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Research Article

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On average theta functions of certain quadratic forms as sums of Eisenstein series

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Abstract: Let Q be an integral positive definite quadratic form of level N in $2k$ variables. Further, we assume that $(-1)^k N$ is a fundamental discriminant. We express the average theta function of Q as an explicit sum of Eisenstein series, which are Hecke eigenforms by applying Atkin-Lehner involution.

Keywords: average theta function, Eisenstein series, Atkin-Lehner involution

MSC 2020: 11E45, 11F11, 11F25, 11F27

1 Introduction

Let Q be an integral positive definite quadratic form of level N in r variables and A be its Hessian matrix. Then it is well known that the theta function

$$\Theta_Q(\tau) = \sum_{\mathbf{x}\in\mathbb{Z}^r} e^{2\pi i Q(\mathbf{x})\tau} \quad (\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\})$$

associated with Q is a modular form of weight $r/2$ for $\Gamma_0(N)$ with a certain Nebentypus [1, Theorem 10.1], and it can be uniquely written as follows:

$$\Theta_Q(\tau) = E_Q(\tau) + f_Q(\tau),$$

where $E_Q(\tau)$ and $f_Q(\tau)$ are an Eisenstein series and a cusp form, respectively. By the work of Siegel [2–4] and Weil [5], which is known as the Siegel-Weil formula, the Eisenstein part $E_Q(\tau)$ can be expressed as follows:

$$E_Q(\tau) = \Theta(\mathrm{gen}(Q),\tau) \coloneqq \left(\sum_{[Q']\in\mathrm{gen}(Q)} \frac{1}{o(Q')}\right)^{-1} \sum_{[Q']\in\mathrm{gen}(Q)} \frac{\Theta_{Q'}(\tau)}{o(Q')},$$

where $o(Q')$ is the number of isometries on Q' and $[Q']$ is the isometry class containing Q' in the genus $\mathrm{gen}(Q)$ of Q. The Eisenstein part $\Theta(\mathrm{gen}(Q),\tau)$ is called the average theta function of Q. Walling [6] provided an explicit expression of the generalized average theta function (introduced by Siegel) as a sum of the Siegel-Eisenstein series when r is even. As a corollary of Walling's work, when the weight $r/2 \geq 3$, one can obtain an explicit expression of $\Theta(\mathrm{gen}(Q),\tau)$ as a linear combination of Eisenstein series defined at each cusp. Walling did not consider the case $r = 2, 4$ because the convergence questions for Eisenstein series of weight 1 and 2 are more delicate. Recently, when Q is a primitive binary quadratic form and $-N$ is a fundamental discriminant, Guerzhoy and Kane [7] gave an explicit formula for $\Theta(\mathrm{gen}(Q),\tau)$ as a sum of Eisenstein series, which are Hecke eigenforms by using the algebraic theory of binary quadratic forms.

In this article, we shall generalize and improve the results of the studies by Walling [6] and Guerzhoy and Kane [7] to certain quadratic forms of arbitrary even rank r by applying the theory of modular forms. More

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precisely, we assume that $r = 2k$ is even, $(-1)^k N$ is a fundamental discriminant and the Kronecker symbols $\left(\frac{(-1)^k N}{\cdot}\right)$ and $\left(\frac{(-1)^k \det(A)}{\cdot}\right)$ coincide. We shall express $\Theta(\mathrm{gen}(Q), \tau)$ of any integral weight k as an explicit linear combination of Eisenstein series, which are also Hecke eigenforms by applying the action of the Atkin-Lehner involution (Theorem 4.5).

2 Modular forms and the Atkin-Lehner involution

Let k be a positive integer and $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$. For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}) = \{\alpha \in \mathrm{GL}_2(\mathbb{R}) \mid \det(\alpha) > 0\}$, we define the slash operator $\cdot|_k\alpha$ on the functions $f(\tau)$ on $\mathbb{H}$ by

$$f(\tau)|_k\alpha = \det(\alpha)^{k/2}(c\tau + d)^{-k} f(\alpha \cdot \tau),$$

where α acts on $\mathbb{H}$ as the fractional linear transformation $\alpha \cdot \tau = (a\tau + b)/(c\tau + d)$. For a positive integer N, put $\Gamma_0(N) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\right\}$. For a Dirichlet character χ modulo N, we define a character χ of $\Gamma_0(N)$ to be

$$\chi(\gamma) = \chi(d) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

A holomorphic function $f(\tau)$ on $\mathbb{H}$ is a modular form of weight k for $\Gamma_0(N)$ associated with the character χ if
(i) $f(\tau)|_k\gamma = \chi(\gamma) f(\tau)$ for all $\gamma \in \Gamma_0(N)$,
(ii) $f(\tau)$ is holomorphic at every cusp.

From the definition, $f(\tau)$ has a Laurent series expansion with respect to $q = e^{2\pi i\tau}$ of the form

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n \quad (a(n) \in \mathbb{C}),$$

which is called the Fourier expansion of $f(\tau)$. The $a(n)$ is said to be the n-th Fourier coefficient of $f(\tau)$. If a modular form vanishes at every cusp, then it is called a cusp form. We denote by $\mathcal{M}_k(N, \chi)$, $\mathcal{S}_k(N, \chi)$, and $\mathcal{E}_k(N, \chi)$ the space of all modular forms, cusp forms and Eisenstein series of weight k for $\Gamma_0(N)$ associated with the character χ, respectively. As well known, $\mathcal{M}_k(N, \chi) = \mathcal{E}_k(N, \chi) \oplus \mathcal{S}_k(N, \chi)$ and $\mathcal{M}_k(N, \chi) = \{0\}$ if $\chi(-1) \neq (-1)^k$ ([8, Theorem 2.1.7(1) and Lemma 4.3.2(1)]).

