DE GRUYTER Open Mathematics 2023; 21: 20230125

Research Article

Zijie Qin* and Lin Chen

Linear maps preserving equivalence or asymptotic equivalence on Banach space

https://doi.org/10.1515/math-2023-0125
received February 17, 2023; accepted September 7, 2023

Abstract: Let X be a complex Banach space with dimension at least two and $B(X)$ the algebra of all bounded linear operators on X. We show that a bijective linear map Φ preserves asymptotic equivalence if and only if it preserves equivalence, and in turn, if and only if there exist invertible bounded linear operators T and S such that either $\Phi(A) = TAS$ or $\Phi(A) = TA^*S$ for all $A \in B(X)$.

Keywords: linear preservers, asymptotic equivalence, equivalence relations, Banach spaces

MSC 2020: 47B49

1 Introduction and main result

The problem of characterizing linear maps on operator algebras preserving certain properties, subsets, or relations has attracted the attention of many authors in the last few decades (see [1,2] and references therein). In this article, we will deal with linear maps preserving certain equivalence relations.

Throughout this article, all algebras and vector spaces will be over the complex field $\mathbb{C}$. Let X be a complex Banach space with topological dual X^*. We denote by $B(X)$ the algebra of all bounded linear operators on X. Recall that two operators A and B are equivalent, denoted by $A \simeq B$, if $A = TBS$ for some invertible operators $T, S \in B(X)$. A map Φ from $B(X)$ into itself is said to be equivalence preserving if $\Phi(A) \simeq \Phi(B)$ whenever $A \simeq B$. In the finite-dimensional case, Horn et al. [3] characterized linear maps preserving equivalence on the algebra of all $n \times n$ matrices. For the infinite-dimensional case, Petek and Radić [4] proved that if X is an infinite-dimensional reflexive complex Banach space, then linear bijections $\Phi : B(X) \to B(X)$ preserve equivalence if and only if there exist bounded invertible linear operators T, S such that either $\Phi(A) = TAS$, for all $A \in B(X)$, or $\Phi(A) = TA^*S$, for all $A \in B(X)$, where A^* denotes the adjoint of A. Later, in [5], they showed that if X is an infinite-dimensional complex Banach space, then the relation $A - B \simeq C$ if and only if $\Phi(A) - \Phi(B) \simeq \Phi(C)$ is enough to determine the structure of surjective maps Φ defined on $B(X)$. Recently, Radić [6] defined another equivalence relation on $B(X)$ and refined the result in [4].

Maps preserving various types of equivalence relations have been considered in the last 20 years. For example, studies in [7–9] studied similarity; the article [10] studied asymptotic similarity; the article [11] studied unitary similarity; the first author of this article in [12] studied involution similarity. We now define another equivalence relation on $B(X)$. Let A and B be in $B(X)$. By $O(A)$, we denote the equivalence orbit of A, i.e.,

$$O(A) = \{B \in B(X) : B \simeq A\}.$$

*** Corresponding author: Zijie Qin,** School of Mathematics and Statistics, Changshu Institute of Technology, Changshu, 215500, P. R. China, e-mail: zijieqin92@163.com

Lin Chen: School of Mathematics and Statistics, Changshu Institute of Technology, Changshu, 215500, P. R. China, e-mail: linchen198112@163.com

It is easy to see that $A \simeq B$ if and only if $\mathcal{O}(A) = \mathcal{O}(B)$. Let $\overline{\mathcal{O}(A)}$ denote the norm closure of $\mathcal{O}(A)$. We say that the two operators A and B are asymptotically equivalent, denoted by $A \simeq_a B$, if $\overline{\mathcal{O}(A)} = \overline{\mathcal{O}(B)}$. It is obvious that asymptotic equivalence is an equivalence relation on $B(X)$. For $A, B \in B(X)$, if $A \simeq B$, then $A \simeq_a B$. A map $\Phi : B(X) \to B(X)$ is said to be asymptotic equivalence preserving if $\Phi(A) \simeq_a \Phi(B)$ whenever $A \simeq_a B$. The purpose of this article is to obtain a complete classification of linear bijections that preserve equivalence or asymptotic equivalence and then generalize the result in [4] to the general Banach space case.

