n, there is a partition of n, i.e.,

$$n = f(k_1) + f(k_2) + \cdots + f(k_\ell),$$

satisfying that

$$1 = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_\ell},$$

where $f(k)$ is an integer-valued positive definite polynomial.

In the case $f(k) = k^2$, Alekseyev [8] proved that for integers $n \geq 8{,}543$, there are positive integers $k_1 > k_2 > \cdots > k_\ell$ satisfying that

$$n = k_1^2 + k_2^2 + \cdots + k_\ell^2 \quad \text{and} \quad 1 = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_\ell}.$$

The main idea of the proof is that there is a recurrence relation that generates a Graham partition of a larger integer from a given Graham partition of an integer. However, this idea seems infeasible to apply to general polynomials other than $f(k) = k^2$.

Here, we consider a twisted version of the original conjecture for the case that $f(k)$ is the kth m-gonal number:

$$f(k) = P_m(k) \coloneqq \frac{(m-2)k^2 - (m-4)k}{2}.$$

To be more specific, we search for a partition of a positive integer n satisfying that

$$\begin{aligned} n &= P_m(k_1) + P_m(k_2) + \cdots + P_m(k_\ell), \\ 1 &= \frac{\chi_p(k_1)}{k_1} + \frac{\chi_p(k_2)}{k_2} + \cdots + \frac{\chi_p(k_\ell)}{k_\ell}. \end{aligned} \tag{1.1}$$

We define $\mathcal{G}_{m,\chi_p}(n)$ to be the set of partitions satisfying equation (1.1). For example,

$$(3, 3, 3, 3, 1, 1, 1) \in \mathcal{G}_{3,\chi_3}(15)$$

as

$$15 = 3 + 3 + 3 + 3 + 1 + 1 + 1 \quad \text{and} \quad 1 = 4 \cdot \frac{\chi_3(2)}{2} + 3 \cdot \frac{\chi_3(1)}{1}.$$

By incorporating the character χ_p, one can find more algorithms to generate a partition of $\mathcal{G}_{m,\chi_p}(n)$ from such a partition of a smaller integer. Here, we prove that $\mathcal{G}_{m,\chi_p}(n)$ is nonempty for a large enough integer n for $m = 3$, 4, or 5.

Theorem 3. *The following statements hold true*:

(1) *For an odd prime p and* $n \geq n_{3,\chi_p} = p^6/4$, *the set* $\mathcal{G}_{3,\chi_p}(n)$ *is not empty.*

(2) *For an odd prime* $p \neq 3, 7$ *and* $n \geq n_{4,\chi_p}$, *the set* $\mathcal{G}_{4,\chi_p}(n)$ *is not empty, where*

$$n_{4,\chi_p} = \begin{cases} 193, & \text{if } p = 5, \\ p^5, & \text{otherwise.} \end{cases}$$

(3) *For an odd prime p and* $n \geq n_{5,\chi_p}$, *the set* $\mathcal{G}_{5,\chi_p}(n)$ *is not empty, where*

$$n_{5,\chi_p} = \begin{cases} 341, & \text{if } p = 5, \\ 9p^5/4, & \text{otherwise.} \end{cases}$$

Here, for simplicity, we roughly state n_{m,χ_p} in terms of p. A sharper bound $\tilde{n}_{m,\chi_p}$ will be given in Section 3 during the proof. As in Theorem 1, the bounds n_{m,χ_p} in Theorem 3 are far from the tight lower bounds. Let N_{m,χ_p} be the smallest integer such that $\mathcal{G}_{m,\chi_p}(n)$ is not empty for all $n \geq N_{m,\chi_p}$. Then, Table 2 provides some values of N_{m,χ_p} and n_{m,χ_p}.

We verify N_{4,χ_5} using the recurrence relations that if $\lambda \in \mathcal{G}_{4,\chi_5}(n)$, then

$$\begin{aligned} 4\lambda \cup (4,1,1) &\in \mathcal{G}_{4,\chi_5}(4n+6), \\ 4\lambda \cup (4,4,4,1,1,1) &\in \mathcal{G}_{4,\chi_5}(4n+15), \\ 4\lambda \cup (4,4,4,4,4,1,1,1,1) &\in \mathcal{G}_{4,\chi_5}(4n+24), \quad \text{and} \\ 4\lambda \cup (16,16,1) &\in \mathcal{G}_{4,\chi_5}(4n+33). \end{aligned}$$

Since there is no known recurrence relation for the case $m \neq 4$, computational cost to verify N_{m,χ_p} could be very expensive, even for small primes p.

