Research Article

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Well-posedness and stability analysis for Timoshenko beam system with Coleman-Gurtin's and Gurtin-Pipkin's thermal laws

https://doi.org/10.1515/math-2023-0127
received February 13, 2023; accepted September 13, 2023

Abstract: In this article, the effect of Coleman-Gurtin's and Gurtin-Pipkin's thermal laws on the displacement of a Timoshenko beam system with suspenders is studied. Using the existing semi-group theory and energy method, the existence and uniqueness of weak global solution, as well as a stability result without imposing any conditions on the coefficient parameters, are established.

Keywords: Timoshenko beam, suspenders, Coleman-Gurtin, Gurtin-Pipkin, well-posedness, stability

MSC 2020: 35B35, 35D30, 35D35, 35Q74, 35K05

1 Introduction

The system describing the motion of a classical Timoshenko-beam of length L (we set $L = \pi$ for simplicity), see [1,2], is defined by

$$\begin{cases} \rho A v_{tt} - S_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho I u_{tt} - M_x + S = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \end{cases} \tag{1.1}$$

where $v = v(x,t)$ is the transverse displacement, $u = u(x,t)$ is the rotation angle of the beam, ρ, A, and I are, respectively, mass density, cross-sectional area of beam, and inertial moment of the cross section. The constitutive laws S and M (shear force and bending moment, respectively) are given by

$$S = kGA(v_x + u), \quad M = EIu_x, \tag{1.2}$$

where the physical parameters E, G, and k are, respectively, the Young's modulus, shear modulus, and shear correction coefficient. Considering a Timoshenko cable-suspended beam structure such as the suspension bridge, where the roadbed has a negligible sectional dimensions in comparison with its length π (span of the bridge), this structure is modeled in Timoshenko theory through a one-dimensional extensible beam, while the (main) suspension cable models an elastic string which is coupled to the deck using elastic strings. The governing equations of motion describing a Timoshenko-suspended-beam system, see [3–9], are given by

$$\begin{cases} \rho w_{tt} - V_x - Q = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho A v_{tt} - S_x + Q = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho I u_{tt} - M_x + S = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \end{cases} \tag{1.3}$$

where $w = w(x,t)$ represents the vertical displacement of the main cable, $V = \alpha w_x$, $Q = \lambda(v - w)$, and α and λ are, respectively, the elastic modulus of the string and the stiffness of elastic springs. Generally, system (1.3) is

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not exponentially stable, see, for instance, [10] and references therein. Thus, we need some form of dissipative mechanism to achieve asymptotic stability of solutions. Recently, Bochicchio et al. [11] considered system (1.3) with linear frictional damping on the first two equations and heat conduction governed by Fourier's law on the bending moment, namely

$$\begin{cases} \rho w_{tt} - V_x - Q + \gamma_0 w_t = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho A v_{tt} - S_x + Q + \gamma_1 v_t = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho I u_{tt} - M_x + S + m\theta_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_3 \theta_t - \beta\theta_{xx} + m u_{xt} = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \end{cases} \tag{1.4}$$

where $\theta = \theta(x,t)$ is the temperature difference along the longitudinal direction, γ_0 and γ_1 are damping coefficients, and $\rho_3, \beta, m > 0$ are thermal coupling constants. The authors in [11] showed that system (1.4) is exponentially stable. Recently, Mukiawa et al. [12,13] studied (1.3) with Kelvin-Voigt and time-varying delay damping, and established exponential and general decay results. For related results on Timoshenko beam system with heat conduction governed by Fourier's law, see [14–17] and references therein. To overcome the drawback of infinite speed propagation usually related to Fourier's law, a possible model is the heat conduction governed by Cattaneo's law, see [18–20], where $(1.4)_4$ is replaced by the following equations:

$$\begin{cases} \rho_3 \theta_t + q_x + m u_{xt} = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \tau q_t + \sigma q + \theta_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \end{cases} \tag{1.5}$$

where $\tau > 0$ is small, $q = q(x,t)$ is the heat flux, and $\rho_3, \tau, \sigma > 0$ are coupling constants. There are several results in the literature for Timoshenko beam system with Cattaneo's law (second sound), see [21–26] and references therein.

Another interesting type of heat conduction law for the heat flux is the Gurtin-Pipkin thermal law, see [27], with the constitutive equation

$$\beta q(t) + \int_0^\infty h(s)\theta_x(x, t-s)\,\mathrm{d}s = 0, \tag{1.6}$$

where the memory h is a convex $L^1([0,+\infty))$ function with unit mass $\left(\int_0^\infty h(s)\,\mathrm{d}s = 1\right)$. In this case, system (1.4) takes the form

$$\begin{cases} \rho w_{tt} - V_x - Q + \gamma_0 w_t = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho A v_{tt} - S_x + Q + \gamma_1 v_t = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho I u_{tt} - M_x + S + m\theta_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_3 \theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty h(s)\theta_{xx}(x,t-s)\,\mathrm{d}s + m u_{xt} = 0, & \text{in } (0,\pi) \times \mathbb{R}_+. \end{cases} \tag{1.7}$$

For results related to (1.6), Dell'Oro and Pata [28] studied

$$\begin{cases} \rho_1 u_{tt} - k_1(u_x + v)_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_2 v_{tt} - b v_{xx} + k_1(u_x + v) + m\theta_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_3 \theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty h(s)\theta_{xx}(x,t-s)\,\mathrm{d}s + m v_{xt} = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \end{cases} \tag{1.8}$$

with $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_1 = kGA$, and $b = EI$. They proved that an exponential stability result provided $\chi_h = 0$, where

$$\chi_h = \left(\frac{\rho_1}{k_1\rho_3} - \frac{\beta}{h(0)}\right)\left(\frac{\rho_1}{k_1} - \frac{\rho_2}{b}\right) - \frac{\beta}{h(0)}\frac{\rho_1 m^2}{k_1 b \rho_3}.$$

Also, we refer the reader to recent Timoshenko and Laminated beams results with Gurtin-Pipkin thermal law in [29,30]. As clearly deduced in [28], the Fourier thermal law,

$$\beta q + \theta_x = 0, \tag{1.9}$$

and the Cattaneo (second sound) thermal law,

$$\tau q_t + \beta q + \theta_x = 0, \tag{1.10}$$

can be recovered from (1.6) by defining the memory function h in (1.6) as follows:

$$h_\delta(s) = \frac{1}{\delta} h\left(\frac{s}{\delta}\right), \quad \delta > 0 \tag{1.11}$$

and

$$h_\tau(s) = \frac{\beta}{\tau} e^{-s\frac{\beta}{\tau}}, \quad \tau > 0, \tag{1.12}$$

respectively.

A much more interesting thermal law is the Coleman-Gurtin heat conduction law, see [31], with the constitutive equation given by

$$\beta q(t) + (1-\eta)\theta_x + \eta \int_0^\infty h(s)\theta_x(x, t-s)\mathrm{d}s = 0, \quad \eta \in (0,1), \tag{1.13}$$

where $\eta = 1$ and $\eta = 0$ correspond to the Gurtin-Pipkin case and the Fourier case, respectively. This entails replacing $(1.7)_4$ with

$$\rho_3\theta_t - \frac{(1-\eta)}{\beta}\theta_{xx} - \frac{\eta}{\beta}\int_0^\infty h(s)\theta_{xx}(x, t-s)\mathrm{d}s + mu_{xt} = 0, \quad \text{in } (0,\pi)\times\mathbb{R}_+. \tag{1.14}$$

For results related to (1.13), see Santos et al. [20] and references therein.

In the present work, motivated by the results in [11,28,32], we consider a suspension, Timoshenko beam, when the shear force and the bending moment are subjected to Coleman-Gurtin thermal law, namely

$$\begin{cases} \rho w_{tt} - \alpha w_{xx} - \lambda(v-w) + \gamma_0 w_t = 0, \\ \rho_1 v_{tt} - k(v_x+u)_x + \lambda(v-w) + m_1\theta_x = 0, \\ \rho_2 u_{tt} - bu_{xx} + k(v_x+u) - m_1\theta + m_2\vartheta_x = 0, \\ \rho_3\theta_t - \beta_1(1-\eta)\theta_{xx} - \beta_1\eta\displaystyle\int_0^{+\infty} h_1(s)\theta_{xx}(x,t-s)\mathrm{d}s + m_1(v_{xt}+u_t) = 0, \\ \rho_4\vartheta_t - \beta_2(1-\eta)\vartheta_{xx} - \beta_2\eta\displaystyle\int_0^{+\infty} h_2(s)\vartheta_{xx}(x,t-s)\mathrm{d}s + m_2u_{xt} = 0, \end{cases} \tag{1.15}$$

where $(x,t)\in(0,\pi)\times\mathbb{R}_+$, $\theta = \theta(x,t)$, and $\vartheta = \vartheta(x,t)$ are the temperature differences along the longitudinal and vertical directions, respectively. The physical parameters $\rho, \alpha, \lambda, \gamma_0, \rho_1, k, m_1, m_2, \rho_2, b, \rho_3, \rho_4, \beta_1, \beta_2$ are all positive and $\eta \in (0,1)$. We supplement system (1.15) with the boundary conditions

$$\begin{cases} w(0,t) = v_x(0,t) = u(0,t) = \theta(0,t) = \vartheta(0,t) = 0, & t\in\mathbb{R}_+, \\ w(\pi,t) = v(\pi,t) = u(\pi,t) = \theta_x(\pi,t) = \vartheta_x(\pi,t) = 0, & t\in\mathbb{R}_+ \end{cases} \tag{1.16}$$

and the initial data

$$\begin{cases} w(x,0) = w_0(x), \quad v(x,0) = v_0(x), \quad u(x,0) = u_0(x), & x\in(0,\pi), \\ w_t(x,0) = w_1(x), \quad v_t(x,0) = v_1(x), \quad u_t(x,0) = u_1(x), & x\in(0,\pi), \\ \theta(x,-t) = \theta_0(x,t), \quad \vartheta(x,-t) = \vartheta_0(x,t), & x\in(0,\pi), \quad t>0. \end{cases} \tag{1.17}$$

The relaxation functions h_1 and h_2 are positive non-increasing functions, which are specified later. The main focus of this study is to investigate the well-posedness and the asymptotic behavior of solutions of systems (1.15)–(1.17). It is important to note here that, due to the damping term w_t on the first equation in system (1.15), the stability result in this work is established without imposing any condition such as the “equal wave of speed propagation” or “stability number” usually associated with Timoshenko system.