Now, let N be a positive integer and χ a Dirichlet character modulo N. For a positive divisor e of N, we say that e is an exact divisor of N and write $e||N$ if $\gcd(e, N/e) = 1$. For $e||N$, we may uniquely express χ as $\chi = \chi^{[e]}\chi^{[N/e]}$, where $\chi^{[e]}$ and $\chi^{[N/e]}$ are Dirichlet characters modulo e and N/e, respectively. For each $e||N$, put $W_e = \begin{pmatrix} ae & b \\ cN & de \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z})$ with $\det(W_e) = e$, that is, $ade^2 - bcN = e$. The slash operator $\cdot|_k W_e$ on $f \in \mathcal{M}_k(N, \chi)$ is called an Atkin-Lehner involution. Then the following is well known.

Proposition 2.1. *Let $e||N$ and χ be a Dirichlet character modulo N.*
(i) χ is primitive if and only if $\chi^{[e]}$ and $\chi^{[N/e]}$ are primitive.
(ii) Let $W_e = \begin{pmatrix} ae & b \\ cN & de \end{pmatrix}$, $\widehat{W_e} = \begin{pmatrix} a'e & b' \\ c'N & d'e \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z})$ with $\det(W_e) = \det(\widehat{W_e}) = e$, $b' \equiv 1 \pmod e$, and $a' \equiv 1 \pmod{N/e}$. For $f \in \mathcal{M}_k(N, \chi)$, we have

$$f|_k W_e \in \mathcal{M}_k(N, \overline{\chi}^{[e]}\chi^{[N/e]}) \quad \text{and} \quad f|_k W_e = \overline{\chi}^{[e]}(b)\overline{\chi}^{[N/e]}(a) f|_k \widehat{W_e}.$$

Proof. For (i), see [9, Lemma 9.3]. For (ii), see [10, Proposition 1.1]. □

Remark 2.2. By Proposition 2.1, the actions of W_e and $\widehat{W_e}$ only differ by a nonzero constant. In this article, we will specially choose $a = c = 1$ so that $W_e = \begin{pmatrix} e & b \\ N & de \end{pmatrix}$ and $de - b(N/e) = 1$.

3 Eisenstein series

For a nonnegative integer k and a nontrivial primitive Dirichlet character χ modulo N, we let $B_{k,\chi}$ be a generalized Bernoulli number defined by

$$\sum_{a=1}^{N-1} \chi(a) \frac{xe^{ax}}{e^{Nx} - 1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{x^k}{k!}.$$

Further, we let $L(s, \chi)$ be the Dirichlet L-function defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (s \in \mathbb{C}).$$

Then it is well known that the aforementioned series converges for $\mathrm{Re}(s) > 1$ and extends to an entire function, $L(1 - k, \chi) = -B_{k,\chi}/k$ and $B_{k,\chi} \neq 0$ if and only if $\chi(-1) = (-1)^k$ ([11, Chapter XIV, Corollary of Theorem 2.2 and Theorem 2.3]).

Proposition 3.1. *Denote by χ_0 the trivial character modulo 1. Let χ_1 and χ_2 be primitive Dirichlet characters modulo L and M, respectively. Let k be a positive integer satisfying $\chi_1(-1)\chi_2(-1) = (-1)^k$. Define*

$$E_{k,\chi_1,\chi_2}(\tau) = b_0 + \sum_{n=1}^{\infty} \left(\sum_{d>0, d|n} \chi_1(n/d)\chi_2(d)d^{k-1} \right) q^n \in \mathbb{C}[[q]] \quad (\tau \in \mathbb{H}),$$

where

$$b_0 = \begin{cases} 0, & \text{if } L > 1, \\ -B_{k,\chi_1\chi_2}/2k, & \text{if } L = 1. \end{cases}$$

Then except for the case when $k = 2$ and $\chi_1 = \chi_2 = \chi_0$, $E_{k,\chi_1,\chi_2}(\tau)$ defines an element of $\mathcal{M}_k(LM, \chi_1\chi_2)$, which is called an Eisenstein series. Furthermore, the $E_{k,\chi_1,\chi_2}(\tau)$ are Hecke eigenforms.

Proof. See [8, Theorem 4.7.1] or [12, Theorems 5.8 and 5.10]. □

Proposition 3.2. *Let k be a positive integer and χ a nontrivial primitive Dirichlet character modulo N with $\chi(-1) = (-1)^k$. Let*

$$\begin{aligned} X_{k,\chi} = \{(\chi_1, \chi_2) \mid\ & \chi_1 \textit{ and } \chi_2 \textit{ are primitive Dirichlet characters modulo } L \text{ and } M, \text{ respectively}, \\ & LM = N \textit{ and } \chi_1\chi_2 = \chi\} \end{aligned}$$

and

$$Y_{1,\chi} = \{\{\chi_1, \chi_2\} \mid (\chi_1, \chi_2) \in X_{1,\chi}\}.$$

Then $\mathcal{B} = \{E_{k,\chi_1,\chi_2}(\tau) \mid (\chi_1, \chi_2) \in X_{k,\chi}\}$ is a basis of $\mathcal{E}_k(N, \chi)$. In particular, we have

$$\{E_{1,\chi_1,\chi_2}(\tau) \mid \{\chi_1, \chi_2\} \in Y_{1,\chi}\} = \{E_{1,\chi_1,\chi_2}(\tau) \mid (\chi_1, \chi_2) \in X_{1,\chi}\}$$

since $E_{1,\chi_1,\chi_2}(\tau) = E_{1,\chi_2,\chi_1}(\tau)$.

Proof. See [8, Theorem 4.7.2] or [12, Theorem 5.9]. □

Remark 3.3. Let k be a positive integer and χ a nontrivial primitive Dirichlet character modulo N with $\chi(-1) = (-1)^k$. Then by Proposition 2.1 (i), one may easily verify that

$$\begin{aligned} \{M \in \mathbb{Z}^+ \mid M \| N\} &\longrightarrow X_{k,\chi} \\ M &\mapsto (\chi^{[N/M]}, \chi^{[M]}) \end{aligned}$$

is a bijection. Thus, we may express the basis $\mathcal{B}$ in Proposition 3.2 as $\mathcal{B} = \{E_{k,\chi^{[N/M]},\chi^{[M]}}(\tau) \mid M \| N\}$.