The rest of this article is organized as follows. In Section 2, we prove Theorem 1 and Corollary 2; in Section 3, we prove Theorem 3 and give a sharper lower bound $\tilde{n}_{m,\chi_p}$ than n_{m,χ_p}. We conclude the article with some remarks in Section 4.

2 Proof of Theorem 1

We first provide a famous result of Sylvester [9,10], which we use to find the lower bound of an integer n having a twisted Graham partition. Let $F(a_1, a_2)$ be the Frobenius number of positive integers a_1 and a_2, that is, the largest integer b such that

$$a_1x_1 + a_2x_2 = b$$

does not hold for nonnegative integers x_1 and x_2.

Lemma 4. (Sylvester [9, 10]) *If* $(a_1, a_2) = 1$, *then* $F(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$.

The following two lemmas are useful to determine the lower bound n_{χ_p} of n with nonempty set $\mathcal{G}_{\chi_p}(n)$.

Table 2: Lower bounds in Theorem 3

m	p	N_{m,χ_p}	n_{m,χ_p}
3	3	24	183
4	5	96	193
5	3	78	547

Lemma 5. (Treviño [11, Theorem 1.2]) *Let $p > 3$ be an odd prime and $k \geq 2$ be an integer such that $k|p-1$. Let $g(p,k)$ be the least kth power nonresidue modulo p. Then,*

$$g(p,k) \leq \begin{cases} 1.1p^{1/4}\log p, & \text{if } k = 2 \quad \text{and} \quad p \equiv 3 \pmod 4, \\ 0.9p^{1/4}\log p, & \text{otherwise.} \end{cases}$$

Lemma 6. (Treviño [12, 13, Theorem A]) *If χ is any nonprincipal Dirichlet character to the prime modulus p, which is constant on $(N, N+H]$, then*

$$H < \left\{ \frac{\pi}{2}\sqrt{\frac{e}{3}} + o(1) \right\} p^{1/4}\log p,$$

where the $o(1)$ term depends only on p. Furthermore,

$$H \leq \begin{cases} 1.55p^{1/4}\log p, & \text{for } p \geq 10^{13}, \\ 3.38p^{1/4}\log p, & \text{for all odd } p. \end{cases}$$

Since the proof of Theorem 1 is lengthy, we first consider the case $p = 5$ as a proposition.

Proposition 7. *The set $\mathcal{G}_{\chi_5}(n)$ is not empty if and only if n is congruent to 1 modulo 5.*

Proof. We first assume that n is a positive integer congruent to 1 modulo 5. Let k be a nonnegative integer such that $n = 5k + 1$. Then, $\lambda = (5, \ldots, 5, 1)$ is a partition of n with $k + 1$ parts, which is in $\mathcal{G}_{\chi_5}(n)$.

Next, assume that $\lambda \in \mathcal{G}_{\chi_5}(n)$. We may rewrite the parts λ_i of λ as a_i, b_i, c_i, d_i, and e_i such that

$$a_i \equiv 1, \quad b_i \equiv 2, \quad c_i \equiv 3, \quad d_i \equiv 4, \quad e_i \equiv 0 \pmod 5.$$

Then, we have

$$n = \sum_{i=1}^{r_1} a_i + \sum_{i=1}^{r_2} b_i + \sum_{i=1}^{r_3} c_i + \sum_{i=1}^{r_4} d_i + \sum_{i=1}^{r_5} e_i \tag{2.1}$$

and

$$1 = \sum_{i=1}^{r_1} \frac{1}{a_i} - \sum_{i=1}^{r_2} \frac{1}{b_i} - \sum_{i=1}^{r_3} \frac{1}{c_i} + \sum_{i=1}^{r_4} \frac{1}{d_i}. \tag{2.2}$$

Let

$$N = \prod_{i=1}^{r_1} a_i \prod_{i=1}^{r_2} b_i \prod_{i=1}^{r_3} c_i \prod_{i=1}^{r_4} d_i,$$

and then, $N \equiv (-1)^{r_3+r_4} 2^{r_2+r_3} \pmod 5$.