The rest of the work is organized as follows: In Section 2, we state some assumptions and set up our problem (1.15)–(1.17) in appropriate spaces. In Section 3, we prove the existence and uniqueness of the result for the systems (1.15)–(1.17). In Section 4, we study the asymptotic behavior of solution of systems (1.15)–(1.17).

2 Problem and functional setting

Throughout this work, C or c is a positive constant that may change within lines. For the kernels h_1 and h_2, we assume

(A0) $h_1, h_2 : [0, +\infty) \longrightarrow (0, +\infty)$ are non-increasing $C^2([0, +\infty))$ and convex summable functions satisfying

$$\lim_{s\to+\infty} h_i(s) = 0, \quad \text{and} \quad \int_0^{+\infty} h_i(s)\mathrm{d}s = 1, \quad i = 1, 2 \tag{2.1}$$

and there exists $\xi_1, \xi_2 > 0$ such that

$$-h_i''(s) \le \xi_1 h_i'(s), \quad \forall s \in (0, +\infty), \quad i = 1, 2. \tag{2.2}$$

By setting

$$g_1(s) = -h_1'(s) \quad \text{and} \quad g_2(s) = -h_2'(s) \tag{2.3}$$

assumption $(A0)$ ensues the following:

(A) $g_1, g_2 : [0, +\infty) \longrightarrow (0, +\infty)$ are non-increasing $C^1([0, +\infty))$ and convex summable functions satisfying

$$g_i^0 = \int_0^{+\infty} g_i(s)\mathrm{d}s = h_i(0) > 0, \quad \text{and} \quad \int_0^{+\infty} s g_i(s)\mathrm{d}s = 1, \quad i = 1, 2 \tag{2.4}$$

and there exists $\xi_1, \xi_2 > 0$ such that

$$g_i'(s) \le -\xi_i g_i(s), \quad \forall s \in (0, +\infty), \ i = 1, 2. \tag{2.5}$$

Now using similar ideas as in Dafermos [33], we define the relative displacement for history functions for $x \in (0, \pi)$, $s, t > 0$ as follows:

$$\sigma = \sigma(x, t, s) := \int_{t-s}^{t} \theta(x, r)\mathrm{d}r \quad \text{and} \quad \zeta = \zeta(x, t, s) := \int_{t-s}^{t} \vartheta(x, r)\mathrm{d}r, \quad x \in (0, \pi), \ s, t > 0. \tag{2.6}$$

Then, using the boundary conditions (1.16), we have

$$\sigma(0, t, s) = \sigma_x(\pi, t, s) = \zeta(0, t, s) = \zeta_x(\pi, t, s) = 0$$

and routine calculations lead to

$$\begin{aligned} \sigma_t + \sigma_s - \theta = 0, &\quad x \in (0, \pi), s, t > 0, \\ \zeta_t + \zeta_s - \vartheta = 0, &\quad x \in (0, \pi), s, t > 0 \end{aligned} \tag{2.7}$$

with

$$\sigma(x, t, 0) = \zeta(x, t, 0) = 0, \quad \forall t \ge 0 \tag{2.8}$$

and initial conditions

$$\begin{aligned} \sigma(x, 0, s) &= \int_0^s \theta_0(x, r)\mathrm{d}r := \sigma_0(x, s), \\ \zeta(x, 0, s) &= \int_0^s \vartheta_0(x, r)\mathrm{d}r := \zeta_0(x, s), \ s > 0, \end{aligned} \tag{2.9}$$

where σ_0 and ζ_0 represent the history of θ and ϑ, respectively. Thus, systems (1.15)–(1.17) and on account of (2.3) and (2.6)–(2.9) become

$$\begin{cases} \rho w_{tt} - \alpha w_{xx} - \lambda(v - w) + \gamma_0 w_t = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_1 v_{tt} - k(v_x + u)_x + \lambda(v - w) + m_1\theta_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_2 u_{tt} - bu_{xx} + k(v_x + u) - m_1\theta + m_2\vartheta_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_3\theta_t - \beta_1(1-\eta)\theta_{xx} - \beta_1\eta \int_0^{+\infty} g_1(s)\sigma_{xx}(x,s)ds + m_1(v_x + u)_t = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \sigma_t + \sigma_s - \theta = 0, & \text{in } (0,\pi) \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \rho_4\vartheta_t - \beta_2(1-\eta)\vartheta_{xx} - \beta_2\eta \int_0^{+\infty} g_2(s)\zeta_{xx}(x,s)ds + m_2 u_{xt} = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \zeta_t + \zeta_s - \vartheta = 0, & \text{in } (0,\pi) \times \mathbb{R}_+ \times \mathbb{R}_+ \end{cases} \tag{2.10}$$

with the boundary conditions

$$\begin{cases} w(0,t) = u(0,t) = v_x(0,t) = \theta(0,t) = \vartheta(0,t) = 0, & t \in \mathbb{R}_+, \\ w(\pi,t) = u(\pi,t) = v(\pi,t) = \theta_x(\pi,t) = \vartheta_x(\pi,t) = 0, & t \in \mathbb{R}_+, \\ \sigma(0,t,s) = \sigma_x(\pi,t,s) = \zeta(0,t,s) = \zeta_x(\pi,t,s) = 0, & s,t \in \mathbb{R}_+ \end{cases} \tag{2.11}$$

and the initial data

$$\begin{cases} w(x,0) = w_0(x), \quad v(x,0) = v_0(x), \quad u(x,0) = u_0(x), & x \in (0,\pi), \\ w_t(x,0) = w_1(x), \quad v_t(x,0) = v_1(x), \quad u_t(x,0) = u_1(x), & x \in (0,\pi), \\ \theta(x,0) = \theta_0(x), \quad \sigma(x,0,s) = \sigma_0(x,s), & x \in (0,\pi), s > 0, \\ \vartheta(x,0) = \vartheta_0(x), \quad \zeta(x,0,s) = \zeta_0(x,s), & x \in (0,\pi), s > 0. \end{cases} \tag{2.12}$$

Let $\Phi = (w, \varphi, v, \psi, u, \phi, \theta, \sigma, \vartheta, \zeta)^T$, where $\varphi = w_t$, $\psi = v_t$, and $\phi = u_t$. Then, the semi-group formulation of systems (2.10)–(2.12) is given by

$$(P)\begin{cases} \Phi_t + \mathcal{A}\Phi = 0, \\ \Phi(0) = \Phi_0, \end{cases} \tag{2.13}$$

where $\Phi_0 = (w_0, w_1, v_0, v_1, u_0, u_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0)^T$, and the linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -\varphi \\ -\frac{\alpha}{\rho}w_{xx} - \frac{\lambda}{\rho}(v - w) + \frac{\gamma_0}{\rho}\varphi \\ -\psi \\ -\frac{k}{\rho_1}(v_x + u)_x + \frac{\lambda}{\rho_1}(v - w) + \frac{m_1}{\rho_1}\theta_x \\ -\phi \\ -\frac{b}{\rho_2}u_{xx} + \frac{k}{\rho_2}(v_x + u) - \frac{m_1}{\rho_2}\theta + \frac{m_2}{\rho_2}\vartheta_x \\ -\frac{\beta_1(1-\eta)}{\rho_3}\theta_{xx} - \frac{\beta_1\eta}{\rho_3}\int_0^{+\infty} g_1(s)\sigma_{xx}(x,s)ds + \frac{m_1}{\rho_3}(\psi_x + \phi) \\ \sigma_s - \theta \\ -\frac{\beta_2(1-\eta)}{\rho_4}\vartheta_{xx} - \frac{\beta_2\eta}{\rho_4}\int_0^{+\infty} g_2(s)\zeta_{xx}(x,s)ds + \frac{m_2}{\rho_4}\phi_x \\ \zeta_s - \vartheta \end{pmatrix}.$$

Let $\langle,\rangle$ and $\|.\|$ denote the inner product and the norm in $L^2(0,\pi)$, respectively, and we define Sobolev spaces as follows:

$$H_a^1 := \{\varpi \in H^1(0,\pi)/\varpi(0) = 0\}, \quad H_b^1 := \{\varpi \in H^1(0,\pi)/\varpi(\pi) = 0\},$$
$$H_a^2 := \{\phi \in H^2(0,\pi)/\varpi_x \in H_a^1(0,\pi)\}, \quad H_b^2 := \{\phi \in H^2(0,\pi)/\varpi_x \in H_b^1(0,\pi)\}.$$

Next we introduce the weighted Hilbert space of $H_a^1(0,\pi)$-real valued functions on $(0,+\infty)$ by

$$\mathcal{M}_{g_i} := L^2_{g_i}(\mathbb{R}_+; H_a^1(0,\pi)) = \left\{\varpi : \mathbb{R}_+ \longrightarrow H_a^1(0,\pi) / \int_0^{+\infty} g_i(s)\|\varpi_x(s)\|^2 \mathrm{d}s < \infty\right\}, \quad i = 1,2$$

and equip it with the inner product

$$(\varpi, \hat{\varpi})_{\mathcal{M}_{g_i}} := \int_0^{+\infty} g_i(s)\langle \varpi_x(s), \hat{\varpi}_x(s)\rangle \mathrm{d}s, \quad i = 1,2$$

and norm

$$\|\varpi\|^2_{\mathcal{M}_{g_i}} = \int_0^{+\infty} g_i(s)\|\varpi_x(s)\|^2 \mathrm{d}s, \quad i = 1,2.$$

Also, we define

$$\mathcal{D}(\mathcal{M}_{g_i}) := \{\varpi \in \mathcal{M}_{g_i}/\varpi_s \in \mathcal{M}_{g_i} \quad \text{and} \quad \varpi(x,0) = 0\}, \quad i = 1,2.$$