Now, for a positive integer k, let χ_1 and χ_2 be Dirichlet characters modulo L and M, respectively, such that $\chi_1(-1)\chi_2(-1) = (-1)^k$. For $\tau \in \mathbb{H}$ and a complex number s, an Eisenstein series with parameter s is defined by

$$E_k(\tau, s; \chi_1, \chi_2) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi_1(m)\chi_2(n)}{(m\tau + n)^k |m\tau + n|^{2s}}.$$

Then $E_k(\tau, s; \chi_1, \chi_2)$ is holomorphic on $k + 2\mathrm{Re}(s) > 2$. Furthermore, it is analytically continued to a holomorphic function

$$H_k(\tau, s; \chi_1, \chi_2) \text{ for } \begin{cases} \text{the whole } s\text{-plane}, & \text{if } \chi_2 \text{ is nontrivial}, \\ \mathrm{Re}(s) > (1-k)/2, & \text{if } \chi_2 \text{ is trivial and } k \geq 2, \\ \mathrm{Re}(s) > -1/2, & \text{if } \chi_2 \text{ is trivial and } k = 1 \end{cases} \tag{3.1}$$

[8, Theorem 7.2.9 and Corollary 7.2.10]. Thus, $H_k(\tau, s; \chi_1, \chi_2)$ is holomorphic at $s = 0$, and the function

$$E_k(\tau; \chi_1, \chi_2) \coloneqq H_k(\tau, 0; \chi_1, \chi_2)$$

becomes a holomorphic function of τ on $\mathbb{H}$ except for the case when $k = 2$, and both χ_1 and χ_2 are trivial [8, Corollary 7.2.14].

Proposition 3.4. *Let χ_1 and χ_2 be primitive Dirichlet characters modulo L and M, respectively, satisfying $\chi_1(-1)\chi_2(-1) = (-1)^k$. Except for the case when $k = 2$ and $\chi_1 = \chi_2 = \chi_0$, we have*

$$E_k(M\tau; \chi_1, \overline{\chi}_2) = A_{k,\overline{\chi}_2} E_{k,\chi_1,\chi_2}(\tau),$$

where

$$A_{k,\varepsilon} = \frac{2(-2\pi i)^k G(\varepsilon)}{N^k (k-1)!} \quad \text{and} \quad G(\varepsilon) = \sum_{a=0}^{N-1} \varepsilon(a) e^{2\pi i a/N}$$

for a primitive Dirichlet character ε modulo N.

Proof. See [8, (7.1.13) and Lemma.7.2.19]. □

Weisinger [13, Proposition 14] derived the action of the Atkin-Lehner involution on the Eisenstein series. Young [14, Section 9.1] obtained the same result for the nonholomorphic Eisenstein series. We shall provide a slightly different proof of Weisinger's result by adopting the ideas of Young as follows.

Theorem 3.5. *Let k be a positive integer. Let χ_1 and χ_2 be primitive Dirichlet characters modulo L and M, respectively, such that $\chi_1(-1)\chi_2(-1) = (-1)^k$ and $\gcd(L, M) = 1$. Further, let $N = LM$ and $\chi = \chi_1\chi_2$. For each $e \| N$, we obtain the action of the Atkin-Lehner involution $W_e = \begin{pmatrix} e & b \\ N & de \end{pmatrix}$ on $E_{k,\chi_1,\chi_2}(\tau)$ as follows:*

$$E_{k,\chi_1,\chi_2}(\tau)|_k W_e = \frac{G(\psi_2)}{G(\overline{\chi}_2)} e^{-k/2} e_M^k \overline{\chi}^{[N/e]}(e_M) \chi^{[e]}(L/e_L) \chi_1^{[e_L]}(-1) E_{k,\psi_1,\overline{\psi}_2}(\tau), \tag{3.2}$$

where $e_L = \gcd(e, L)$, $e_M = \gcd(e, M)$, $\psi_1 = \chi_1^{[L/e_L]}\overline{\chi}_2^{[e_M]}$, *and* $\psi_2 = \chi_1^{[e_L]}\overline{\chi}_2^{[M/e_M]}$. *Furthermore, the constant term of the Fourier expansion of* $E_{k,\chi_1,\chi_2}(\tau)|_k W_e$ *is nonzero if and only if* $e = L$. *When* $e = L$ *and* χ *is real, the constant term is given by*

$$\frac{G(\chi)}{G(\chi^{[M]})} L^{-k/2} \chi^{[L]}(-1)\left(-\frac{B_{k,\chi}}{2k}\right).$$

Proof. Let $e||N$ and $W_e = \begin{pmatrix} e & b \\ N & de \end{pmatrix}$ be the Atkin-Lehner involution. Put $e_L = \gcd(e, L)$, $e_M = \gcd(e, M)$, and $\delta = \begin{pmatrix} e_M & L/e_L \\ bM/e_M & de_L \end{pmatrix}$. Then we have

$$e = e_L e_M \quad \text{and} \quad \det(\delta) = de - b\frac{N}{e} = 1$$

since $\gcd(L, M) = 1$ and $\det(W_e) = e$. Thus, we obtain an invertible linear change of variables

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \delta \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} me_M + nL/e_L \\ mbM/e_M + nde_L \end{pmatrix} \quad \text{or, equivalently} \quad \begin{pmatrix} m \\ n \end{pmatrix} = \delta^{-1}\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} m'de_L - n'L/e_L \\ -m'bM/e_M + n'e_M \end{pmatrix}$$

on $\mathbb{Z}^2 - \{(0, 0)\}$, and one may verify that

$$m(M(W_e \cdot \tau)) + n = (N\tau + de)^{-1}((mMe + nN)\tau + (mMb + nde)) = e_M(N\tau + de)^{-1}\left(m'\left(\frac{Me_L}{e_M}\tau\right) + n'\right). \tag{3.3}$$