Multiplying both sides of equation (2.2) by N, we see that

$$(-1)^{r_3+r_4} 2^{r_2+r_3} \equiv (-1)^{r_3+r_4} 2^{r_2+r_3}(r_1 - r_4) - (-1)^{r_3+r_4} 2^{r_2+r_3-1}(r_2 - r_3) \pmod 5,$$

which implies that $(r_1 - r_4) + 2(r_2 - r_3) \equiv 1 \pmod 5$. Because equation (2.1) gives

$$n \equiv (r_1 - r_4) + 2(r_2 - r_3) \pmod 5,$$

we complete our proof. □

We now turn to the case that p is an odd prime other than 5.

Proof of Theorem 1. We first claim that, for an odd prime $p \neq 5$, there is an integer $\alpha \in \{2, 3, \dots, p-1\}$ such that

$$\chi_p(\alpha) = -1 \quad \text{and} \quad \alpha^2 + 1 \not\equiv 0 \pmod{p}. \tag{2.3}$$

Indeed, for $p = 3$, we can take $\alpha = 2$. When $p \geq 7$, there are $(p-1)/2 \geq 3$ quadratic nonresidues modulo p, while there are at most two solutions for $x^2 + 1 \equiv 0 \pmod{p}$.

From now on, we may assume α as the smallest integer satisfying equation (2.3). Lemma 4 gives that $F(\alpha^2 + 1, p) = \alpha^2(p-1) - 1$. For $n \geq \alpha^2(p-1) + 1$, there are nonnegative integers j_1 and j_2 such that

$$n = (\alpha^2 + 1)j_1 + pj_2 + 1.$$

The partition

$$\lambda = (\underbrace{\alpha, \alpha, \dots, \alpha}_{\alpha j_1}) \cup (\underbrace{1, 1, \dots, 1}_{j_1+1}) \cup (\underbrace{p, p, \dots, p}_{j_2})$$

is a twisted Graham partition of n because

$$\sum_i \frac{\chi_p(\lambda_i)}{\lambda_i} = \alpha j_1 \cdot \frac{\chi_p(\alpha)}{\alpha} + (j_1 + 1)\cdot \frac{\chi_p(1)}{1} + j_2 \cdot \frac{\chi_p(p)}{p} = 1. \tag{2.4}$$

Now we estimate α to obtain n_{χ_p}. First, suppose that $p \equiv 3 \pmod 4$. Note that there is no solution for $x^2 + 1 \equiv 0 \pmod{p}$. By Lemma 5, the smallest quadratic nonresidue α satisfies that

$$\alpha < 1.1p^{\frac{1}{4}} \log p.$$

Next, suppose that $p \equiv 1 \pmod 4$. Lemma 5 says that the smallest quadratic nonresidue q_1 satisfies the following:

$$q_1 < 0.9p^{\frac{1}{4}} \log p.$$

If q_1 is not a solution of $x^2 + 1 \equiv 0 \pmod{p}$, then it is the desired α. Otherwise, we obtain the second-smallest quadratic nonresidue q_2. If $q_2 = q_1 + 1$, then

$$q_1, q_2 < 0.9p^{\frac{1}{4}} \log p + 1 < \frac{p-1}{2}.$$

If both q_1 and q_2 are solutions of $x^2 + 1 \equiv 0 \pmod{p}$, then

$$2q_1 + 1 < 1.8p^{\frac{1}{4}} \log p + 1 < p - 1,$$

which is a contradiction.

If $q_2 > q_1 + 1$, then

$$\chi_p(q_1 + 1) = \cdots = \chi_p(q_2 - 1) = -1,$$

so that

$$q_2 - q_1 - 1 < 3.38p^{\frac{1}{4}} \log p$$

by Lemma 6. Hence, we obtain

$$q_2 < 4.28p^{\frac{1}{4}} \log p + 1 < \frac{p-1}{2},$$

so that not all of q_1 and q_2 are solutions of $x^2 + 1 \equiv 0 \pmod{p}$. So, we can take $\alpha = q_1$ or q_2.

At any case, we have $\alpha \leq 4.28p^{\frac{1}{4}} \log p + 1$. We arrive at

$$F(\alpha^2 + 1, p) + 2 \leq \left(4.28p^{\frac{1}{4}} \log p + 1\right)^2 (p-1) + 1 \leq 18.8(\log p)^2 p^{3/2} =: n_{\chi_p}. \qquad \square$$

Now we see if there is a partition $\lambda \in \mathcal{G}_{\chi_p}(n)$ consisting of only odd parts. To this end, we use the following lemmas.

Lemma 8. (Somer [14, Lemma 2]) *Let $N_1(p)$ be the number of odd quadratic residues modulo p. If p is congruent to* 1 *modulo* 4, *then* $N_1(p) = (p-1)/4$.