Our main result reads as follows.

Theorem 1.1. *Let X be a complex Banach space with* $\dim X \geq 2$. *Suppose $\Phi : B(X) \to B(X)$ is a surjective linear map satisfying*

$$A \simeq B \Rightarrow \Phi(A) \simeq_a \Phi(B),$$

for all $A, B \in B(X)$. Then, one of the following statements holds.

(1) *There exist invertible bounded linear operators $T, S : X \to X$ such that*:

$$\Phi(A) = TAS,$$

for all $A \in B(X)$.

(2) *There exist invertible bounded linear operators $T : X^* \to X$ and $S : X \to X^*$ such that*:

$$\Phi(A) = TA^*S,$$

for all $A \in B(X)$.

(3) $\Phi(F) = 0$, *for every finite-rank operator F in $B(X)$.*

2 Preliminary results

In this section, we show some results that will be used to prove our main result. Let X be a complex Banach space. By $\mathcal{F}(X)$, we denote the ideal of all finite-rank operators in $B(X)$. For $A \in B(X)$, the notations $\ker(A)$, $\operatorname{ran}(A)$, and $\operatorname{rank}(A)$ will stand for the kernel, the range, and the rank of A, respectively. For nonzero vector $x \in X$ and nonzero functional $f \in X^*$, the rank-one operator $x \otimes f$ is defined as the map $y \mapsto f(y)x$, $y \in X$.

The following lemma comes from [4, Lemma 2.2].

Lemma 2.1. *Let X be a complex Banach space and $A, B \in B(X)$. Assume that A is of rank one and B is nonzero. If there exist two nonzero scalars λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$ such that* $\operatorname{rank}(A + \lambda_i B) = 1$, *for* $i = 1, 2$, *then* $\operatorname{rank}(B) = 1$.

The following result gives a characterization of rank-one non-increasing additive maps.

Proposition 2.2. [13] *Let X be a complex Banach space with* $\dim X \geq 2$. *Suppose $\Phi : \mathcal{F}(X) \to \mathcal{F}(X)$* is an additive map. If Φ *maps rank-one operators to operators of rank at most one, then one of the following statements holds.*

(1) *There exist a functional $g \in X^*$ and an additive map $\tau : \mathcal{F}(X) \to X$ such that $\Phi(A) = \tau(A) \otimes g$, for every $A \in \mathcal{F}(X)$.*
(2) *There exist a vector $y \in X$ and an additive map $\varphi : \mathcal{F}(X) \to X^*$ such that $\Phi(A) = y \otimes \varphi(A)$, for every $A \in \mathcal{F}(X)$.*
(3) *There exist additive maps $T : X \to X$ and $S : X^* \to X^*$ such that $\Phi(x \otimes f) = Tx \otimes Sf$, for every rank-one operator $x \otimes f \in \mathcal{F}(X)$.*
(4) *There exist additive maps $T : X^* \to X$ and $S : X \to X^*$ such that $\Phi(x \otimes f) = Tf \otimes Sx$, for every rank-one operator $x \otimes f \in \mathcal{F}(X)$.*

Remark 2.3. If Φ is linear, it is clear that τ, φ, T, and S are linear. Moreover, if Φ is injective, then T and S are injective too.

Recall that a linear map $\Phi : B(X) \to B(X)$ is said to be invertibility preserving if $\Phi(A)$ is invertible for every invertible operator $A \in B(X)$. The following proposition characterizes linear maps preserving invertibility on $B(X)$, which is crucial for proving Theorem 1.1.

Proposition 2.4. [14] *Let X be a complex Banach space. Suppose that $\Phi : B(X) \to B(X)$ is a linear bijective map. If Φ preserves invertibility, then one of the following statements holds.*

1. *There exist invertible bounded linear operators $T, S : X \to X$ such that $\Phi(A) = TAS$, for all $A \in B(X)$.*
2. *There exist invertible bounded linear operators $T : X^* \to X$ and $S : X \to X^*$ such that $\Phi(A) = TA^*S$, for all $A \in B(X)$.*

We will close this section with the following lemmas.