Let $\psi_1 = \chi_1^{[L/e_L]}\overline{\chi}_2^{[e_M]}$ and $\psi_2 = \chi_1^{[e_L]}\overline{\chi}_2^{[M/e_M]}$. Note that $\chi_1 = \chi^{[L]}$ and $\chi_2 = \chi^{[M]}$. Since

$$1 = de - b\frac{N}{e} = de_L e_M - b\frac{M}{e_M}\frac{L}{e_L}, \quad \chi_1^{[e_L]}\chi_2^{[e_M]} = \chi^{[e]} \quad \text{and} \quad \chi_1^{[L/e_L]}\chi_2^{[M/e_M]} = \chi^{[N/e]},$$

we obtain that $de_L \equiv e_M^{-1} \pmod{L/e_L}$, $-bM/e_M \equiv (L/e_L)^{-1} \pmod{e_M}$ and

$$\begin{aligned}
\chi_1(m)\overline{\chi}_2(n) &= \chi_1(m'de_L - n'L/e_L)\overline{\chi}_2(-m'bM/e_M + n'e_M) \\
&= \chi_1^{[L/e_L]}(m'de_L)\chi_1^{[e_L]}(-n'L/e_L)\overline{\chi}_2^{[e_M]}(-m'bM/e_M)\overline{\chi}_2^{[M/e_M]}(n'e_M) \\
&= \chi_1^{[L/e_L]}(m'e_M^{-1})\chi_1^{[e_L]}(-1)\chi_1^{[e_L]}(n'L/e_L)\overline{\chi}_2^{[e_M]}(m'(L/e_L)^{-1})\overline{\chi}_2^{[M/e_M]}(n'e_M) \\
&= (\chi_1^{[L/e_L]}(m')\overline{\chi}_2^{[e_M]}(m'))(\chi_1^{[e_L]}(n')\overline{\chi}_2^{[M/e_M]}(n')) \\
&\quad \cdot(\overline{\chi}_1^{[L/e_L]}(e_M)\overline{\chi}_2^{[M/e_M]}(e_M))(\chi_1^{[e_L]}(L/e_L)\chi_2^{[e_M]}(L/e_L))\chi_1^{[e_L]}(-1) \\
&= \psi_1(m')\psi_2(n')\overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1).
\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), for complex numbers s satisfying (3.1), we derive that

$$\begin{aligned}
&H_k(M\tau, s; \chi_1, \overline{\chi}_2)|_k W_e \\
&\quad = E_k(M\tau, s; \chi_1, \overline{\chi}_2)|_k W_e \\
&\quad = e^{k/2}(N\tau + de)^{-k}E_k(M(W_e \cdot \tau), s; \chi_1, \overline{\chi}_2) \\
&\quad = e^{k/2}(N\tau + de)^{-k} \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi_1(m)\overline{\chi}_2(n)}{(m(M(W_e \cdot \tau)) + n)^k|m(M(W_e \cdot \tau)) + n|^{2s}} \\
&\quad = e^{k/2}e_M^{-k}\overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1)e_M^{-2s}|N\tau + de|^{-2s} \\
&\qquad \cdot \sum_{(m',n')\in\mathbb{Z}^2-\{(0,0)\}} \frac{\psi_1(m')\psi_2(n')}{(m'(Me_L/e_M)\tau + n')^k|m'(Me_L/e_M)\tau + n|^{2s}} \\
&\quad = e^{k/2}e_M^{-k}\overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1)e_M^{-2s}|N\tau + de|^{-2s}E_k((Me_L/e_M)\tau, s; \psi_1, \psi_2) \\
&\quad = e^{k/2}e_M^{-k}\overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1)e_M^{-2s}|N\tau + de|^{-2s}H_k((Me_L/e_M)\tau, s; \psi_1, \psi_2).
\end{aligned}$$

Note that Me_L/e_M is the conductor of ψ_2. From Proposition 3.4 and the aforementioned equation, we deduce that

$$\begin{aligned}E_{k,\chi_1,\chi_2}(\tau)|_k W_e &= A^{-1}_{k,\overline{\chi}_2} E_k(M\tau;\, \chi_1, \overline{\chi}_2)|_k W_e \\ &= A^{-1}_{k,\overline{\chi}_2} H_k(M\tau, 0;\, \chi_1, \overline{\chi}_2)|_k W_e \\ &= A^{-1}_{k,\overline{\chi}_2} e^{k/2} e_M^{-k} \overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1) H_k((Me_L/e_M)\tau, 0;\, \psi_1, \psi_2) \\ &= A^{-1}_{k,\overline{\chi}_2} e^{k/2} e_M^{-k} \overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1) E_k((Me_L/e_M)\tau;\, \psi_1, \psi_2) \\ &= \frac{A_{k,\psi_2}}{A_{k,\overline{\chi}_2}} e^{k/2} e_M^{-k} \overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1) E_{k,\psi_1,\overline{\psi}_2}(\tau) \\ &= \frac{G(\psi_2)}{G(\overline{\chi}_2)} e^{-k/2} e_M^{k} \overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1) E_{k,\psi_1,\overline{\psi}_2}(\tau)\end{aligned}$$

because

$$\frac{A_{k,\psi_2}}{A_{k,\overline{\chi}_2}} e^{k/2} e_M^{-k} = \frac{2(-2\pi i)^k G(\psi_2)}{(Me_L/e_M)^k (k-1)!} \cdot \frac{M^k (k-1)!}{2(-2\pi i)^k G(\overline{\chi}_2)} \cdot e^{k/2} e_M^{-k} = \frac{G(\psi_2)}{G(\overline{\chi}_2)} e^{-k/2} e_M^{k}.$$