Lemma 9. (Knapp et al. [15, Theorem 2]) *Let p be a prime* ≥29 *and v_p be the smallest odd quadratic nonresidue modulo p. Then,*

$$v_p \le \sqrt{p-12},$$

except for $p = 59, 109,$ *or* 131.

Now we are ready to prove Corollary 2.

Proof of Corollary 2. It is suffices to choose an odd integer $\alpha \in \{3, \dots, p-2\}$ satisfying equation (2.3). Note that all parts of λ are 1, α or p which are odd. Hence we construct a twisted Graham partition λ only with odd parts. After taking such an α, one can proceed exactly as in the proof of Theorem 1.

Suppose that $p \equiv 1 \pmod 4$, and then $p \ge 13$ as $p \ne 5$. Then, Lemma 8 gives that the number of odd quadratic nonresidues modulo p is

$$\frac{p-1}{2} - N_1(p) = \frac{p-1}{4} \ge 3,$$

where $N_1(p)$ is given in Lemma 8. Because there are at most two solutions for $x^2 + 1 \equiv 0 \pmod p$, there is an odd integer α satisfying equation (2.3).

Suppose that $p \equiv 3 \pmod 4$. We may take $\alpha = p-4$ because $p-4$ is odd,

$$\chi_p(p-4) = \chi_p(-1) = -1 \text{ and } (p-4)^2 + 1 \equiv 17 \not\equiv 0 \pmod p.$$

Let q be the smallest odd quadratic nonresidue modulo p. Then, by Lemma 9, we find that $q \le \sqrt{p-12}$ when $p \ge 137$. Since $q^2 < p-1$, it is clear that $q^2 + 1 \not\equiv 0 \pmod p$. By checking primes up to 131, we verify that we can choose $\alpha \le \sqrt{p-1} + 6$. Therefore, we can conclude that

$$F(\alpha^2+1, p) + 2 \le (\sqrt{p-1} + 6)^2(p-1) + 2 \le p(\sqrt{p} + 6)^2$$

is desired. □

3 Proof of Theorem 3

Before proving Theorem 3, we start with a lemma.

Lemma 10. *Let p be an odd prime, and let m* ≥ 3 *be an integer. Assume that there is a positive integer α satisfying the following conditions*:

(*a*) *α is a quadratic nonresidue modulo p.*

(*b*) $\alpha P_m(\alpha) + 1$ *is relatively prime to* $P_m(tp)$ *for a positive integer t.*

Then, the set $\mathcal{G}_{m,\chi_p}(n)$ *is not empty for* $n \ge \alpha P_m(\alpha)(P_m(tp) - 1) + 1$.

Proof. Suppose that α is a positive integer satisfying (a) and (b) and also that

$$n \ge \alpha P_m(\alpha)(P_m(tp) - 1) + 1.$$

By assumption (b), we have

$$\alpha P_m(\alpha)(P_m(tp) - 1) + 1 = F(\alpha P_m(\alpha) + 1, P_m(tp)) + 2,$$

and there are nonnegative integers j_1 and j_2 such that

$$n-1=(aP_m(a)+1)j_1+P_m(tp)j_2.$$

Define the partition λ of n as follows:

$$\lambda=(\underbrace{P_m(a),P_m(a),\ldots,P_m(a)}_{aj_1})\cup(\underbrace{P_m(1),P_m(1),\ldots,P_m(1)}_{j_1+1})\cup(\underbrace{P_m(tp),P_m(tp),\ldots,P_m(tp)}_{j_2}).$$

Then, $\lambda\in\mathcal{G}_{m,\chi_p}(n)$, because

$$\sum_i\frac{\chi_p(k_i)}{k_i}=aj_1\cdot\frac{\chi_p(a)}{a}+(j_1+1)\cdot\frac{\chi_p(1)}{1}+j_2\cdot\frac{\chi_p(tp)}{p}=1.$$ □

For each case $m=3$, 4, or 5, we present a separate proposition to provide more details on lower bounds.

Proposition 11. *For an odd prime p and an integer $n\geq\tilde{n}_{3,\chi_p}$, there is a partition in $\mathcal{G}_{3,\chi_p}(n)$, where*

$$\tilde{n}_{3,\chi_p}=\begin{cases}aP_3(a)(aP_3(a)+5)+1, & \text{if}\quad p\equiv 1\pmod 8,\\ 6P_3(p)-5, & \text{if}\quad p\equiv 5\pmod 8\quad\text{and}\quad (p+1,7)=1,\\ 6P_3(2p)-5, & \text{if}\quad p\equiv 5\pmod 8\quad\text{and}\quad (p+1,7)=7,\\ (p-1)P_3(p-1)(P_3(p)-1)+1, & \text{if}\quad p\equiv 3\pmod 4,\end{cases}$$

where a is the smallest positive integer satisfying (a) and (b) in Lemma 10 with $m=3$ and $t=1$.