Lemma 2.5. [14] *Let X be a complex Banach space and $x \in X$ and $f \in X^*$. Then, $I + x \otimes f$ is invertible in $B(X)$ if and only if $f(x) \neq -1$.*

Lemma 2.6. *Let X be a complex Banach space and $A, B \in B(X)$ with $A \simeq_a B$. If A is invertible, then B is invertible as well.*

Proof. Since $A \in \overline{O(B)}$, there exist sequences $\{T_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ of invertible operators in $B(X)$ such that $A = \lim_{n\to\infty} T_n B S_n$. As A is invertible and bounded, it is clear that $\|A^{-1}\| > 0$. Now, we can observe that there exists a positive integer N such that $\|A - T_n B S_n\| < \|A^{-1}\|^{-1}$, for every $n > N$. Note that

$$\|A^{-1}(A - T_{N+1}BS_{N+1})\| \leq \|A^{-1}\| \cdot \|A - T_{N+1}BS_{N+1}\| < 1,$$

so

$$T_{N+1}BS_{N+1} = A - (A - T_{N+1}BS_{N+1}) = A(I - A^{-1}(A - T_{N+1}BS_{N+1}))$$

is invertible. Hence, B is invertible. The proof is complete. □

Lemma 2.7. *Let X be a complex Banach space. Assume that $\{T_n\}_{n=1}^{\infty}$ is a sequence of rank-one operators in $B(X)$. If $\{T_n\}_{n=1}^{\infty}$ converges to a nonzero operator $T \in B(X)$, then T is of rank one.*

Proof. We first claim that there exists a positive integer N such that $m < \|T_n\| < M$, for all $n > N$, where $0 < m < M$. Actually, since $\lim_{n\to\infty} T_n = T$, we have $\lim_{n\to\infty} \|T_n\| = \|T\|$. Note that $T \neq 0$. So, for $\|T\| > 0$, there exists a positive integer N such that

$$\big|\|T_n\| - \|T\|\big| < \frac{1}{2}\|T\|,$$

for all $n > N$, which further implies that $\frac{1}{2}\|T\| < \|T_n\| < \frac{3}{2}\|T\|$, for all $n > N$. Set $m = \frac{1}{2}\|T\|$ and $M = \frac{3}{2}\|T\|$, establishing the claim.

Now suppose that $T_n = x_n \otimes f_n$, where $x_n \in X$ and $f_n \in X^*$ with $\|f_n\| = 1$. By the claim, there exist positive numbers m and M such that $m < \|x_n\| < M$, for all $n > N$. Assume, on the contrary, that $\operatorname{rank}(T) \geq 2$. Then, there exist unit vectors $y_1, y_2 \in X$ such that Ty_1 and Ty_2 are linearly independent. Since $\lim_{n\to\infty} T_n = T$, we see that $\lim_{n\to\infty} T_n y_i = Ty_i$, for $i = 1, 2$. It follows that

$$\lim_{n\to\infty} f_n(y_i)x_n = Ty_i, \quad i = 1, 2. \tag{2.1}$$

Note that $\|x_n \otimes f_n\| < M$. So, for $i = 1, 2$, $\{f_n(y_i)\}_{n=1}^{\infty}$ is a bounded sequence. Hence, it has a convergent subsequence. Without loss of generality, we may assume that there exists a nonzero scalar λ_i such that $\lim_{n\to\infty} f_n(y_i) = \lambda_i$, for $i = 1, 2$. This together with (2.1) gives $\{Ty_1, Ty_2\}$ is a linearly dependent set, a contradiction. The proof is complete. □

3 Proof of the main result

In this section, we will complete the proof of Theorem 1.1. Throughout this section, X is a complex Banach space with $\dim X \geq 2$ and $\Phi : B(X) \to B(X)$ is assumed to satisfy the hypotheses in Theorem 1.1. For clarity, we will organize the proof into a series of lemmas.