Note that the constant term of Fourier expansion of $E_{k,\chi_1,\chi_2}(\tau)|_k W_e$ is nonzero if and only if the constant term of $E_{k,\psi_1,\overline{\psi}_2}(\tau)$ is nonzero. In this case, from the definition, we should have $\psi_1 = \chi_0$ and $e = L$. When $e = L$ and χ is real, we have $\psi_2 = \chi$, $e_L = L$ and $e_M = 1$. Thus, one may verify that the constant term is

$$\begin{aligned}\frac{G(\psi_2)}{G(\overline{\chi}_2)} e^{-k/2} e_M^{k} \overline{\chi}^{[N/e]}(e_M)\chi^{[e]}(L/e_L)\chi_1^{[e_L]}(-1)\left(-\frac{B_{k,\psi_1\overline{\psi}_2}}{2k}\right) &= \frac{G(\chi)}{G(\chi_2)} L^{-k/2} \chi_1(-1)\left(-\frac{B_{k,\chi}}{2k}\right) \\ &= \frac{G(\chi)}{G(\chi^{[M]})} L^{-k/2} \chi^{[L]}(-1)\left(-\frac{B_{k,\chi}}{2k}\right)\end{aligned}$$

since $\chi_1 = \chi^{[L]}$ and $\chi_2 = \chi^{[M]}$. This finishes the proof. □

Remark 3.6.

(i) In Theorem 3.5, we may replace $E_{k,\chi_1,\chi_2}(\tau)$ by $E_{k,\chi^{[L]},\chi^{[M]}}(\tau)$.

(ii) The constant in (3.2) is slightly different from that of [13, Proposition 14] because we used the matrix of the form $W_e = \begin{pmatrix} e & b \\ N & de \end{pmatrix}$ instead of $\widehat{W_e} = \begin{pmatrix} a'e & b' \\ c'N & d'e \end{pmatrix}$ with $\det(\widehat{W_e}) = e$, $b' \equiv 1 \pmod{e}$ and $a' \equiv 1 \pmod{N/e}$.

4 Average theta function as a sum of Eisenstein series

For positive integers k and N such that $(-1)^k N \equiv 0$ or $1 \pmod 4$, let $\chi_{(-1)^k N}$ be the Dirichlet character defined by the Kronecker symbol, that is,

$$\chi_{(-1)^k N}(\cdot) = \left(\frac{(-1)^k N}{\cdot}\right).$$

Then $(-1)^k N$ is a fundamental discriminant if and only if $\chi_{(-1)^k N}$ is a primitive Dirichlet character modulo N [9, Theorem 9.13]. Moreover, it is well known that

$$G(\chi_{(-1)^k N}) = \sum_{a=0}^{N-1} \chi_{(-1)^k N}(a) e^{2\pi i a/N} = \sqrt{(-1)^k N} \tag{4.1}$$

([8, Lemma 4.8.1.(1)]). Let Q be an integral positive definite quadratic form in $2k$ variables and $A = \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}\right)$ the Hessian matrix of Q so that

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A \mathbf{x} \quad \text{for}\, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{2k} \end{pmatrix} \in \mathbb{Z}^{2k}.$$

Further, let N be the level of Q, which is the smallest positive integer such that NA^{-1} is integral and has even diagonal entries. For a nonnegative integer n, let

$$r_Q(n) = \#\{\mathbf{x} \in \mathbb{Z}^{2k} \mid Q(\mathbf{x}) = n\}$$

be the representation number of n by Q and denote the generating function $\Theta_Q(\tau)$ of $r_Q(n)$ by

$$\Theta_Q(\tau) = \sum_{\mathbf{x}\in\mathbb{Z}^{2k}} e^{2\pi i Q(\mathbf{x})\tau} = \sum_{n=0}^{\infty} r_Q(n)q^n \quad (\tau \in \mathbb{H}),$$

which is called the theta function associated with Q. Then it is well known that the theta function $\Theta_Q(\tau)$ belongs to $\mathcal{M}_k(N, \chi_{(-1)^k\det(A)}) = \mathcal{E}_k(N, \chi_{(-1)^k\det(A)}) \oplus \mathcal{S}_k(N, \chi_{(-1)^k\det(A)})$ (see [8, Corollary 4.9.5(3)]). Let P be another integral positive definite quadratic form in $2k$ variables with Hessian matrix B. Let R be the ring $\mathbb{Z}$ of rational integers or the ring $\mathbb{Z}_p$ of p-adic integers for a prime p. Two forms Q and P are said to be equivalent over R if there is a matrix $S \in \mathrm{GL}_{2k}(R)$ such that $S^TAS = B$. We denote by $Q \sim P$ if Q and P are equivalent over $\mathbb{Z}$. We also denote by $Q \sim_p P$ if Q and P are equivalent over $\mathbb{Z}_p$. We say that Q and P are in the same genus if $Q \sim_p P$ for all prime p and denote by $\mathrm{gen}(Q) = \{Q' \mid Q \sim_p Q' \text{for all prime} p\}/\sim$. Let $o(Q) = \#\{S \in \mathrm{GL}_{2k}(\mathbb{Z}) \mid S^TAS = A\}$. Then the average theta function $\Theta(\mathrm{gen}(Q), \tau)$ of Q is defined by

$$\Theta(\mathrm{gen}(Q), \tau) = \left(\sum_{[Q']\in\mathrm{gen}(Q)} \frac{1}{o(Q')}\right)^{-1} \sum_{[Q']\in\mathrm{gen}(Q)} \frac{\Theta_{Q'}(\tau)}{o(Q')} = \sum_{n=0}^{\infty} r(\mathrm{gen}(Q), n)q^n,$$

where $[Q']$ is the equivalence class containing Q' in the genus $\mathrm{gen}(Q)$. Then it is well known that $\Theta(\mathrm{gen}(Q), \tau) \in \mathcal{E}_k(N, \chi_{(-1)^k\det(A)})$ and $\Theta_Q(\tau) - \Theta(\mathrm{gen}(Q), \tau) \in \mathcal{S}_k(N, \chi_{(-1)^k\det(A)})$ (see [2–5]). Thus, there is a cusp form $f_Q(\tau) \in \mathcal{S}_k(N, \chi_{(-1)^k\det(A)})$ so that