Now we introduce the Hilbert space

$$\mathcal{H} := H_0^1 \times L^2 \times H_b^1 \times L^2 \times H_0^1 \times L^2 \times L^2 \times \mathcal{M}_{g_1} \times L^2 \times \mathcal{M}_{g_2}$$

and equipped it with the inner product

$$\begin{aligned}\langle \Phi, \hat{\Phi}\rangle_{\mathcal{H}} :=\ & \alpha \int_0^\pi w_x \hat{w}_x \mathrm{d}x + \rho \int_0^\pi \varphi\hat{\varphi}\mathrm{d}x + \lambda \int_0^\pi (v - w)(\hat{v} - \hat{w})\mathrm{d}x + k \int_0^\pi (v_x + u)(\hat{v}_x + \hat{u})\mathrm{d}x + \rho_1 \int_0^\pi \psi\hat{\psi}\mathrm{d}x \\ & + b \int_0^\pi u_x \hat{u}_x \mathrm{d}x + \rho_2 \int_0^\pi \phi\hat{\phi}\mathrm{d}x + \rho_3 \int_0^\pi \theta\hat{\theta}\mathrm{d}x + \beta_1\eta \int_0^{+\infty} g_1(s) \int_0^\pi \sigma_x(s)\hat{\sigma}_x(s)\mathrm{d}x\mathrm{d}s \\ & + \rho_4 \int_0^\pi \vartheta\hat{\vartheta}\mathrm{d}x + \beta_2\eta \int_0^{+\infty} g_2(s) \int_0^\pi \zeta_x(s)\hat{\zeta}_x(s)\mathrm{d}x\mathrm{d}s\end{aligned}$$

for any $\Phi = (w,\varphi,v,\psi,u,\phi,\theta,\sigma,\vartheta,\zeta)^T$, $\hat{\Phi} = (\hat{w},\hat{\varphi},\hat{v},\hat{\psi},\hat{u},\hat{\phi},\hat{\theta},\hat{\sigma},\hat{\vartheta},\hat{\zeta})^T \in \mathcal{H}$, and norm

$$\begin{aligned}\|\Phi\|^2_{\mathcal{H}} :=\ & \alpha\|w_x\|^2 + \rho\|\varphi\|^2 + \lambda\|v - w\|^2 + k\|v_x + u\|^2 + \rho_1\|\psi\|^2 \\ & + b\|u_x\|^2 + \rho_2\|\phi\|^2 + \rho_3\|\theta\|^2 + \beta_1\eta\|\sigma\|^2_{\mathcal{M}_{g_1}} + \rho_4\|\vartheta\|^2 + \beta_2\eta\|\zeta\|^2_{\mathcal{M}_{g_2}},\end{aligned}$$

for any $\Phi = (w,\varphi,v,\psi,u,\phi,\theta,\sigma,\vartheta,\zeta)^T \in \mathcal{H}$.

The domain of the linear operator $\mathcal{A}$ in (2.13) is defined as follows:

$$\mathcal{D}(\mathcal{A}) := \left\{ (w,\varphi,v,\psi,u,\phi,\theta,\sigma,\vartheta,\zeta) \in \mathcal{H} \;\middle|\; \begin{array}{l} w, u \in H^2 \cap H_0^1, \quad \varphi, \phi \in H_0^1, \\ v \in H_b^2, \quad \psi \in H_b^1, \quad \theta, \vartheta \in H^2 \cap H_a^1, \\ \sigma \in \mathcal{D}(\mathcal{M}_{g_1}), \quad \sigma_x \in H^1, \\ \zeta \in \mathcal{D}(\mathcal{M}_{g_2}), \quad \zeta_x \in H^1, \\ \int_0^{+\infty} g_1(s)\sigma_{xx}(s)\mathrm{d}s \in L^2, \\ \int_0^{+\infty} g_2(s)\zeta_{xx}(s)\mathrm{d}s \in L^2 \end{array} \right\}.$$

Remark 2.1. Using (2.5), we can deduce that

$$\langle -\varpi_s, \varpi\rangle_{\mathcal{M}_{g_i}} \le -\frac{\xi_i}{2}\|\varpi\|^2_{\mathcal{M}_{g_i}}, \quad \forall \varpi \in \mathcal{D}(\mathcal{M}_{g_i}), \quad i = 1,2. \tag{2.14}$$

Indeed, for any $\varpi \in \mathcal{D}(\mathcal{M}_{g_i})$ and on account of $\lim_{s\to+\infty} g_i(s)\|\varpi_x(s)\|^2 = 0$ and $\varpi_x(0) = 0$, we have

$$\begin{aligned}\langle -\varpi_s, \varpi\rangle_{\mathcal{M}_{g_i}} &= -\frac{1}{2}\int_0^{+\infty} g_i(s)\frac{\mathrm{d}}{\mathrm{d}s}(\|\varpi_x(s)\|^2)\mathrm{d}s\\ &= \frac{1}{2}\int_0^{+\infty} g_i'(s)\|\varpi_x(s)\|^2\mathrm{d}s - \frac{1}{2}\int_0^{+\infty}\frac{\mathrm{d}}{\mathrm{d}s}(g_i(s)\|\varpi_x(s)\|^2)\mathrm{d}s\\ &= \frac{1}{2}\int_0^{+\infty} g_i'(s)\|\varpi_x(s)\|^2\mathrm{d}s\\ &\le -\frac{\xi_i}{2}\int_0^{+\infty} g_i(s)\|\varpi_x(s)\|^2\mathrm{d}s = -\frac{\xi_i}{2}\|\varpi\|^2_{\mathcal{M}_{g_i}}.\end{aligned}$$

3 Well-posedness

In this section, we show that systems (2.10)–(2.12) have a global and unique weak solution for $\eta = 1$ (Gurtin-Pipkin thermal law) and $0 < \eta < 1$ (Coleman-Gurtin thermal law). The case of $\eta = 0$ (Fourier's law) can be established in a similar way, see [32]. First, establish some needed lemmas.

Lemma 3.1. *Suppose condition* (A) *holds and* $0 < \eta \le 1$, *then the linear operator*

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

defined in (2.13) *is monotone.*

Proof. Let $\Phi = (w, \varphi, v, \psi, u, \phi, \theta, \sigma, \vartheta, \zeta) \in \mathcal{D}(\mathcal{A})$, then using integration by parts and the boundary conditions (2.11), we have

$$\begin{aligned}\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} &= -\alpha\int_0^{\pi} w_x\varphi_x\mathrm{d}x + \int_0^{\pi}[-\alpha w_{xx} - \lambda(v-w) + \gamma_0\varphi]\varphi\mathrm{d}x\\ &\quad + \lambda\int_0^{\pi}(\varphi-\psi)(v-w)\mathrm{d}x - k\int_0^{\pi}(\psi_x+\phi)(v_x+u)\mathrm{d}x\\ &\quad + \int_0^{\pi}[-k(v_x+u)_x + \lambda(v-w) + m_1\theta_x]\psi\mathrm{d}x - b\int_0^{\pi}\phi_x u_x\mathrm{d}x\\ &\quad + \int_0^{\pi}[-bu_{xx} + k(v_x+u) - m_1\theta + m_2\vartheta_x]\phi\mathrm{d}x\\ &\quad + \int_0^{\pi}\left[-\beta_1(1-\eta)\theta_{xx} - \beta_1\eta\int_0^{+\infty} g(s)\sigma_{xx}(x,s)\mathrm{d}s + m_1(\psi_x+\phi)\right]\theta\mathrm{d}x\\ &\quad + \beta_1\eta\int_0^{+\infty} g_1(s)\int_0^{\pi}(\sigma_s(x,s) - \theta(x,t))_x\sigma_x(x,s)\mathrm{d}x\mathrm{d}s\\ &\quad + \int_0^{\pi}\left[-\beta_2(1-\eta)\vartheta_{xx} - \beta_2\eta\int_0^{+\infty} g(s)\zeta_{xx}(x,s)\mathrm{d}s + m_2\phi_x\right]\vartheta\mathrm{d}x\\ &\quad + \beta_2\eta\int_0^{+\infty} g_2(s)\int_0^{\pi}(\zeta_s(x,s) - \vartheta(x,t))_x\zeta_x(x,s)\mathrm{d}x\mathrm{d}s,\\ &= \gamma_0\|\varphi\|_2^2 + \beta_1(1-\eta)\|\theta_x\|^2 + \frac{\beta_1\eta}{2}\int_0^{+\infty} g_1(s)\frac{\mathrm{d}}{\mathrm{d}s}(\|\sigma_x(s)\|^2)\mathrm{d}s\\ &\quad + \beta_2(1-\eta)\|\vartheta_x\|^2 + \frac{\beta_2\eta}{2}\int_0^{+\infty} g_2(s)\frac{\mathrm{d}}{\mathrm{d}s}(\|\zeta_x(s)\|^2)\mathrm{d}s.\end{aligned}$$

On account of Remark 2.1, we obtain

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} \geq \gamma_0\|\varphi\|_2^2 + \beta_1(1-\eta)\|\theta_x\|^2 + \frac{\beta_1\eta\xi_1}{2}\|\sigma\|^2_{\mathcal{M}_{g_1}} + \beta_2(1-\eta)\|\vartheta_x\|^2 + \frac{\beta_1\eta\xi_2}{2}\|\zeta\|^2_{\mathcal{M}_{g_2}}.$$

Thus, $\mathcal{A}$ is monotone. □

Lemma 3.2. *Suppose condition* (A) *holds and* $0 < \eta \leq 1$, *then the linear operator*

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

defined in (2.13) *is maximal.*

Proof. Let $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8, f^9, f^{10}) \in \mathcal{H}$. We find

$$\Phi = (w, \varphi, v, \psi, u, \phi, \theta, \sigma, \vartheta, \zeta) \in \mathcal{D}(\mathcal{A})$$

such that Φ solves the stationary problem

$$\Phi + \mathcal{A}\Phi = F. \tag{3.1}$$

Now, from (3.1) we have

$$\begin{cases} w - \varphi = f^1, \\ (\rho + \gamma_0)\varphi - \alpha w_{xx} - \lambda(v - w) = \rho f^2, \\ v - \psi = f^3, \\ \rho_1\psi - k(v_x + u)_x + \lambda(v - w) + m_1\theta_x = \rho_1 f^4, \\ u - \phi = f^5, \\ \rho_2\phi - bu_{xx} + k(v_x + u) - m_1\theta + m_2\vartheta_x = \rho_2 f^6, \\ \rho_3\theta - \beta_1(1-\eta)\theta_{xx} - \beta_1\eta\displaystyle\int_0^{+\infty} g_1(s)\sigma_{xx}(x, s)\mathrm{d}s + m_1(\psi_x + \phi) = \rho_3 f^7, \\ \sigma + \sigma_s - \theta = f^8, \\ \rho_4\vartheta - \beta_2(1-\eta)\vartheta_{xx} - \beta_2\eta\displaystyle\int_0^{+\infty} g_2(s)\zeta_{xx}(x, s)\mathrm{d}s + m_2\phi_x = \rho_4 f^9, \\ \zeta + \zeta_s - \vartheta = f^{10}. \end{cases} \tag{3.2}$$

Multiplying $(3.2)_8$ and $(3.2)_{10}$ by e^y, and integrating the results over $(0, s)$, we obtain

$$\begin{aligned} \sigma(s) &= (1 - e^{-s})\theta + \int_0^s e^{y-s} f^8(y)\mathrm{d}y, \\ \zeta(s) &= (1 - e^{-s})\vartheta + \int_0^s e^{y-s} f^{10}(y)\mathrm{d}y. \end{aligned} \tag{3.3}$$