Lemma 3.1. Φ *maps rank-one operators to operators of rank at most one.*

Proof. Let $B \in B(X)$ be of rank one. By the surjectivity of Φ, there exists an $A \in B(X)$ such that $\Phi(A) = B$. Since $A \neq 0$, there exists an $x \in X$ such that $Ax \neq 0$. Take a nonzero functional $f \in X^*$ such that $f(x) = 0$. Let $\lambda \in \mathbb{C}$ be arbitrary. Then, $I - \lambda x \otimes f$ is invertible by Lemma 2.5. It follows that

$$A \simeq A(I - \lambda x \otimes f) = A - \lambda Ax \otimes f.$$

This gives us

$$\overline{O(B)} = \overline{O(B - \lambda\Phi(Ax \otimes f))}. \tag{3.1}$$

So $B - \lambda\Phi(Ax \otimes f) \in \overline{O(B)}$. Then, there exist sequences $\{T_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ of invertible operators in $B(X)$ such that $B - \lambda\Phi(Ax \otimes f) = \lim_{n\to\infty} T_n B S_n$. Note that B is of rank one. By Lemma 2.7, $\operatorname{rank}(B - \lambda\Phi(Ax \otimes f)) = 1$ for every $\lambda \in \mathbb{C}$. Actually, if there exists $\lambda \in \mathbb{C}$ such that $B - \lambda\Phi(Ax \otimes f) = 0$, by equation (3.1), $B = 0$, a contradiction. Now, we consider two cases.

Case 1: $\Phi(Ax \otimes f) \neq 0$. By Lemma 2.1, $\Phi(Ax \otimes f)$ is of rank one. For any rank-one operator $F \in B(X)$, we have $F \simeq Ax \otimes f$. It follows that $\overline{O(\Phi(F))} = \overline{O(\Phi(Ax \otimes f))}$. By a similar argument, we obtain that $\operatorname{rank}(\Phi(F)) = 1$.

Case 2: $\Phi(Ax \otimes f) = 0$. For every rank-one operator $F \in B(X)$, we have $F \simeq Ax \otimes f$. This leads to $\Phi(F) \simeq_a \Phi(Ax \otimes f) = 0$, which implies that $\Phi(F) = 0$, for every rank-one operator $F \in B(X)$.

So, Φ maps rank-one operators to operators of rank at most one. The proof is complete. □

Note that every finite-rank operator in $B(X)$ can be written as a sum of rank-one operators in $B(X)$. By the proof of Lemma 3.1, we have either Φ is rank-one preserving or $\Phi(\mathcal{F}(X)) = 0$. In what follows, we may assume that Φ is a rank-one preserving map.

Lemma 3.2. Φ *is injective.*

Proof. Suppose that $\Phi(A) = 0$, for some operator $A \in B(X)$. If $A \neq 0$, then there exists a nonzero vector $x \in X$ such that $Ax \neq 0$. Since $\dim X \geq 2$, we can take a nonzero functional $f \in X^*$ such that $f(x) = 0$. By Lemma 2.5, $I + x \otimes f$ is invertible in $B(X)$. It follows from $A \simeq A(I + x \otimes f)$ that

$$0 = \Phi(A) \simeq_a \Phi(A + Ax \otimes f) = \Phi(Ax \otimes f),$$

which implies that $\Phi(Ax \otimes f) = 0$, which is a contradiction. So, $A = 0$. □

In the sequel, we assume that Φ is injective.

Lemma 3.3. *One and only one of the following statements holds.*

(1) *There exist injective linear maps* $T : X \to X$ *and* $S : X^* \to X^*$ *such that* $\Phi(x \otimes f) = Tx \otimes Sf$, *for every rank-one operator* $x \otimes f \in B(X)$.

(2) *There exist injective linear maps* $T : X^* \to X$ *and* $S : X \to X^*$ *such that* $\Phi(x \otimes f) = Tf \otimes Sx$, *for every rank-one operator* $x \otimes f \in B(X)$.