$$\Theta_Q(\tau) = \Theta(\mathrm{gen}(Q), \tau) + f_Q(\tau).$$

To find the action of the Atkin-Lehner involution on $\Theta_Q(\tau)$, we need to consider the congruent theta functions. For $h \in \mathbb{Z}^{2k}$ with $Ah \in N\mathbb{Z}^{2k}$, let

$$\theta(\tau;\, h, A, N) = \sum_{\substack{m\in\mathbb{Z}^{2k} \\ m\equiv h \pmod N}} e\left(\frac{A[m]}{2N^2}\tau\right),$$

where $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$ and $A[\mathbf{x}] = \mathbf{x}^TA\mathbf{x}$ for column vectors $\mathbf{x}$ of size $2k$. From the definition, we have $\Theta_Q(\tau) = \theta(\tau;\, 0, A, N)$.

Lemma 4.1. *Let $h \in \mathbb{Z}^{2k}$ with $Ah \in N\mathbb{Z}^{2k}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$, we have*

$$\theta(\gamma\cdot\tau;\, h, A, N)|_k\gamma = \det(A)^{-1/2}c^{-k}(-i)^k \sum_{\substack{l\in\mathbb{Z}^{2k}/N\mathbb{Z}^{2k} \\ Al\equiv 0 \pmod N}} \Phi_\gamma^A(h, l)\theta(\tau;\, l, A, N),$$

where

$$\Phi_\gamma^A(h, l) = \sum_{\substack{g\in\mathbb{Z}^{2k}/cN\mathbb{Z}^{2k} \\ g\equiv h \pmod N}} e\left(\frac{aA[g] + 2l^TAg + dA[l]}{2cN^2}\right).$$

Proof. See [1, page 5]. ☐

Lemma 4.2. *Let $h \in \mathbb{Z}^{2k}$ with $Ah \in N\mathbb{Z}^{2k}$. For each $e\|N$, we have the action of the Atkin-Lehner involution $W_e = \begin{pmatrix} e & b \\ N & de \end{pmatrix}$ on $\theta(\tau;\, h, A, N)$ as follows:*

$$\theta(\tau;\ h, A, N)|_k W_e = \det(A)^{-1/2} N^{-k} e^{3k/2} (-i)^k \sum_{\substack{l \in \mathbb{Z}^{2k}/N\mathbb{Z}^{2k} \\ Al \equiv 0 \ (\mathrm{mod}\ N)}} \Phi_\gamma^A(h, l) \theta(e\tau; l, A, N),$$

where $\gamma = \begin{pmatrix} 1 & b \\ N/e & de \end{pmatrix}$.

Proof. Put $\gamma = \begin{pmatrix} 1 & b \\ N/e & de \end{pmatrix}$. Then $\gamma = \in \mathrm{SL}_2(\mathbb{Z})$ since $\det(W_e) = e$. Note that

$$W_e = \begin{pmatrix} 1 & b \\ N/e & de \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = \gamma \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, from Lemma 4.1, we deduce that

$$\begin{aligned} \theta(\tau;\ h, A, N)|_k W_e &= (\theta(\tau;\ h, A, N)|_k \gamma)|_k \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \\ &= \left(\det(A)^{-1/2} (N/e)^{-k} (-i)^k \sum_{\substack{l \in \mathbb{Z}^{2k}/N\mathbb{Z}^{2k} \\ Al \equiv 0 \ (\mathrm{mod}\ N)}} \Phi_\gamma^A(h, l) \theta(\tau; l, A, N) \right) \Bigg|_k \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \\ &= \det(A)^{-1/2} N^{-k} e^{3k/2} (-i)^k \sum_{\substack{l \in \mathbb{Z}^{2k}/N\mathbb{Z}^{2k} \\ Al \equiv 0 \ (\mathrm{mod}\ N)}} \Phi_\gamma^A(h, l) \theta(e\tau; l, A, N). \end{aligned}$$

□

Before we go further, we need the following definition and lemma.

Definition 4.3. Let A be a $2k \times 2k$ symmetric positive definite integral matrix with even diagonal entries. For positive integers L, M, we denote by $G^A(L, M)$ a Gauss sum defined by

$$G^A(L, M) = \sum_{g \in \mathbb{Z}^{2k}/M\mathbb{Z}^{2k}} e\left(\frac{LA[g]}{2M} \right).$$

Lemma 4.4. *Let the notation be as mentioned earlier and assume that Q is a binary quadratic form.*

(i) *For a given integer m, Q properly represents at least one positive integer, say a, relatively prime to m. Moreover, we have $Q \sim ax^2 + bxy + cy^2$ for some $b, c \in \mathbb{Z}$.*

(ii) *Let a and M be positive integers such that M is odd and $\gcd(a, M) = 1$, then we have*

$$G^{2a}(1, M) = \sum_{x=0}^{M-1} e\left(\frac{ax^2}{M} \right) = \varepsilon_M \left(\frac{a}{M} \right) \sqrt{M},$$

where $\varepsilon_M = 1$ or i if $M \equiv 1 \ (\mathrm{mod}\ 4)$ or $M \equiv 3 \ (\mathrm{mod}\ 4)$, respectively.