Proof. First, let $p\equiv 1\pmod 8$ and a be the smallest quadratic nonresidue modulo p. Then, a is a prime number greater than 2 because $\chi_p(2)=1$. Note that $aP_3(a)+1=a^2(a+1)/2+1\equiv 1,2,$ or $4\pmod 5$, and thus

$$(aP_3(a)+1,aP_3(a)+6)=(aP_3(a)+1,5)=1.$$

Therefore, for $n\geq F(aP_3(a)+1,aP_3(a)+6)+2$, there exist nonnegative integers j_1 and j_2 such that

$$\begin{aligned}n&=(aP_3(a)+1)j_1+(aP_3(a)+6)j_2+1\\&=(j_1+1)P_3(1)+2j_2P_3(2)+(aj_1+aj_2)P_3(a).\end{aligned}$$

We note that

$$\lambda=(\underbrace{P_3(1),P_3(1),\ldots,P_3(1)}_{j_1+1})\cup(\underbrace{P_3(2),P_3(2),\ldots,P_3(2)}_{2j_2})\cup(\underbrace{P_3(a),P_3(a),\ldots,P_3(a)}_{aj_1+aj_2})$$

is in $\mathcal{G}_{3,\chi_p}(n)$ as it satisfies

$$\sum\frac{\chi_p(k_i)}{k_i}=(j_1+1)\cdot\frac{\chi_p(1)}{1}+2j_2\cdot\frac{\chi_p(2)}{2}+(aj_1+aj_2)\cdot\frac{\chi_p(a)}{a}=1.$$

Now we assume that $p\equiv 5\pmod 8$ and $(p+1,7)=1$. Then, we choose $a=2$ in Lemma 10 with $t=1$ because $(2P_3(2)+1,P_3(p))=(7,p(p+1)/2)=1$ and 2 is always a quadratic nonresidue modulo p. If $p\equiv 5\pmod 8$ and $(p+1,7)=7$, then we choose $a=2$ in Lemma 10 with $m=3$ and $t=2$. Thus, we have a desirable partition for all n greater than

$$F(2P_3(2)+1,P_3(2p))+2.$$

Finally, we focus on $p\equiv 3\pmod 4$. For $a=p-1$, let $d=(aP_3(a)+1,P_3(p))$. Then, d is a divisor of p^2 because

$$p(aP_3(a)+1)-(p^2-3p+2)P_3(p)=p^2.$$

Since d is a divisor of $aP_3(a)+1$, which is equivalent to 1 modulo p, we have that $d=1$. Moreover $\chi_p(p-1)=-1$, $a=p-1$, satisfies conditions (a) and (b) in Lemma 10. □

Now we investigate $\mathcal{G}_{4,\chi_p}(n)$. When $p = 3$ or $p = 7$, we obtain a sufficient and necessary condition for the nonemptyness of $\mathcal{G}_{4,\chi_p}(n)$.

Proposition 12. *The following statements hold true*:
(1) *The set* $\mathcal{G}_{4,\chi_3}(n)$ *is not empty if and only if* $n \equiv 1 \pmod 9$.
(2) *The set* $\mathcal{G}_{4,\chi_7}(n)$ *is not empty if and only if* $n \equiv 1 \pmod 7$ *and* $n \neq 15, 22$.

Proof. We first prove (1). If $n = 9k + 1$ for a nonnegative integer k, then the partition $\lambda = (P_4(3), \ldots, P_4(3), 1)$ is in $\mathcal{G}_{4,\chi_3}(n)$.