Proof. By Lemma 3.1, one of the four cases in Proposition 2.2 holds. It is sufficient to show that Case 1 and Case 2 do not hold. For this, we first assume that Case 1 holds. Then, there exist a nonzero functional $g \in X^*$ and a linear map $\tau : \mathcal{F}(X) \to X$ such that $\Phi(x \otimes f) = \tau(x \otimes f) \otimes g$, for every rank-one operator $x \otimes f \in B(X)$. Take and fix $g_0 \in X^*$ such that g_0 and g are linearly independent and then choose a nonzero vector $x_0 \in X$. By the surjectivity of Φ, there exists an $A \in B(X)$ such that $\Phi(A) = x_0 \otimes g_0$. Note that $\operatorname{rank}(A) \geq 2$. Hence, there exist $x_1, x_2 \in X$ such that Ax_1 and Ax_2 are linearly independent. Take nonzero functionals $f_1, f_2 \in X^*$ such that $f_1(x_1) = f_2(x_2) = 0$. Then,

$$A \simeq A(I + x_i \otimes f_i) = A + Ax_i \otimes f_i, \ \ i = 1, 2.$$

It follows that

$$\overline{O(x_0 \otimes g_0)} = \overline{O(x_0 \otimes g_0 + \tau(Ax_i \otimes f_i) \otimes g)}, \ \ i = 1, 2.$$

By Lemma 2.7, we have $\operatorname{rank}(x_0 \otimes g_0 + \tau(Ax_i \otimes f_i) \otimes g) = 1$, for $i = 1, 2$. Since g_0 and g are linearly independent, both $\{x_0, \tau(Ax_1 \otimes f_1)\}$ and $\{x_0, \tau(Ax_2 \otimes f_2)\}$ are linearly dependent, which implies that $\Phi(Ax_1 \otimes f_1)$ and $\Phi(Ax_2 \otimes f_2)$ are linearly dependent. Because Φ is injective and linear, Ax_1 and Ax_2 are linearly dependent, which is a contradiction. By a similar way, we show that Case 2 does not hold. The proof is complete. □

According to Lemma 3.3, we further assume that there exist injective linear maps $T : X \to X$ and $S : X^* \to X^*$ such that

$$\Phi(x \otimes f) = Tx \otimes Sf, \tag{3.2}$$

for every rank-one operator $x \otimes f \in B(X)$.

Lemma 3.4. *T and S are surjective.*

Proof. We only show that S is surjective. Suppose, on the contrary, that there exists a nonzero functional $f_0 \in X^* \setminus \operatorname{ran}(S)$. Fix a nonzero vector $x_0 \in X$. Since Φ is surjective, there exists an $A \in B(X)$ such that $\Phi(A) = x_0 \otimes f_0$. Note that $\operatorname{rank}(A) \geq 2$. So, there exist $x_1, x_2 \in X$ such that Ax_1 and Ax_2 are linearly independent. Take $f_1, f_2 \in X^*$ such that $f_1(x_1) = f_2(x_2) = 1$. Then,

$$A \simeq A(I + x_i \otimes f_i) = A + Ax_i \otimes f_i, \ \ i = 1, 2.$$

This together with equation (3.2) gives us

$$\overline{O(x_0 \otimes f_0)} = \overline{O(x_0 \otimes f_0 + TAx_i \otimes Sf_i)}, \ \ i = 1, 2.$$

By Lemma 2.7, $\operatorname{rank}(x_0 \otimes f_0 + TAx_i \otimes Sf_i) = 1$, for $i = 1, 2$. Since both $\{f_0, Sf_1\}$ and $\{f_0, Sf_2\}$ are linearly independent sets, both $\{x_0, TAx_1\}$ and $\{x_0, TAx_2\}$ are linearly dependent sets, which implies that TAx_1 and TAx_2 are linearly dependent. It follows from the linearity and the injectivity of T that Ax_1 and Ax_2 are linearly dependent, which is a contradiction. By a similar argument, we show that T is surjective too. The proof is complete. □

Since Φ is surjective, in the sequel, we assume that $\Phi(U) = I$, for some nonzero operator $U \in B(X)$.