Proof. See the study by Cox [15, Lemmas 2.25 and 2.3] for (i). For (ii), see, [16, Equation (3.38)]. □

Theorem 4.5. *Let k be a positive integer and Q an integral positive definite quadratic form in $2k$ variables with Hessian matrix A. Let N be the level of Q. Assume that $(-1)^k N$ is a fundamental discriminant and $\chi_{(-1)^k N} = \chi_{(-1)^k \det(A)}$. Put $\chi = \chi_{(-1)^k N}$. Write $\Theta(\mathrm{gen}(Q), \tau)$ as a sum of Eisenstein series*

$$\Theta(\mathrm{gen}(Q), \tau) = \delta \sum_{M || N} c(M) E_{k, \chi^{[N/M]}, \chi^{[M]}}(\tau), \tag{4.2}$$

where $\delta = 1$ or $1/2$ if $k \geq 2$ or $k = 1$, respectively, and $c(M) \in \mathbb{C}$. Then we have

$$c(M) = -\frac{2k}{B_{k,\chi}} \det(A)^{-1/2} N^{(2k-1)/2} M^{(1-4k)/2} i^{k-k^2} \chi^{[M]}(-1)^{-1/2} G^A(1, M) \tag{4.3}$$

for $M\|N$. In particular, when $k = 1$ and Q is primitive (i.e., the coefficients are coprime), let Q represent $a \in \mathbb{Z}^+$ with $\gcd(a, N) = 1$. Then we have

$$\Theta(\mathrm{gen}(Q), \tau) = -\frac{2\delta(N)}{B_{1,\chi}} \sum_{\substack{M\|N \\ M \text{ odd}}} \chi^{[M]}(a) E_{1,\chi^{[N/M]},\chi^{[M]}}(\tau), \tag{4.4}$$

where $\delta(N) = 1$ or $1/2$ if N is even or odd, respectively.

Proof. Let $\delta = 1$ or $1/2$ if $k \geq 2$ or $k = 1$, respectively. By Proposition 3.2 and Remark 3.3, there are $c(M) \in \mathbb{C}$ such that

$$\Theta(\mathrm{gen}(Q), \tau) = \delta \sum_{M\|N} c(M) E_{k,\chi^{[N/M]},\chi^{[M]}}(\tau)$$

for $M\|N$. Note that when $k = 1$, the term $\delta = 1/2$ occurs in (4.2) since $E_{1,\chi^{[N/M]},\chi^{[M]}}(\tau) = E_{1,\chi^{[M]},\chi^{[N/M]}}(\tau)$. Let $M\|N$ and $L = N/M$. To avoid any confusion, we change the variable M in (4.2) to J and express $\Theta_Q(\tau)$ as follows:

$$\Theta_Q(\tau) = \Theta(\mathrm{gen}(Q), \tau) + f_Q(\tau) = \delta \sum_{J\|N} c(J) E_{k,\chi^{[N/J]},\chi^{[J]}}(\tau) + f_Q(\tau), \tag{4.5}$$

where $f_Q(\tau) \in \mathcal{S}_k(N, \chi)$. From Lemma 4.2, by applying the Atkin-Lehner involution $W_L = \begin{pmatrix} L & b \\ N & dL \end{pmatrix}$ on the left-hand side of (4.5), we obtain

$$\begin{aligned} \Theta_Q(\tau)|_k W_L &= \theta(\tau;\, 0, A, N)|_k W_L \\ &= \det(A)^{-1/2} N^{-k} L^{3k/2} (-i)^k \sum_{\substack{l \in \mathbb{Z}^{2k}/N\mathbb{Z}^{2k} \\ Al \equiv 0 \pmod{N}}} \Phi_\gamma^A(0, l) \theta(L\tau; l, A, N), \end{aligned}$$

where $\gamma = \begin{pmatrix} 1 & b \\ M & dL \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and the constant term is given by

$$\begin{aligned} \det(A)^{-1/2} N^{-k} L^{3k/2} (-i)^k \Phi_\gamma^A(0, 0) &= \det(A)^{-1/2} N^{-k} L^{3k/2} (-i)^k \sum_{\substack{g \in \mathbb{Z}^{2k}/MN\mathbb{Z}^{2k} \\ g \equiv 0 \pmod{N}}} e\left(\frac{A[g]}{2MN^2}\right) \\ &= \det(A)^{-1/2} N^{-k} L^{3k/2} (-i)^k \sum_{g \in \mathbb{Z}^{2k}/M\mathbb{Z}^{2k}} e\left(\frac{A[g]}{2M}\right) \\ &= \det(A)^{-1/2} N^{-k} L^{3k/2} (-i)^k G^A(1, M). \end{aligned} \tag{4.6}$$

From Theorem 3.5, $E_{k,\chi^{[N/J]},\chi^{[J]}}(\tau)|_k W_L$ has a nonzero constant term if and only if $L = N/J$, that is, $J = M$. Since $\chi = \chi_{(-1)^k N}$ is real, by Theorem 3.5, the constant term of the action of W_L on the right-hand side of (4.5) is given by

$$c(M) \frac{G(\chi)}{G(\chi^{[M]})} L^{-k/2} \chi^{[L]}(-1) \left(-\frac{B_{k,\chi}}{2k}\right). \tag{4.7}$$

Since $\chi^{[L]}(-1)\chi^{[M]}(-1) = \chi(-1) = (-1)^k$ and $\sqrt{\chi(-1)} = \sqrt{\chi_{(-1)^k N}(-1)} = \sqrt{(-1)^k} = i^{k^2}$, by (4.1), we have

$$\frac{G(\chi)}{G(\chi^{[M]})} L^{-k/2} \chi^{[L]}(-1) = \frac{\sqrt{\chi(-1)}\sqrt{N}}{\sqrt{\chi^{[M]}(-1)}\sqrt{M}} L^{-k/2} \chi^{[L]}(-1) = L^{-(k-1)/2} i^{k^2} (-1)^k \chi^{[M]}(-1)^{1/2}. \tag{4.8}$$

By combining (4.6), (4.7), and (4.8), we obtain that

$$\begin{aligned} c(M) &= (\det(A)^{-1/2}N^{-k}L^{3k/2}(-i)^k G^A(1,M))\cdot \left(L^{(k-1)/2}i^{-k^2}(-1)^k\chi^{[M]}(-1)^{-1/2}\left(-\frac{2k}{B_{k,\chi}}\right)\right) \\ &= -\frac{2k}{B_{k,\chi}}\det(A)^{-1/2}N^{-k}L^{(4k-1)/2}i^{k-k^2}\chi^{[M]}(-1)^{-1/2}G^A(1,M) \\ &= -\frac{2k}{B_{k,\chi}}\det(A)^{-1/2}N^{-k}\left(\frac{N}{M}\right)^{(4k-1)/2}i^{k-k^2}\chi^{[M]}(-1)^{-1/2}G^A(1,M) \\ &= -\frac{2k}{B_{k,\chi}}\det(A)^{-1/2}N^{(2k-1)/2}M^{(1-4k)/2}i^{k-k^2}\chi^{[M]}(-1)^{-1/2}G^A(1,M). \end{aligned}$$