Substituting $(3.2)_1$, $(3.2)_3$, and $(3.2)_5$ in $(3.2)_2$, $(3.2)_4$, and $(3.2)_6$, respectively, we arrive at

$$\begin{cases}(\rho+\gamma_0)w-aw_{xx}-\lambda(v-w)=\underbrace{\rho f^1+\gamma_0 f^1+\rho f^2}_{J_1},\\ \rho_1 v-k(v_x+u)_x+\lambda(v-w)+m_1\theta_x=\underbrace{\rho_1 f^3+\rho_1 f^4}_{J_2},\\ \rho_2 u-bu_{xx}+k(v_x+u)-m_1\theta+m_2\vartheta_x=\underbrace{\rho_2 f^5+\rho_2 f^6}_{J_3},\\ \rho_3\theta-C_{\eta,\beta_1,g_1}\theta_{xx}+m_1(v_x+u)=\underbrace{m_1 f_x^3+m_1 f^5+\rho_3 f^7+\beta_1\eta\int_0^{+\infty}g_1(s)\left(\int_0^s e^{y-s}f_{xx}^8(y)\mathrm{d}y\right)\mathrm{d}s}_{J_4},\\ \rho_4\vartheta-C_{\eta,\beta_2,g_2}\vartheta_{xx}+m_2u_x=\underbrace{m_2 f_x^5+\rho_4 f^9+\beta_1\eta\int_0^{+\infty}g_2(s)\left(\int_0^s e^{y-s}f_{xx}^{10}(y)\mathrm{d}y\right)\mathrm{d}s}_{J_5},\end{cases} \tag{3.4}$$

where

$$C_{\eta,\beta_i,g_i}=\beta_i(1-\eta)+\beta_i\eta\int_0^{+\infty}g_i(s)(1-e^{-s})\mathrm{d}s>0,\quad i=1,2.$$

We note that the last terms in J_4 and J_5 are in $H^{-1}(0,\pi)$. In fact, for

$$q\in H_a^1(0,\pi)\quad\text{with }\|q_x\|\le 1,$$

we have

$$\begin{aligned}\left|\left\langle\int_0^{+\infty}g_1(s)\left(\int_0^s e^{y-s}f_{xx}^8(y)\mathrm{d}y\right)\mathrm{d}s,q\right\rangle\right|&=\left|\left\langle\int_0^{+\infty}g_1(s)\left(\int_0^s e^{y-s}f_x^8(y)\mathrm{d}y\right)\mathrm{d}s,q_x\right\rangle\right|\\&\le\int_0^{+\infty}g_1(s)e^{-s}\left(\int_0^s e^y\|f_x^8(y)\|\mathrm{d}y\right)\mathrm{d}s\\&=\int_0^{+\infty}e^y\|f_x^8(y)\|\left(\int_y^{+\infty}e^{-s}g_1(s)\mathrm{d}s\right)\mathrm{d}y\\&\le\int_0^{+\infty}g_1(y)e^y\|f_x^8(y)\|\int_y^{+\infty}e^{-s}\mathrm{d}s\mathrm{d}y\\&=\int_0^{+\infty}g_1(y)\|f_x^8(y)\|\mathrm{d}y<\infty.\end{aligned}$$

Similarly,

$$\int_0^{+\infty}g_2(s)\left(\int_0^s e^{y-s}f_{xx}^{10}(y)\mathrm{d}y\right)\mathrm{d}s\in H^{-1}(0,\pi).$$

Next using the weak formulation of (3.4), we define the following bilinear form $\mathscr{B}$ on $\mathbb{H}\times\mathbb{H}$ and linear form $\mathscr{L}$ on $\mathbb{H}$, where $\mathbb{H}:=H_0^1\times H_b^1\times H_0^1\times L^2\times L^2$, as follows:

$$\mathscr{B}((w,v,u,\theta,\vartheta),(w^*,v^*,u^*,\theta^*,\vartheta^*)) := (\rho+\gamma_0)\int_0^\pi ww^*\mathrm{d}x + \alpha\int_0^\pi w_x w_x^*\mathrm{d}x + \lambda\int_0^\pi (v-w)(v^*-w^*)\mathrm{d}x$$
$$+ \rho_1\int_0^\pi vv^*\mathrm{d}x + k\int_0^\pi (v_x+u)(v_x^*+u^*)\mathrm{d}x + m_1\int_0^\pi \theta_x v^*\mathrm{d}x$$
$$+ \rho_2\int_0^\pi uu^*\mathrm{d}x + b\int_0^\pi u_x u_x^*\mathrm{d}x - m_1\int_0^\pi \theta u^*\mathrm{d}x + m_2\int_0^\pi \vartheta_x u^*\mathrm{d}x$$
$$+ \rho_3\int_0^\pi \theta\theta^*\mathrm{d}x + C_{\eta,\beta_1,g_1}\int_0^\pi \theta_x\theta_x^*\mathrm{d}x + m_1\int_0^\pi (v_x+u)\theta^*\mathrm{d}x$$
$$+ \rho_4\int_0^\pi \vartheta\vartheta^*\mathrm{d}x + C_{\eta,\beta_2,g_2}\int_0^\pi \vartheta_x\vartheta_x^*\mathrm{d}x + m_2\int_0^\pi u_x\theta^*\mathrm{d}x$$

and

$$\mathscr{L}((w^*,v^*,u^*,\theta^*,\vartheta^*)) := \int_0^\pi (\rho f^1 + \gamma_0 f^1 + \rho f^2)w^*\mathrm{d}x + \int_0^\pi (\rho_1 f^3 + \rho_1 f^4)v^*\mathrm{d}x$$
$$+ \int_0^\pi (\rho_2 f^5 + \rho_2 f^6)u^*\mathrm{d}x + \int_0^\pi (m_1 f_x^3 + m_1 f^5 + \rho_3 f^7)\theta^*\mathrm{d}x$$
$$+ \beta_1\eta\int_0^\pi \theta_x^*\int_0^{+\infty} g_1(s)\left(\int_0^s e^{y-s} f_x^8(y)\mathrm{d}y\right)\mathrm{d}s\mathrm{d}x$$
$$+ \int_0^\pi (m_2 f_x^5 + \rho_4 f^9)\vartheta^*\mathrm{d}x + \beta_2\eta\int_0^\pi \vartheta_x^*\int_0^{+\infty} g_2(s)\left(\int_0^s e^{y-s} f_x^{10}(y)\mathrm{d}y\right)\mathrm{d}s\mathrm{d}x$$

for every $(w,v,u,\theta,\vartheta),(w^*,v^*,u^*,\theta^*,\vartheta^*) \in \mathbb{H}$.

When $\mathbb{H}$ is endowed with the following norm:

$$\|(w,v,u,\theta,\vartheta)\|_{\mathbb{H}}^2 = \rho\|w\|^2 + \alpha\|w_x\|^2 + \lambda\|v-w\|^2 + \rho_1\|v\|^2 + k_1\|v_x+u\|^2 + \rho_2\|u\|^2 + k_2\|u_x\|^2 + \rho_3\|\theta\|^2 + \rho_4\|\vartheta\|^2,$$

it is easy to see that $\mathscr{B}$ is a bounded and coercive bilinear form on $\mathbb{H}\times\mathbb{H}$, and $\mathscr{L}$ is a bounded linear form on $\mathbb{H}$. Thus, by the Lax-Milgram theorem, there exists a unique $(w,v,u,\theta)\in\mathbb{H}$ such that

$$\mathscr{B}((w,v,u,\theta,\vartheta),(w^*,v^*,u^*,\theta^*,\vartheta^*)) = \mathscr{L}((w^*,v^*,u^*,\theta^*,\vartheta^*)), \quad \forall(w^*,v^*,u^*,\theta^*,\vartheta^*)\in\mathbb{H}.$$

From $(3.2)_1$, $(3.2)_3$, and $(3.2)_5$, it follows that

$$\varphi\in H_0^1, \quad \psi\in H_b^1 \quad \text{and} \quad \phi\in H_0^1,$$

respectively. Then, using regularity theory, it follows from $(3.4)_1$, $(3.4)_2$, $(3.4)_3$, $(3.4)_4$, and $(3.4)_5$ that

$$w,u\in H^2\cap H_0^1, \quad v\in H_b^2\cap H_b^1, \theta,\vartheta\in H^2\cap H_a^1.$$

Also, from (3.3), substituting ψ,ϕ,θ, and ϑ, we have

$$\sigma\in\mathcal{D}\big(\mathcal{M}_{g_1}\big), \quad \zeta\in\mathcal{D}\big(\mathcal{M}_{g_2}\big).$$

Finally, from $(3.2)_7$ and $(3.2)_9$, we obtain that

$$\int_0^{+\infty} g(s)\sigma_{xx}(s)\mathrm{d}s, \int_0^{+\infty} g(s)\zeta_{xx}(s)\mathrm{d}s \in L^2.$$

Therefore, $\Phi = (w,\varphi,v,\psi,u,\phi,\theta,\sigma,\vartheta,\zeta)\in\mathcal{D}(\mathcal{A})$ and satisfies (3.1). Thus, the operator $\mathcal{A}$ is maximal. □

On account of Lemmas 3.1 and 3.2, we apply the semi-group theory for linear operator (Hille-Yosida theory, see Pazy [34]). We have the following existence and uniqueness result.

Theorem 3.1. *Let* $\Phi_0 = (w_0, w_1, v_0, v_1, u_0, u_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0) \in \mathcal{H}$ *be given. Suppose condition* (A) *holds and* $0 < \eta \le 1$, *then the Cauchy problem* (2.13) *has a unique local weak solution*

$$\Phi \in C([0, T_m), \mathcal{H}), \quad \text{for some } T_m > 0.$$

Remark 3.1. We can show that (see (4.2)) the solution

$$\Phi = (w, w_t, v, v_t, u, u_t, \theta, \sigma, \vartheta, \zeta)$$

of (1.15)–(1.17) given by Theorem 3.1 satisfies

$$\|\Phi(t)\|_{\mathcal{H}}^2 \le C\|\Phi_0\|_{\mathcal{H}}^2, \quad \forall t \ge 0,$$

where $C > 0$ is a constant independent of t. Thus, the solution Φ is global, that is, if $\Phi_0 \in \mathcal{H}$, then

$$\Phi \in C([0, \infty), \mathcal{H}).$$

Now due to the density of $\mathcal{D}(\mathcal{A})$ in $\mathcal{H}$, see [35], we can announce the following more regular result.