Assume that there is a partition $\lambda \in \mathcal{G}_{4,\chi_3}(n)$. We rewrite the parts of λ as $P_4(a_j^{(i)})$ such that

$$a_j^{(i)} \equiv i \pmod 9, \quad (i = 0, \cdots, 8).$$

Then,

$$n = \sum_{i=0}^{8} \sum_{j=1}^{r_i} (a_j^{(i)})^2 \tag{3.1}$$

and

$$1 = \sum_{j=1}^{r_1} \frac{1}{a_j^{(1)}} - \sum_{j=1}^{r_2} \frac{1}{a_j^{(2)}} + \sum_{j=1}^{r_4} \frac{1}{a_j^{(4)}} - \sum_{j=1}^{r_5} \frac{1}{a_j^{(5)}} + \sum_{j=1}^{r_7} \frac{1}{a_j^{(7)}} - \sum_{j=1}^{r_8} \frac{1}{a_j^{(8)}}. \tag{3.2}$$

Let N be the positive integer defined as

$$N = \prod_{j=1}^{r_1} a_j^{(1)} \prod_{j=1}^{r_2} a_j^{(2)} \prod_{j=1}^{r_4} a_j^{(4)} \prod_{j=1}^{r_5} a_j^{(5)} \prod_{j=1}^{r_7} a_j^{(7)} \prod_{j=1}^{r_8} a_j^{(8)}.$$

Multiplying both sides of equation (3.2) by N, we have that

$$\begin{aligned}(-1)^{r_5+r_7+r_8} 2^{r_2+r_7} 4^{r_4+r_5} \equiv\ & (-1)^{r_5+r_7+r_8} r_1 2^{r_2+r_7} 4^{r_4+r_5} + (-1)^{r_5+r_7+r_8+1} r_2 2^{r_2+r_7-1} 4^{r_4+r_5} \\ & + (-1)^{r_5+r_7+r_8} r_4 2^{r_2+r_7} 4^{r_4+r_5-1} + (-1)^{r_5+r_7+r_8} r_5 2^{r_2+r_7} 4^{r_4+r_5-1} \\ & + (-1)^{r_5+r_7+r_8+1} r_7 2^{r_2+r_7-1} 4^{r_4+r_5} + (-1)^{r_5+r_7+r_8} r_8 2^{r_2+r_7} 4^{r_4+r_5} \pmod 9\end{aligned}$$

and

$$1 \equiv r_1 + 4r_2 + 7r_4 + 7r_5 + 4r_7 + r_8 \pmod 9.$$

In equation (3.1), we see that

$$n \equiv r_1 + r_8 + 4(r_2 + r_7) + 7(r_4 + r_5) \pmod 9.$$

Therefore, $n \equiv 1 \pmod 9$.

Next we prove (2). The proof for $p = 7$ is similar to the proof for (1). Suppose that n is a positive integer congruent to 1 modulo 7. In each residue class modulo 49, we can construct a partition λ in $\mathcal{G}_{4,\chi_7}(n)$ as follows:

$$\lambda = \begin{cases} (P_4(1)) \cup (P_4(7), \ldots, P_4(7)), & \text{if } n \equiv 1 \pmod{49}, \\ (P_4(2), P_4(2)) \cup (P_4(7), \ldots, P_4(7)), & \text{if } n \equiv 8 \pmod{49}, \\ (P_4(4), P_4(4), P_4(4), P_4(4)) \cup (P_4(7), \ldots, P_4(7)), & \text{if } n \equiv 15 \pmod{49}, \\ (P_4(2), P_4(2), P_4(2), P_4(3), P_4(3), P_4(3), P_4(4), P_4(4)) \cup (P_4(7), \ldots, P_4(7)), & \text{if } n \equiv 22 \pmod{49}, \\ (P_4(1), P_4(1), P_4(3), P_4(3), P_4(3)) \cup (P_4(7), \ldots, P_4(7)), & \text{if } n \equiv 29 \pmod{49}, \\ (P_4(2), P_4(4), P_4(4)) \cup (P_4(7), \ldots, P_4(7)), & \text{if } n \equiv 36 \pmod{49}, \\ (P_4(2), P_4(2), P_4(2), P_4(2), P_4(3), P_4(3), P_4(3)) \cup (P_4(7), \ldots, P_4(7)), & \text{if } n \equiv 43 \pmod{49}. \end{cases}$$

By an exhaustive search on partitions of 15 and 22, we can check that $\mathcal{G}_{4,\chi_7}(15)$ and $\mathcal{G}_{4,\chi_7}(22)$ are empty.