This proves (4.3). Now, we suppose that $k = 1$ and Q is primitive. Then we have $N = \det(A)$ since Q is primitive and $-N$ is a fundamental discriminant. Let $M||N$ be odd. Since $E_{1,\chi^{[N/M]},\chi^{[M]}}(\tau) = E_{1,\chi^{[M]},\chi^{[N/M]}}(\tau)$, we have

$$\frac{1}{2}\sum_{M||N} c(M)E_{1,\chi^{[N/M]},\chi^{[M]}}(\tau) = \begin{cases} \displaystyle\sum_{\substack{M||N \\ M\text{ odd}}} c(M)E_{1,\chi^{[N/M]},\chi^{[M]}}(\tau), & \text{if } N \text{ even}, \\ \displaystyle\frac{1}{2}\sum_{\substack{M||N \\ M\text{ odd}}} c(M)E_{1,\chi^{[N/M]},\chi^{[M]}}(\tau), & \text{if } N \text{ odd}. \end{cases} \tag{4.9}$$

Since Q represents $a \in \mathbb{Z}^+$ with $\gcd(a, N) = 1$, we may assume that $Q = ax^2 + bxy + cy^2$ by Lemma 4.4 (i). Note that $N = \det(A) = 4ac - b^2$. Since $\gcd(a, N) = 1$ and $M||N$ is odd, there is an even $\overline{2a} \in \mathbb{Z}$ such that $2a\overline{2a} \equiv 1 \pmod{M}$. Take $S = \begin{pmatrix} 1 & -b\overline{2a} \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then we have

$$S^TAS \equiv \begin{pmatrix} 1 & 0 \\ -b\overline{2a} & 1 \end{pmatrix}\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\begin{pmatrix} 1 & -b\overline{2a} \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 2a & 0 \\ 0 & N\overline{2a} \end{pmatrix} \pmod{M}.$$

Thus, from Lemma 4.4 (ii), we obtain that

$$\begin{aligned} G^A(1,M) &= \sum_{g\in\mathbb{Z}^2/M\mathbb{Z}^2} e\left(\frac{A[g]}{2M}\right) = \sum_{\binom{x}{y}\in\mathbb{Z}^2/M\mathbb{Z}^2} e\left(\frac{ax^2 + N\overline{2a}y^2/2}{M}\right) \\ &= M\sum_{x=0}^{M-1} e\left(\frac{ax^2}{M}\right) = M\varepsilon_M\left(\frac{a}{M}\right)\sqrt{M} \\ &= M^{3/2}\chi^{[M]}(a)\chi^{[M]}(-1)^{1/2}, \end{aligned} \tag{4.10}$$

since

$$\chi^{[M]}(\cdot) = \chi^{[M]}_{(-1)^kN}(\cdot) = \left(\frac{(-1)^{\frac{M-1}{2}}M}{\cdot}\right), \quad \chi^{[M]}(a) = \left(\frac{a}{M}\right) \quad \text{and} \quad \chi^{[M]}(-1)^{1/2} = \varepsilon_M.$$

Since $N = \det(A)$, by (4.3), (4.9), and (4.10), we derive that

$$c(M) = -\frac{2}{B_{1,\chi}}\det(A)^{-1/2}N^{1/2}M^{-3/2}\chi^{[M]}(-1)^{-1/2}G^A(1,M) = -\frac{2}{B_{1,\chi}}\chi^{[M]}(a)$$

for odd $M||N$ and

$$\Theta(\mathrm{gen}(Q),\tau) = \frac{1}{2}\sum_{M||N} c(M)E_{1,\chi^{[N/M]},\chi^{[M]}}(\tau) = -\frac{2\delta(N)}{B_{1,\chi}}\sum_{\substack{M||N \\ M\text{ odd}}} \chi^{[M]}(a)E_{1,\chi^{[N/M]},\chi^{[M]}}(\tau).$$

This finishes the proof. ☐

Remark 4.6. As we mentioned in Section 1, Guerzhoy and Kane [7] obtained the same result as in (4.4) by using the algebraic theory of binary quadratic forms.

In general cases, one may find the n-th Fourier coefficient $r(\mathrm{gen}(Q), n)$ of $\Theta(\mathrm{gen}(Q), \tau)$ by using Minkowski-Siegel formula. In our cases, the following corollary gives another way to find $r(\mathrm{gen}(Q), n)$ and is obtained by comparing the Fourier coefficients of both sides of (4.2) and (4.4).

Corollary 4.7. *Let the notations and assumptions be as in Theorem* 4.5. *Let n be a positive integer. Then we have*

$$\begin{aligned} r(\mathrm{gen}(Q), n) = {} & -\frac{2k}{B_{k,\chi}} \det(A)^{-1/2} N^{(2k-1)/2} i^{k-k^2} \sum_{M \| N} M^{(1-4k)/2} \chi^{[M]}(-1)^{-1/2} G^A(1, M) \\ & \times \sum_{d>0, d|n} \chi^{[N/M]}(n/d) \chi^{[M]}(d) d^{k-1} \end{aligned}$$

for $k \geq 2$ *and*

$$r(\mathrm{gen}(Q), n) = -\frac{2\delta(N)}{B_{1,\chi}} \sum_{\substack{M \| N \\ M \text{ odd}}} \chi^{[M]}(a) \sum_{d>0, d|n} \chi^{[N/M]}(n/d) \chi^{[M]}(d)$$

for $k = 1$.

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