Theorem 3.2. *Let* $\Phi_0 = (w_0, w_1, v_0, v_1, u_0, u_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0) \in \mathcal{D}(\mathcal{A})$ *be given. Suppose condition* (A) *holds and* $0 < \eta \le 1$, *then the Cauchy problem* (2.13) *has a unique global weak solution in the class*

$$\Phi \in C([0, \infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, \infty), \mathcal{H}).$$

Moreover, if $\Phi_0 = (w_0, w_1, v_0, v_1, u_0, u_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0) \in \mathcal{D}(\mathcal{A}^\ell)$, $\ell \ge 2$, *then the global weak solution is in the class*

$$\Phi \in \bigcap_{j=0}^{\ell} C^{\ell-j}([0, \infty); \mathcal{D}(\mathcal{A}^j)).$$

4 Stability result

In this section, we study the stability of solution of (2.10)–(2.12) for $0 < \eta \le 1$.

4.1 Coleman-Gurtin thermal law: case $0 < \eta < 1$

The energy functional associated with the solution $\Phi = (w, w_t, v, v_t, u, u_t, \theta, \sigma, \vartheta, \zeta)$ of systems (2.10)–(2.12) is defined by

$$\begin{aligned}\mathcal{E}(t) = &\frac{1}{2}[\rho\|w_t\|^2 + \rho_1\|v_t\|^2 + \rho_2\|u_t\|^2 + \alpha\|w_x\|^2 + \lambda\|(v - w)\|^2 + k\|v_x + u\|^2] \\ &+ \frac{1}{2}[b\|u_x\|^2 + \rho_3\|\theta\|^2 + \|\sigma\|_{\mathcal{M}_{g_1}}^2 + \rho_4\|\vartheta\|^2 + \|\zeta\|_{\mathcal{M}_{g_2}}^2], \quad \forall t \ge 0.\end{aligned} \tag{4.1}$$

First, we provide some lemmas to facilitate the proof of the main stability result in this subsection.

Lemma 4.1. *Under the conditions of Theorem* (3.1), *the energy functional* (4.1) *satisfies*

$$\begin{aligned}\mathcal{E}'(t) = &-\gamma_0\|w_t\|^2 - \beta_1(1 - \eta)\|\theta_x\|^2 + \frac{\beta_1\eta}{2}\int_0^{+\infty} g_1'(s)\|\sigma_x(s)\|^2 ds \\ &- \beta_2(1 - \eta)\|\vartheta_x\|^2 + \frac{\beta_2\eta}{2}\int_0^{+\infty} g_2'(s)\|\zeta_x(s)\|^2 ds \le 0, \quad \forall t \ge 0.\end{aligned} \tag{4.2}$$

Proof. By multiplying $(2.10)_1$, $(2.10)_2$, $(2.10)_3$, $(2.10)_4$, and $(2.10)_6$ by w_t, v_t, u_t, θ, and ϑ, respectively, in $L^2(0,\pi)$, and $(2.10)_5$ and $(2.10)_7$ by σ and ζ in $\mathcal{M}_{g_1}$ and $\mathcal{M}_{g_2}$, respectively, using integration by parts and the boundary conditions (2.11), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\rho\|w_t\|^2 + \alpha\|w_x\|^2 + \lambda\|(v-w)\|^2] - \lambda\langle(v-w), v_t\rangle + \gamma_0\|w_t\|^2 = 0, \tag{4.3}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\rho_1\|v_t\|^2 + k_1\|v_x + u\|^2] - k\langle(v_x+u), u_t\rangle + \lambda\langle(v-w), v_t\rangle - m_1\langle\theta, v_{xt}\rangle = 0, \tag{4.4}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\rho_2\|u_t\|^2 + b\|u_x\|^2] + k\langle(v_x+u), u_t\rangle - m_1\langle\theta, u_t\rangle - m_2\langle\vartheta, u_{xt}\rangle = 0, \tag{4.5}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\rho_3\|\theta\|^2] + \beta_1(1-\eta)\|\theta_x\|^2 + \beta_1\eta\int_0^{+\infty} g_1(s)\langle\sigma_x(s), \theta_x(t)\rangle\mathrm{d}s + m_1\langle\theta, (v_{xt}+u_t)\rangle = 0, \tag{4.6}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\sigma\|^2_{\mathcal{M}_{g_1}}] - \beta_1\eta\int_0^{+\infty} g_1'(s)\|\sigma_x(s)\|^2\mathrm{d}s - \beta_1\eta\int_0^{+\infty} g_1(s)\langle\sigma_x(s), \theta_x(t)\rangle\mathrm{d}s = 0, \tag{4.7}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\rho_4\|\vartheta\|^2] + \beta_2(1-\eta)\|\vartheta_x\|^2 + \beta_2\eta\int_0^{+\infty} g_2(s)\langle\zeta_x(s), \vartheta_x(t)\rangle\mathrm{d}s + m_2\langle\vartheta, u_{xt}\rangle = 0, \tag{4.8}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\zeta\|^2_{\mathcal{M}_{g_2}}] - \beta_2\eta\int_0^{+\infty} g_2'(s)\|\zeta_x(s)\|^2\mathrm{d}s - \beta_1\eta\int_0^{+\infty} g_2(s)\langle\zeta_x(s), \vartheta_x(t)\rangle\mathrm{d}s = 0. \tag{4.9}$$

Adding (4.3)–(4.9) and recalling Remark 2.1, we obtain

$$\begin{aligned}\mathcal{E}'(t) = &-\gamma_0\|w_t\|^2 - \beta_1(1-\eta)\|\theta_x\|^2 + \beta_1\eta\int_0^{+\infty} g_1'(s)\|\sigma_x(s)\|^2\mathrm{d}s \\ &- \beta_2(1-\eta)\|\vartheta_x\|^2 + \beta_2\eta\int_0^{+\infty} g_2'(s)\|\zeta_x(s)\|^2\mathrm{d}s \le 0, \quad \forall t \ge 0.\end{aligned} \tag{4.10}$$

□

Remark 4.1. The lemma above implies that the energy $\mathcal{E}$ in (4.1) is non-increasing and bounded above by $\mathcal{E}(0)$. Also, the computations are done for regular solution. However, the result remains true for weak solution by density argument.

Lemma 4.2. *The functional F_1 defined by*

$$F_1(t) = \rho\langle w_t, w\rangle + \rho_1\langle v_t, v\rangle + \rho_2\langle u_t, u\rangle + \frac{\gamma_0}{2}\|w\|^2,$$

along the solution of systems (2.10)–(2.12) *satisfies the estimate*

$$\begin{aligned}F_1'(t) \le &-\alpha\|w_x\|^2 - \lambda\|v-w\|^2 - \frac{k}{2}\|v_x+u\|^2 - \frac{b}{2}\|u_x\|^2 \\ &+ \rho\|w_t\|^2 + \rho_1\|v_t\|^2 + \rho_2\|u_t\|^2 + C\|\theta\|^2 + C\|\vartheta\|^2, \quad \forall t \ge 0.\end{aligned} \tag{4.11}$$

Proof. Differentiating F_1, using $(2.10)_1$, $(2.10)_2$, and $(2.10)_3$, then applying integration by parts over $(0,\pi)$ and making use of the boundary conditions (2.11), we obtain

$$\begin{aligned}F_1'(t) = &\rho\|w_t\|^2 + \rho_1\|v_t\|^2 + \rho_2\|u_t\|^2 - \alpha\|w_x\|^2 - \lambda\|v-w\|^2 - k\|v_x+u\|^2 \\ &- b\|u_x\|^2 + m_1\langle(v_x+u), \theta\rangle + m_2\langle u_x, \vartheta\rangle.\end{aligned} \tag{4.12}$$

Young's and Poincaré's inequalities lead to

$$\begin{aligned} F_1'(t) \le &-\alpha\|w_x\|^2 - \lambda\|v - w\|^2 - \frac{k}{2}\|v_x + u\|^2 - \frac{b}{2}\|u_x\|^2 \\ &+ \rho\|w_t\|^2 + \rho_1\|v_t\|^2 + \rho_2\|u_t\|^2 + C\|\theta_x\|^2 + C\|\vartheta_x\|^2. \end{aligned} \tag{4.13}$$

□

Lemma 4.3. *For any* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$, *the functional* F_2 *defined by*

$$F_2(t) = -\rho_1\rho_3\langle\theta, \bar{v}_t\rangle, \quad \text{where } \bar{v}_t(x,t) = \int_0^x v_t(y,t)\mathrm{d}y\mathrm{d}x,$$

along the solution of systems (2.10)–(2.12) *satisfies the estimates*

$$\begin{aligned} F_2'(t) \le &-\frac{m_1\rho_1}{2}\|v_t\|^2 + \varepsilon_1\|v_x + u\|^2 + \varepsilon_2\|v - w\|^2 + C\|u_t\|^2 + C\|\sigma\|^2_{\mathcal{M}_{g_1}} \\ &+ C\left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\|\theta_x\|^2, \quad \forall t \ge 0. \end{aligned} \tag{4.14}$$

Proof. Differentiation of F_2, using $(2.10)_2$ and $(2.10)_4$, integration by parts, and boundary conditions (2.11), we arrive at

$$\begin{aligned} F_2'(t) &= -\rho_1\rho_3\langle\theta, \bar{v}_{tt}\rangle - \rho_1\rho_3\langle\theta_t, \bar{v}_t\rangle \\ &= -m_1\rho_1\|v_t\|^2 + m_1\rho_1\langle u_t, \bar{v}_t\rangle - \rho_3 k\langle v_x + u, \theta\rangle + \rho_3\lambda\langle\theta, \overline{(v - w)}\rangle \\ &\quad + \rho_1\beta_1(1-\eta)\langle\theta_x, v_t\rangle + \rho_1\beta_1\eta\int_0^{+\infty} g_1(s)\langle\sigma_x(s), v_t(t)\rangle\mathrm{d}s + \rho_3 m_1\|\theta\|^2. \end{aligned} \tag{4.15}$$