Now we assume that $\lambda \in \mathcal{G}_{4,\chi_7}(n)$. Then, we may rewrite the parts of λ as $P_4(a_j^{(i)})$ such that

$$a_j^{(i)} \equiv i \pmod 7), (i = 0, \cdots, 6)$$

and

$$n = \sum_{i=0}^{6} \sum_{j=1}^{r_i} P_4(a_j^{(i)}).$$

Multiplying

$$N = \prod_{j=1}^{r_1} a_j^{(1)} \prod_{j=1}^{r_2} a_j^{(2)} \prod_{j=1}^{r_3} a_j^{(3)} \prod_{j=1}^{r_4} a_j^{(4)} \prod_{j=1}^{r_5} a_j^{(5)} \prod_{j=1}^{r_6} a_j^{(6)}$$

on both sides of

$$1 = \sum_{j=1}^{r_1} \frac{1}{a_j^{(1)}} + \sum_{j=1}^{r_2} \frac{1}{a_j^{(2)}} - \sum_{j=1}^{r_3} \frac{1}{a_j^{(3)}} + \sum_{j=1}^{r_4} \frac{1}{a_j^{(4)}} - \sum_{j=1}^{r_5} \frac{1}{a_j^{(5)}} - \sum_{j=1}^{r_6} \frac{1}{a_j^{(6)}}, \tag{3.3}$$

we complete our proof. □

For an odd prime p other than 3 and 7, the set $\mathcal{G}_{4,\chi_p}(n)$ is not empty for a sufficiently large integer n.

Proposition 13. *For an odd prime $p \neq 3$ and 7 and a positive integer $n \geq \tilde{n}_{4,\chi_p}$, there is a partition in $\mathcal{G}_{4,\chi_p}(n)$, where*

$$\tilde{n}_{4,\chi_p} = \begin{cases} 193, & \text{if } p = 5, \\ \alpha P_4(\alpha)(P_4(p) - 1) + 1, & \text{if } p \geq 11, \end{cases}$$

where α is the smallest positive integer satisfying (a) and (b) in Lemma 10 with $m = 4$ and $t = 1$.

Proof. When $p = 5$, we let $\alpha = 2$, $m = 4$, and $t = 1$ to satisfy two conditions in Lemma 10 and hence $\tilde{n}_{4,\chi_5} = \alpha P_m(\alpha)(P_m(tp) - 1) + 1 = 193$.

When $p \geq 11$, there are $(p-1)/2 \geq 5$ quadratic nonresidues. However, the number of solutions to $x^3 + 1 \equiv 0 \pmod p$ is at most three. Therefore, we can deduce that there is an integer α in Lemma 10 with $m = 4$ and $t = 1$. Hence, we find that $\tilde{n}_{4,\chi_p} = \alpha P_4(\alpha)(P_4(p) - 1) + 1$. □

Now we give a lower bound for the existence of twisted pentagonal Graham partitions.

Proposition 14. *For an odd prime p and a positive integer $n \geq \tilde{n}_{5,\chi_p}$, there is a partition in $\mathcal{G}_{5,\chi_p}(n)$, where*

$$\tilde{n}_{5,\chi_p} = \begin{cases} \alpha P_5(\alpha)(\alpha P_5(\alpha) + 9) + 1, & \text{if } p \equiv 1 \pmod 8, \\ 10P_5(p) - 9, & \text{if } p \equiv 5 \pmod 8 \text{ and } (11, 6p-1) = 1, \\ 10P_5(2p) - 9, & \text{if } p \equiv 5 \pmod 8 \text{ and } (11, 6p-1) = 11, \\ (p-1)P_5(p-1)(P_5(p) - 1) + 1, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where α is the smallest positive integer satisfying (a) and (b) in Lemma 10 with $m = 5$ and $t = 1$.

As the proof of Proposition 14 is similar to the proof of Propositions 11 and 13, we only give a brief sketch.

Proof. Let $p \equiv 1 \pmod 8$ and α be the smallest quadratic nonresidue modulo p. Then, $\alpha \geq 3$ is a prime number. Note that $(\alpha P_5(\alpha) + 1, \alpha P_5(\alpha) + 10) = (\alpha P_5(\alpha) + 1, 9) = 1$ because $\alpha P_5(\alpha) + 1 \not\equiv 0 \pmod 3$. Therefore, we find that

$$\tilde{n}_{5,\chi_p} = \alpha P_5(\alpha)(\alpha P_5(\alpha) + 9) + 1$$

as there are nonnegative integers j_1 and j_2 such that

$$n = (\alpha P_5(\alpha) + 1)j_1 + (\alpha P_5(\alpha) + 10)j_2 + 1 = (j_1 + 1)P_5(1) + 2j_2P_5(2) + \alpha(j_1 + j_2)P_5(\alpha).$$