Lemma 3.5. *Let $x \in X$ and $f \in X^*$. Then,*

$$(Sf)(TUx) = f(x) \tag{3.3}$$

and

$$(SU^*f)(Tx) = f(x). \tag{3.4}$$

Proof. Take any nonzero vector $x \in X$ and nonzero functional $f \in X^*$ such that $f(x) = 0$. Let λ be in $\mathbb{C}$. Then, $I + \lambda x \otimes f$ is invertible by Lemma 2.5. It follows that

$$U \simeq U(I + \lambda x \otimes f) = U + \lambda Ux \otimes f.$$

This together with equation (3.2) gives

$$I \simeq_a I + \lambda TUx \otimes Sf.$$

By Lemma 2.6, $I + \lambda TUx \otimes Sf$ is invertible. So, we have $\lambda(Sf)(TUx) \neq -1$ by Lemma 2.5. Hence, by the arbitrariness of λ, we have

$$(Sf)(TUx) = 0, \tag{3.5}$$

for every $x \in X$ and $f \in X^*$ with $f(x) = 0$.

In what follows, we will show that there exists a constant $\mu \in \{0, 1\}$ such that

$$(Sf)(TUx) = \mu f(x), \tag{3.6}$$

for every $x \in X$ and $f \in X^*$. Choose $x_1 \in X$ and $f_1 \in X^*$ such that $f_1(x_1) = 1$. For every $\lambda_1 \in \mathbb{C}\backslash\{1\}$, $I - \lambda_1 x_1 \otimes f_1$ is invertible by Lemma 2.5. Set $\mu = (Sf_1)(TUx_1)$. By the same method as for the proof of equation (3.5), where, instead of $I + \lambda x \otimes f$, we use invertible operator $I - \lambda_1 x_1 \otimes f_1$, we obtain that $\lambda_1\mu \neq 1$ for all $\lambda_1 \in \mathbb{C}\backslash\{1\}$. This implies that $\mu = 0$ or $\mu = 1$.

Now, we shall consider two cases according to the dimension of X.

Case 1: X is finite-dimensional. Assume that $\dim X = n$ $(2 \leq n < \infty)$. Let $\{x_1, x_2, \ldots, x_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ be the bases of X and X^* satisfying $f_i(x_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. For $i \neq j$, since $(f_i + f_j)(x_i - x_j) = 0$, applying equation (3.5) gives $(S(f_i + f_j))(TU(x_i - x_j)) = 0$. This together with $(Sf_i)(TUx_j) = 0$ gives us $(Sf_i)(TUx_i) = (Sf_j)(TUx_j) = \mu$. For any $x \in X$, $f \in X^*$, we can write $x = \sum_{i=1}^{n}\alpha_i x_i$ and $f = \sum_{j=1}^{n}\beta_j f_j$, where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{C}$. If follows that

$$(Sf)(TUx) = \left(S\left(\sum_{j=1}^{n}\beta_j f_j\right)\right)\left(TU\left(\sum_{i=1}^{n}\alpha_i x_i\right)\right) = \mu f(x).$$

Case 2: X is infinite-dimensional. Since T and S are the linear maps, it is sufficient to show that $(Sf)(TUx) = \mu$ for every $x \in X$ and $f \in X^*$ with $f(x) = 1$. To do this, take a nonzero vector $y \in X$ such that $f(y) = f_1(y) = 0$ and $y \notin \operatorname{span}\{x, x_1\}$. Then, we can find a functional $g \in X^*$ such that $g(y) = 1$ and $g(x) = g(x_1) = 0$. Since $f_1(y) = g(x_1) = (f_1 + g)(x_1 - y) = 0$, by equation (3.5), we see that $(Sg)(TUy) = \mu$. By a similar argument, we obtain that $(Sf)(TUx) = (Sg)(TUy)$, which yields that $(Sf)(TUx) = \mu$.