Young's, Cauchy-Schwarz, and Poincaré's inequalities yield

$$\begin{aligned} F_2'(t) \le &-m_1\rho_1\|v_t\|^2 + \frac{m_1\rho_1}{6}\|v_t\|^2 + \frac{5m_1\rho_1}{6}\|u_t\|^2 + \varepsilon_1\|v_x + u\|^2 + \frac{(\rho_3 k)^2}{4\varepsilon_1}\|\theta\|^2 \\ &+ \varepsilon_2\|v - w\|^2 + \frac{(\lambda\rho_3)^2}{4\varepsilon_2}\|\theta\|^2 + \frac{m_1\rho_1}{6}\|v_t\|^2 + \frac{5\rho_1(\beta_1(1-\eta))^2}{6m_1}\|\theta_x\|^2 \\ &+ \frac{m_1\rho_1}{6}\|v_t\|^2 + \frac{5\rho_1(\beta_1\eta)^2}{6m_1}\|\sigma\|^2_{\mathcal{M}_{g_1}} + m_1\rho_3\|\theta\|^2 \\ \le &-\frac{m_1\rho_1}{2}\|v_t\|^2 + \varepsilon_1\|v_x + u\|^2 + \varepsilon_2\|v - w\|^2 + C\|u_t\|^2 + C\|\sigma\|^2_{\mathcal{M}_{g_1}} \\ &+ C\left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\|\theta_x\|^2. \end{aligned}$$

Hence, we obtain (4.14). □

Lemma 4.4. *For any* $\varepsilon_3 > 0$ *and* $\varepsilon_4 > 0$, *the functional* F_3 *defined by*

$$F_3(t) = -\rho_2\rho_4\langle\vartheta, \bar{u}_t\rangle, \quad \text{where } \bar{u}_t(x,t) = \int_0^x u_t(y,t)\mathrm{d}y\mathrm{d}x,$$

along the solution of systems (2.10)–(2.12), *satisfies the estimate*

$$\begin{aligned} F_3'(t) \le &-\frac{m_2\rho_2}{2}\|u_t\|^2 + \varepsilon_3\|v_x + u\|^2 + \varepsilon_4\|u_x\|^2 + C\|\theta_x\|^2 + C\|\zeta\|^2_{\mathcal{M}_{g_2}} \\ &+ C\left(1 + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4}\right)\|\vartheta_x\|^2, \quad \forall t \ge 0. \end{aligned} \tag{4.16}$$

Proof. Differentiation of F_3, using $(2.10)_3$, and $(2.10)_6$, integration by parts and boundary conditions (2.11) lead to

$$\begin{aligned} F_3'(t) &= -\rho_2\rho_4\langle\vartheta, \bar{u}_{tt}\rangle - \rho_2\rho_4\langle\vartheta_t, \bar{u}_t\rangle \\ &= -m_2\rho_2\|u_t\|^2 + \rho_2\beta_2(1-\eta)\langle\vartheta_x, u_t\rangle + \rho_4 k\langle\overline{v_x + u}, \vartheta\rangle - \rho_4 m_1\langle\vartheta, \bar{\theta}\rangle \\ &\quad + \rho_4 b\langle\vartheta, u_x\rangle + \rho_2\beta_2\eta\int_0^{+\infty} g_2(s)\langle\zeta_x(s), u_t(t)\rangle ds + \rho_4 m_2\|\vartheta\|^2. \end{aligned} \tag{4.17}$$

Using Young's, Cauchy-Schwarz, and Poincaré's inequalities, we obtain

$$\begin{aligned} F_3'(t) &\le -m_2\rho_2\|u_t\|^2 + \frac{m_2\rho_2}{4}\|u_t\|^2 + \frac{3(\rho_2\beta_2(1-\eta))^2}{4m_2\rho_2}\|\vartheta_x\|^2 + \varepsilon_3\|v_x + u\|^2 \\ &\quad + \frac{(k\rho_4)^2}{4\varepsilon_3}\|\vartheta\|^2 + \frac{m_1\rho_4}{2}\|\theta\|^2 + \frac{m_1\rho_4}{2}\|\vartheta\|^2 + \varepsilon_4\|u_x\|^2 + \frac{(b\rho_4)^2}{4\varepsilon_4}\|\vartheta\|^2 \\ &\quad + \frac{m_2\rho_2}{4}\|u_t\|^2 + \frac{3(\rho_2\beta_2\eta)^2}{6m_2\rho_2}\|\zeta\|^2_{\mathcal{M}_{g_2}} + m_2\rho_4\|\vartheta\|^2 \\ &\le -\frac{m_2\rho_2}{2}\|u_t\|^2 + \varepsilon_3\|v_x + u\|^2 + \varepsilon_4\|u_x\|^2 + C\|\theta_x\|^2 + C\|\zeta\|^2_{\mathcal{M}_{g_2}} \\ &\quad + C\left(1 + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4}\right)\|\vartheta_x\|^2. \end{aligned}$$

□

The main stability result in this subsection is as follows.

Theorem 4.1. *Let* $\Phi_0 = (w_0, w_1, v_0, v_1, u_0, u_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0) \in \mathcal{D}(\mathcal{A})$ *be given. Suppose condition* (*A*) *holds and* $0 < \eta < 1$, *then the energy functional* $\mathcal{E}(t)$ *defined in* (4.1) *decays exponentially as time approaches infinity. That is, there exist two constants* $K, \mu > 0$ *such that*

$$\mathcal{E}(t) \le K e^{-\mu t}, \quad \forall t \ge 0. \tag{4.18}$$

Proof. We construct a suitable Lyapunov functional $\mathcal{L}$ such that

$$\Lambda_1\mathcal{E}(t) \le \mathcal{L}(t) \le \Lambda_2\mathcal{E}(t), \quad \forall t \ge 0 \tag{4.19}$$

for some $\Lambda_1, \Lambda_2 > 0$ and show that $\mathcal{L}$ satisfies for some $\delta > 0$

$$\mathcal{L}'(t) \le -\delta\mathcal{L}(t), \quad \forall t \ge 0, \tag{4.20}$$

from which, we obtain

$$\mathcal{L}(t) \le \mathcal{L}(0)e^{-\varpi t}, \quad \forall t \ge 0, \tag{4.21}$$

for some $\varpi > 0$. The exponential decay of $\mathcal{L}$ in (4.21) will then imply the exponential decay of the energy functional $\mathcal{E}(t)$. To achieve (4.19)–(4.21), we define $\mathcal{L}$ as follows:

$$\mathcal{L}(t) := n\mathcal{E} + n_1F_1(t) + n_2F_2(t) + n_3F_3(t), \quad t \ge 0, \tag{4.22}$$

for some $n, n_1, n_2, n_3 > 0$, which are specified later. Routine computations, applying Young's, Cauchy-Schwarz, and Poincaré's inequalities give (4.19) easily. Using the assumption (2.5), and Lemmas 4.1–4.4, it follows from (4.22) that

$$\begin{aligned}\mathcal{L}'(t) \le &-[\gamma_0 n - \rho n_1]\|w_t\|^2 - \rho_1\left[\frac{m_1}{2}n_2 - n_1\right]\|v_t\|^2 - \left[\frac{m_2\rho_2}{2}n_3 - Cn_2 - \rho_2 n_1\right]\|u_t\|^2 - \alpha n_1\|w_x\|^2\\ &- [\lambda n_1 - \varepsilon_2 n_2]\|v - w\|^2 - \left[\frac{k}{2}n_1 - \varepsilon_1 n_2 - \varepsilon_3 n_3\right]\|v_x + u\|^2 - \left[\frac{b}{2}n_1 - \varepsilon_4 n_3\right]\|u_x\|^2\\ &- \left[\beta_1(1-\eta)n - Cn_1 - Cn_2\left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) - Cn_3\right]\|\theta_x\|^2 - \left[\frac{\beta_1\eta\xi_1}{2}n - Cn_2\right]\|\sigma\|^2_{\mathcal{M}_{g_1}}\\ &- \left[\beta_2(1-\eta)n - Cn_1 - Cn_3\left(1 + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4}\right)\right]\|\vartheta_x\|^2 - \left[\frac{\beta_2\eta\xi_2}{2}n - Cn_3\right]\|\zeta\|^2_{\mathcal{M}_{g_2}}.\end{aligned} \tag{4.23}$$

By choosing

$$n_1 = 1, \quad \varepsilon_1 = \frac{k}{8n_2}, \quad \varepsilon_2 = \frac{\lambda}{2n_2}, \quad \varepsilon_3 = \frac{k}{8n_3}, \quad \varepsilon_4 = \frac{b}{4n_3},$$

we arrive at

$$\begin{aligned}\mathcal{L}'(t) \le &-[\gamma_0 n - \rho]\|w_t\|^2 - \rho_1\left[\frac{m_1}{2}n_2 - 1\right]\|v_t\|^2 - \left[\frac{m_2\rho_2}{2}n_3 - Cn_2 - \rho_2\right]\|u_t\|^2 - \alpha\|w_x\|^2 - \frac{\lambda}{2}\|v - w\|^2\\ &- \frac{k}{4}\|v_x + u\|^2 - \frac{b}{4}\|u_x\|^2 - \left[\beta_1(1-\eta)n - Cn_2\left(1 + \frac{8n_2}{k} + \frac{2n_2}{\lambda}\right) - Cn_3 - C\right]\|\theta_x\|^2\\ &- \left[\frac{\beta_1\eta\xi_1}{2}n - Cn_2\right]\|\sigma\|^2_{\mathcal{M}_{g_1}} - \left[\beta_2(1-\eta)n - Cn_3\left(1 + \frac{8n_2}{k} + \frac{4n_3}{b}\right) - C\right]\|\vartheta_x\|^2 - \left[\frac{\beta_2\eta\xi_2}{2}n - Cn_3\right]\|\zeta\|^2_{\mathcal{M}_{g_2}}.\end{aligned} \tag{4.24}$$

Now we choose n_2 large such that

$$\frac{m_1}{2}n_2 - 1 > 0.$$

Next we select n_3 large enough such that

$$\frac{m_2\rho_2}{2}n_3 - Cn_2 - \rho_2 > 0.$$

Finally, we choose n very large enough so that (4.19) remains valid and

$$\gamma_0 n - \rho > 0, \quad \beta_1(1-\eta)n - Cn_2\left(1 + \frac{8n_2}{k} + \frac{2n_2}{\lambda}\right) - Cn_3 - C > 0,$$

$$\frac{\beta_1\eta\xi_1}{2}n - Cn_2 > 0, \quad \beta_2(1-\eta)n - Cn_3\left(1 + \frac{8n_2}{k} + \frac{4n_3}{b}\right) - C > 0,$$

$$\frac{\beta_2\eta\xi_2}{2}n - Cn_3 > 0.$$

On account of Poincaré's inequality, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \le &-\delta(\rho\|w_t\|^2\rho_1 + \|v_t\|^2 + \rho_2\|u_t\|^2 + \alpha\|w_x\|^2 + \lambda\|v - w\|^2 + k\|v_x + u\|^2)\\ &- \delta(b\|u_x\|^2 + \rho_3\|\theta\|^2 + \beta_1\eta\|\sigma\|^2_{\mathcal{M}_{g_1}} + \rho_4\|\vartheta\|^2 + \beta_2\eta\|\zeta\|^2_{\mathcal{M}_{g_2}})\end{aligned} \tag{4.25}$$

for some $\delta > 0$. Thus, it follows from (4.1) and (4.19) that

$$\mathcal{L}'(t) \le -\delta_1\mathcal{L}(t), \quad \forall t \ge 0, \tag{4.26}$$

for some positive constant δ_1. Integrating (4.26) over $(0, t)$ yields for some $\varpi > 0$

$$\mathcal{L}(t) \le \mathcal{L}(0)e^{-\varpi t}, \quad \forall t \ge 0. \tag{4.27}$$