Therefore, we can construct a partition $\lambda \in \mathcal{G}_{5,\chi_p}(n)$ as follows:

$$\lambda = (\underbrace{P_5(\alpha), \dots, P_5(\alpha)}_{\alpha(j_1+j_2)}) \cup (\underbrace{P_5(1), \dots, P_5(1)}_{j_1+1}) \cup (\underbrace{P_5(2), \dots, P_5(2)}_{2j_2}).$$

If $p \equiv 5 \pmod 8$ with $(11, 6p-1) = 1$, then we can choose $\alpha = 2$ with $m = 5$ and $t = 1$ in Lemma 10, which implies

$$\tilde{n}_{5,\chi_p} = F(11, P_5(p)) + 2 = 10P_5(p) - 9.$$

If $p \equiv 5 \pmod 8$ with $(11, 6p-1) \neq 1$, then we can choose $\alpha = 2$ with $m = 5$ and $t = 2$ in Lemma 10, so we derive that

$$\tilde{n}_{5,\chi_p} = F(11, P_5(2p)) + 2 = 10P_5(2p) - 9.$$

Finally, for $p \equiv 3 \pmod 4$, one can check that $(p-1)P_5(p-1) + 1$ is always relative prime to $P_5(p)$. Since $p-1$ is a quadratic nonresidue, we may choose $\alpha = p-1$ in Lemma 10 with $t = 1$. Therefore, we can derive that

$$\tilde{n}_{5,\chi_p} = (p-1)P_5(p-1)(P_5(p) - 1) + 1. \qquad \square$$

To complete the proof of Theorem 3, it suffices to check that the lower bound n_{m,χ_p} is not smaller than $\tilde{n}_{m,\chi_p}$.

Proof of Theorem 3. Since checking the inequality $\tilde{n} \leq n$ is tedious, we only give the worst case for each m.

For the triangular case, we estimate $\tilde{n}_{3,\chi_p}$ when $p \equiv 1 \pmod 8$. Since $\alpha \leq p-2$ and $p \geq 17$, we see that

$$\tilde{n}_{3,\chi_p} = \alpha P_3(\alpha)(\alpha P_3(\alpha) + 5) + 1 \leq (\alpha P_3(\alpha) + 3)^2 \leq \left(\frac{(p-2)^3}{2} + 3\right)^2 \leq \frac{1}{4}p^6.$$

For the square case, we see that

$$\tilde{n}_{4,\chi_p} = \alpha^3(p^2 - 1) + 1 < p^5.$$

Finally, for the pentagonal case, we estimate $\tilde{n}_{5,\chi_p}$ when $p \equiv 3 \pmod 4$. We find that

$$\tilde{n}_{5,\chi_p} = (p-1)^2\frac{3p-4}{2}\left(\frac{p(3p-1)}{2} - 1\right) + 1 < \frac{9}{4}p^5$$

is desired. $\square$

4 Concluding remarks

For completeness, we give the results on the existence of Graham partitions twisted by the Legendre symbol modulo 2 without proofs:

(1) The set $\mathcal{G}_{\chi_2}(n)$ is not empty if and only if n is odd.
(2) The set $\mathcal{G}_{3,\chi_2}(n)$ is not empty for $n \geq 16$.
(3) The set $\mathcal{G}_{4,\chi_2}(n)$ is not empty for odd $n \geq 25$.
(4) The set $\mathcal{G}_{5,\chi_2}(n)$ is not empty for $n \geq 70$.

Numerical experiments support that the set $\mathcal{G}_{m,\chi_p}(n)$ is not empty for an odd prime p, an integer $m \geq 6$, and a sufficiently large integer n. It would be interesting if one could obtain an effective bound for the nonemptiness of $\mathcal{G}_{m,\chi_p}(n)$. It is also desirable to find an efficient algorithm to find a twisted m-gonal Graham partition of n. In generating functionology,

$$\sum_{n\geq 0} |\mathcal{G}_{m,\chi_p}(n)| q^n = [z] \prod_{k\geq 1} \frac{1}{1 - z^{\chi_p(k)/k} q^{P_m(k)}},$$

where $|G_{m,\chi_p}(n)|$ is the number of elements of $\mathcal{G}_{m,\chi_p}(n)$ and $[z]g(z,q) \in \mathbb{Z}[[z,q]]$ is the coefficient of z of the power series expansion of $g(z,q)$. In this sense, we have studied the positivity of the coefficients of zq^n for the generating function. It would be nice if one could obtain other interesting arithmetic properties of twisted Graham partitions from the above generating function.

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