Hence, the exponential decay of the energy functional $\mathcal{E}(t)$ in (4.18) follows from (4.27) by using (4.19). □

4.2 Gurtin-Pipkin thermal law: case $\eta = 1$

In this subsection, we establish a stability result for systems (2.10)–(2.12) when $\eta = 1$. This translates to the following system:

$$\begin{cases} \rho w_{tt} - \alpha w_{xx} - \lambda(v - w) + \gamma_0 w_t = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_1 v_{tt} - k(v_x + u)_x + \lambda(v - w) + m_1\theta_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_2 u_{tt} - bu_{xx} + k(v_x + u) - m_1\theta + m_2\vartheta_x = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \rho_3\theta_t - \beta_1 \int_0^{+\infty} g_1(s)\sigma_{xx}(x,s)\mathrm{d}s + m_1(v_x + u)_t = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \sigma_t + \sigma_s - \theta = 0, & \text{in } (0,\pi) \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \rho_4\vartheta_t - \beta_2 \int_0^{+\infty} g_2(s)\zeta_{xx}(x,s)\mathrm{d}s + m_2 u_{xt} = 0, & \text{in } (0,\pi) \times \mathbb{R}_+, \\ \zeta_t + \zeta_s - \vartheta = 0, & \text{in } (0,\pi) \times \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \tag{4.28}$$

A quick observation here is that the stability of system (2.10) for $0 < \eta < 1$ (Coleman-Gurtin thermal law) is less difficult to achieve due to the presence of the dissipation given by the terms $-\theta_{xx}$ and $-\vartheta_{xx}$. Indeed, we can recover system (4.28) from (2.10) by considering in (1.15)

$$h_i^\delta(s) := \frac{(1-\eta)}{\delta} h_i\left(\frac{s}{\delta}\right) + \eta h_i(s), \quad \delta > 0, \quad i = 1, 2.$$

Taking limit $\delta \to 0$, in the sense of distribution, we obtain

$$h_i^\delta(s) \longrightarrow (1-\eta)\delta_0 + \eta h_i, \quad i = 1, 2,$$

where δ_0 is a delta function of unit mass. Therefore,

$$\int_0^{+\infty} h_1^\delta(s)\theta_{xx}(x, t-s)\mathrm{d}s \longrightarrow (1-\eta)\theta_{xx} + \eta \int_0^{+\infty} h_1(s)\theta_{xx}(x, t-s)\mathrm{d}s$$

and

$$\int_0^{+\infty} h_2^\delta(s)\vartheta_{xx}(x, t-s)\mathrm{d}s \longrightarrow (1-\eta)\vartheta_{xx} + \eta \int_0^{+\infty} h_2(s)\vartheta_{xx}(x, t-s)\mathrm{d}s.$$

Thus, we obtain (2.10). Also, the energy functional associated with (2.11)–(2.12), and (4.28) is defined by

$$\begin{aligned} \tilde{\mathcal{E}}(t) = & \frac{1}{2}[\rho\|w_t\|^2 + \rho_1\|v_t\|^2 + \rho_2\|u_t\|^2 + \alpha\|w_x\|^2 + \lambda\|(v - w)\|^2 + k\|v_x + u\|^2] \\ & + \frac{1}{2}[b\|u_x\|^2 + \rho_3\|\theta\|^2 + \|\sigma\|^2_{\mathcal{M}_{g_1}} + \rho_4\|\vartheta\|^2 + \|\zeta\|^2_{\mathcal{M}_{g_2}}], \quad \forall t \geq 0, \end{aligned} \tag{4.29}$$

and (4.29) satisfies

$$\tilde{\mathcal{E}}'(t) = -\gamma_0\|w_t\|^2 + \frac{\beta_1}{2}\int_0^{+\infty} g_1'(s)\|\sigma_x(s)\|^2\mathrm{d}s + \frac{\beta_2}{2}\int_0^{+\infty} g_2'(s)\|\zeta_x(s)\|^2\mathrm{d}s \leq 0, \quad \forall t \geq 0. \tag{4.30}$$

Using similar calculations as in Lemmas 4.3 and 4.4, the functionals

$$\tilde{F}_2(t) = -\rho_1\rho_3\langle\theta, \bar{v}_t\rangle, \quad \text{where } \bar{v}_t(x,t) = \int_0^x v_t(y,t)\mathrm{d}y\mathrm{d}x$$

and

$$\tilde{F}_3(t) = -\rho_2\rho_4\langle\vartheta, \bar{u}_t\rangle, \quad \text{where } \bar{u}_t(x,t) = \int_0^x u_t(y,t)\mathrm{d}y\mathrm{d}x$$

satisfy along the solution of (4.28) the following estimates:

$$\tilde{F}_2'(t) \le -\frac{m_1\rho_1}{2}\|v_t\|^2 + \tilde{\varepsilon}_1\|v_x + u\|^2 + \tilde{\varepsilon}_2\|v - w\|^2 + C\|u_t\|^2 + C\|\sigma\|^2_{\mathcal{M}_{g_1}} + C\left(1 + \frac{1}{\tilde{\varepsilon}_1} + \frac{1}{\tilde{\varepsilon}_2}\right)\|\theta\|^2, \quad \forall t \ge 0 \tag{4.31}$$

and

$$\tilde{F}_3'(t) \le -\frac{m_2\rho_2}{2}\|u_t\|^2 + \tilde{\varepsilon}_3\|v_x + u\|^2 + \tilde{\varepsilon}_4\|u_x\|^2 + C\|\theta\|^2 + C\|\zeta\|^2_{\mathcal{M}_{g_2}} + C\left(1 + \frac{1}{\tilde{\varepsilon}_3} + \frac{1}{\tilde{\varepsilon}_4}\right)\|\vartheta\|^2, \quad \forall t \ge 0, \tag{4.32}$$

respectively, for any positive $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, $\tilde{\varepsilon}_3$, and $\tilde{\varepsilon}_4$.

Lemma 4.5. *For any $\varepsilon_5 > 0$ and $\varepsilon_6 > 0$, the functional F_4 defined by*

$$\tilde{F}_4(t) = -\rho_3\int_0^\infty g_1(s)\langle\theta(t), \sigma(s)\rangle\mathrm{d}s,$$

along the solution of (4.28) *satisfies the estimate*

$$\tilde{F}_4'(t) \le -\frac{\rho_3 h_1(0)}{2}\|\theta\|^2 + \varepsilon_5\|u_t\|^2 + \varepsilon_6\|v_t\|^2 - C\int_0^\infty g_1'(s)\|\sigma_x(s)\|^2\mathrm{d}s + C\left(1 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6}\right)\|\sigma\|^2_{\mathcal{M}_{g_1}}, \quad \forall t \ge 0. \tag{4.33}$$

Proof. Differentiation of $\tilde{F}_4$, using $(4.28)_4$ and $(4.28)_5$, integration by parts, and boundary conditions (2.11) and recalling (2.4), we arrive at

$$\begin{aligned}\tilde{F}_4'(t) = &-\rho_3 h_1(0)\|\theta\|^2 + \beta_1\left\|\int_0^\infty g_1(s)\sigma_x(s)\mathrm{d}s\right\|^2 \\ &+ m_1\int_0^\infty g_1(s)\langle u_t(t), \sigma(s)\rangle\mathrm{d}s - m_1\int_0^\infty g_1(s)\langle v_t(t), \sigma_x(s)\rangle\mathrm{d}s \\ &+ \rho_3\int_0^\infty g_1(s)\langle\theta(t), \sigma_s(s)\rangle\mathrm{d}s.\end{aligned} \tag{4.34}$$

Using Young's, Cauchy-Schwarz, and Poincaré's inequalities, we have

$$\beta_1\left\|\int_0^\infty g_1(s)\sigma_x(s)\mathrm{d}s\right\|^2 \le C\|\sigma\|^2_{\mathcal{M}_{g_1}}, \tag{4.35}$$

$$\left|m_1\int_0^\infty g_1(s)\langle u_t(t), \sigma(s)\rangle\mathrm{d}s\right| \le \varepsilon_5\|u_t\|^2 + \frac{C}{\varepsilon_5}\|\sigma\|^2_{\mathcal{M}_{g_1}}, \quad \text{for any } \varepsilon_5 > 0, \tag{4.36}$$

$$\left| -m_1 \int_0^\infty g_1(s) \langle v_t(t), \sigma_x(s) \rangle \mathrm{d}s \right| \le \varepsilon_6 \|v_t\|^2 + \frac{C}{\varepsilon_6} \|\sigma\|^2_{\mathcal{M}_{g_1}}, \quad \text{for any} \quad \varepsilon_6 > 0, \tag{4.37}$$

and using integration by parts with respect to s, we obtain

$$\begin{aligned} \left| \rho_3 \int_0^\infty g_1(s) \langle \theta(t), \sigma_s(s) \rangle \mathrm{d}s \right| &= \left| -\rho_3 \int_0^\infty g_1'(s) \langle \theta(t), \sigma(s) \rangle \mathrm{d}s \right| \\ &\le C\|\theta\| \left(-\int_0^\infty g_1'(s) \|\sigma_x(s)\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \\ &\le \frac{\rho_3 h_1(0)}{2} \|\theta\|^2 - C \int_0^\infty g_1'(s) \|\sigma_x(s)\|^2 \mathrm{d}s. \end{aligned} \tag{4.38}$$

On account of (4.35)–(4.38), it follows from (4.34) that

$$\begin{aligned} \tilde{F}4'(t) \le & -\frac{\rho_3 h_1(0)}{2} \|\theta\|^2 + \varepsilon_5 \|u_t\|^2 + \varepsilon_6 \|v_t\|^2 - C \int_0^\infty g_1'(s) \|\sigma_x(s)\|^2 \mathrm{d}s \\ & + C\left(1 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6}\right) \|\sigma\|^2_{\mathcal{M}_{g_1}}. \end{aligned}$$

Hence, we obtain (4.33). □

Lemma 4.6. *For any $\varepsilon_7 > 0$, the functional $\tilde{F}_5$ defined by*

$$\tilde{F}_5(t) = -\rho_4 \int_0^\infty g_2(s) \langle \vartheta(t), \zeta(s) \rangle \mathrm{d}s,$$

along the solution of (4.28) *satisfies the estimate*

$$\tilde{F}_5'(t) \le -\frac{\rho_4 h_2(0)}{2} \|\vartheta\|^2 + \varepsilon_7 \|u_t\|^2 - C \int_0^\infty g_2'(s) \|\zeta_x(s)\|^2 \mathrm{d}s + C\left(1 + \frac{1}{\varepsilon_7}\right) \|\zeta\|^2_{\mathcal{M}_{g_2}}, \quad \forall t \ge 0. \tag{4.39}$$

Proof. Differentiation of $\tilde{F}_5$, using $(4.28)_6$ and $(4.28)_7$, integration by parts, and boundary conditions (2.11) and recalling (2.4), we arrive at

$$\begin{aligned} \tilde{F}_5'(t) = & -\rho_4 h_2(0) \|\vartheta\|^2 + \beta_2 \left\| \int_0^\infty g_2(s) \zeta_x(s) \mathrm{d}s \right\|^2 - m_2 \int_0^\infty g_1(s) \langle u_t(t), \zeta_x(s) \rangle \mathrm{d}s \\ & + \rho_4 \int_0^\infty g_2(s) \langle \vartheta(t), \zeta_s(s) \rangle \mathrm{d}s. \end{aligned} \tag{4.40}$$

Similar computations as in (4.35)–(4.38) lead to (4.39). □

Now we give the stability result in this subsection.

Theorem 4.2. *Let $\Phi_0 = (w_0, w_1, v_0, v_1, u_0, u_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0) \in \mathcal{D}(\mathcal{A})$ be given. Suppose condition (A) holds and $\eta = 1$, then there exist two constants $\tilde{K}, \tilde{\mu} > 0$ such that the energy functional $\tilde{\mathcal{E}}(t)$ defined in* (4.29) *satisfies*

$$\tilde{\mathcal{E}}(t) \le \tilde{K} e^{-\tilde{\mu} t}, \quad \forall t \ge \quad 0. \tag{4.41}$$

Proof. We set

$$\tilde{L}(t) := N\tilde{\mathcal{E}}(t) + F_1(t) + N_2\tilde{F}_2(t) + N_3\tilde{F}_3(t) + N_4\tilde{F}_4(t) + N_5\tilde{F}_5(t), \quad t \geq 0, \tag{4.42}$$

for some $N, N, N_2, N_3, N_4, N_5 > 0$ to be specified. Routine computations, applying Young's, Cauchy-Schwarz, and Poincaré's inequalities, give

$$\tilde{\alpha}_1\tilde{\mathcal{E}}(t) \leq \tilde{L}(t) \leq \tilde{\alpha}_2\tilde{\mathcal{E}}(t), \quad t \geq 0, \tag{4.43}$$

for some positive constants $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. On account of assumption (2.5) and the estimates in (4.11), (4.30), (4.31), (4.32), (4.33), and (4.39), we arrive at

$$\begin{aligned}
\tilde{L}'(t) \leq &-[\gamma_0 N - \rho]\|w_t\|^2 - \left[\frac{m_1\rho_1}{2}N_2 - \varepsilon_6 N_4 - \rho_1\right]\|v_t\|^2 \\
&- \left[\frac{m_2\rho_2}{2}N_3 - CN_2 - \varepsilon_5 N_4 - \varepsilon_7 N_5 - \rho_2\right]\|u_t\|^2 - \alpha\|w_x\|^2 \\
&- [\lambda - \tilde{\varepsilon}_2 N_2]\|v - w\|^2 - \left[\frac{b}{2} - \tilde{\varepsilon}_4 N_3\right]\|u_x\|^2 - \left[\frac{k}{2} - \tilde{\varepsilon}_1 N_2 - \tilde{\varepsilon}_3 N_3\right]\|v_x + u\|^2 \\
&- \left[\frac{\rho_3 h_1(0)}{2}N_4 - CN_2\left(1 + \frac{1}{\tilde{\varepsilon}_1} + \frac{1}{\tilde{\varepsilon}_2}\right) - CN_3 - C\right]\|\theta\|^2 \\
&- \left[\frac{\beta_1\xi_1}{2}N - C\xi_1 N_4 - CN_2 - CN_4\left(1 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6}\right)\right]\|\sigma\|^2_{\mathcal{M}_{g_1}} \\
&- \left[\frac{\rho_4 h_2(0)}{2}N_5 - CN_3\left(1 + \frac{1}{\tilde{\varepsilon}_3} + \frac{1}{\tilde{\varepsilon}_4}\right) - C\right]\|\vartheta\|^2 \\
&- \left[\frac{\beta_2\xi_2}{2}N - C\xi_2 N_5 - CN_3 - CN_5\left(1 + \frac{1}{\varepsilon_7}\right)\right]\|\zeta\|^2_{\mathcal{M}_{g_2}}.
\end{aligned} \tag{4.44}$$

Now by choosing

$$\tilde{\varepsilon}_1 = \frac{k}{8N_2}, \quad \tilde{\varepsilon}_2 = \frac{\lambda}{2N_2}, \quad \tilde{\varepsilon}_3 = \frac{k}{8N_3}, \quad \tilde{\varepsilon}_4 = \frac{b}{4N_3},$$
$$\varepsilon_5 = \frac{m_2\rho_2}{8N_4}, \quad \varepsilon_6 = \frac{m_1\rho_1}{4N_4}, \quad \varepsilon_7 = \frac{m_2\rho_2}{8N_5},$$

(4.44) takes the form

$$\begin{aligned}
\tilde{L}'(t) \leq &-[\gamma_0 N - \rho]\|w_t\|^2 - \left[\frac{m_1\rho_1}{4}N_2 - \rho_1\right]\|v_t\|^2 - \left[\frac{m_2\rho_2}{4}N_3 - CN_2 - \rho_2\right]\|u_t\|^2 \\
&- \alpha\|w_x\|^2 - \frac{\lambda}{2}\|v - w\|^2 - \frac{b}{4}\|u_x\|^2 - \frac{k}{4}\|v_x + u\|^2 \\
&- \left[\frac{\rho_3 h_1(0)}{2}N_4 - CN_2\left(1 + \frac{8N_2}{k} + \frac{2N_2}{\lambda}\right) - CN_3 - C\right]\|\theta\|^2 \\
&- \left[\frac{\beta_1\xi_1}{2}N - C\xi_1 N_4 - CN_2 - CN_4\left(1 + \frac{8N_4}{m_2\rho_2} + \frac{4N_4}{m_1\rho_1}\right)\right]\|\sigma\|^2_{\mathcal{M}_{g_1}} \\
&- \left[\frac{\rho_4 h_2(0)}{2}N_5 - CN_3\left(1 + \frac{8N_3}{k} + \frac{4N_3}{b}\right) - C\right]\|\vartheta\|^2 \\
&- \left[\frac{\beta_2\xi_2}{2}N - C\xi_2 N_5 - CN_3 - CN_5\left(1 + \frac{8N_5}{m_2\rho_2}\right)\right]\|\zeta\|^2_{\mathcal{M}_{g_2}}.
\end{aligned} \tag{4.45}$$

At this stage, we will specify the remaining parameters. We choose N_2 large such that

$$\frac{m_1\rho_1}{4}N_2 - \rho_1 > 0$$

Second, we select N_3 large enough such that

$$\frac{m_2\rho_2}{4}N_3 - CN_2 - \rho_2 > 0.$$

Third, we choose N_4 and N_5 large enough such that

$$\frac{\rho_3 h_1(0)}{2}N_4 - CN_2\left(1 + \frac{8N_2}{k} + \frac{2N_2}{\lambda}\right) - CN_3 - C > 0$$

and

$$\frac{\rho_4 h_2(0)}{2}N_5 - CN_3\left(1 + \frac{8N_3}{k} + \frac{4N_3}{b}\right) - C > 0.$$

Finally, we choose N very large enough so that (4.43) remains valid and

$$\gamma_0 N - \rho > 0, \quad \frac{\beta_1\xi_1}{2}N - C\xi_1 N_4 - CN_2 - CN_4\left(1 + \frac{8N_4}{m_2\rho_2} + \frac{4N_4}{m_1\rho_1}\right) > 0,$$

$$\frac{\beta_2\xi_2}{2}N - C\xi_2 N_5 - CN_3 - CN_5\left(1 + \frac{8N_5}{m_2\rho_2}\right) > 0.$$

Therefore, we arrive at

$$\begin{aligned}\tilde{L}'(t) \le & -\tilde{\delta}(\rho\|w_t\|^2\rho_1 + \|v_t\|^2 + \rho_2\|u_t\|^2 + \alpha\|w_x\|^2 + \lambda\|v - w\|^2 + k\|v_x + u\|^2) \\ & - \tilde{\delta}(b\|u_x\|^2 + \rho_3\|\theta\|^2 + \|\sigma\|^2_{\mathcal{M}_{g_1}} + \rho_4\|\vartheta\|^2 + \|\zeta\|^2_{\mathcal{M}_{g_2}})\end{aligned} \tag{4.46}$$

for some $\tilde{\delta} > 0$. Thus, it follows from (4.29) and (4.43) that

$$\tilde{L}'(t) \le -\tilde{\delta}_1\tilde{L}(t), \quad \forall t \ge 0, \tag{4.47}$$

for some constant $\tilde{\delta}_1 > 0$. Integrating (4.47) over $(0, t)$ yields for some $\tilde{\varpi} > 0$

$$\tilde{L}(t) \le \tilde{L}(0)(0)e^{-\tilde{\varpi}t}, \quad \forall t \ge 0. \tag{4.48}$$

Hence, the exponential estimate of the energy functional $\tilde{\mathcal{E}}(t)$ in (4.41) follows from (4.48) by using (4.43). This completes the proof. □

Acknowledgment: The author gratefully acknowledges the technical and financial support from the Ministry of Education and the University of Hafr Al Batin, Saudi Arabia.

Funding information: This research work was funded by Institutional fund project # IFP-A-2022-2-1-04.

Conflict of interest: The author declares no potential conflict of interests.

Data availability statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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