So far, equation (3.6) has been established.

Finally, we will show that $\mu = 1$. Assume, on the contrary, that $\mu = 0$. Then, for all $x \in X$ and $f \in X^*$, $(Sf)(TUx) = 0$. Since S is surjective, we have $f(TUx) = 0$, for all $x \in X$ and $f \in X^*$. This implies that $TUx = 0$, for all $x \in X$. By the injectivity of T, $Ux = 0$, for all $x \in X$, which contradicts the fact that $U \neq 0$. ☐

The following lemma comes from Step 5 in [6]. For the sake of completeness, we give the details here.

Lemma 3.6. *T and S are bounded.*

Proof. Let $\Phi(U) = I$. We first show that TU is bounded. Let $\{x_n\}_{n=1}^{\infty} \subseteq X$ be such that $\lim_{n\to\infty}x_n = 0$ and $\lim_{n\to\infty}TUx_n = y_0$, where $y_0 \in X$. It follows from equation (3.3) that for every $f \in X^*$, we have $Sf(y_0) = 0$. Since S is surjective by Lemma 3.4, $y_0 = 0$. By the closed graph theorem, TU is bounded.

Now, we show that S is bounded. By the bijectivity of S and equation (3.3), we obtain $S^{-1}f(x) = f(TUx)$, for every $x \in X$ and $f \in X^*$. Since TU is bounded, we have

$$|S^{-1}f(x)| = |f(TUx)| \leq ||f||\cdot||TU||\cdot||x||,$$

for every $x \in X$ and $f \in X^*$. If follows that $||S^{-1}f|| \leq ||TU||\cdot||f||$, for all $f \in X^*$. So $||S^{-1}|| \leq ||TU||$, and hence, S is bounded. By the similar method, we show that T is bounded by equation (3.4). ☐

Lemma 3.7. *U is invertible.*

Proof. Let us first see that U has dense range. It suffices to show that U^* is injective. For this, let $U^*f = 0$ and $f \in X^*$. By equation (3.4), $f(x) = 0$, for all $x \in X$, i.e., $f = 0$. Hence, U^* is injective.

Now, we show that U is bounded below. Choose any nonzero $x \in X$. Since S is bijective, we can take $f_x \in X^*$ such that $\|S^{-1}f_x\| = 1$ and $(S^{-1}f_x)(x) = \|x\|$. Then, $\|f_x\| = \|SS^{-1}f_x\| \leq \|S\|$. By Lemma 3.6, we obtain

$$\|x\| = |(S^{-1}f_x)(x)| = |f_x(TUx)| \leq \|f_x\|\cdot\|TUx\| \leq \|S\|\cdot\|T\|\cdot\|Ux\|.$$

By the arbitrariness of x, U is bounded below, which completes the proof. □

The proof of Theorem 1.1. Let $A \in B(X)$ be invertible. Then, $A^{-1}AU = U$. Since U is invertible by Lemma 3.7, $A \simeq U$. It follows that $\Phi(A) \simeq_a I$. Then, by Lemma 2.6, $\Phi(A)$ is invertible. Applying Proposition 2.4, we complete the proof.

4 Applications

In this section, we will give some applications of Theorem 1.1. First, we generalize the result in [4] to the general Banach space case.

Corollary 4.1. *Let X be a complex Banach space with* $\dim X \geq 2$. *Then, a linear bijection $\Phi : B(X) \to B(X)$ preserves equivalence if and only if it is of the form either* (1) *or* (2) *in Theorem* 1.1.

Proof. The sufficiency is obvious. It remains to show the necessity. For this, suppose that $A \simeq B$, for $A, B \in B(X)$. It follows that $\Phi(A) \simeq \Phi(B)$. This leads to $\Phi(A) \simeq_a \Phi(B)$. Since Φ is bijective, applying Theorem 1.1, we complete the proof. □

The following corollary characterizes bijective linear maps preserving asymptotic equivalence on Banach space.

Corollary 4.2. *Let X be a complex Banach space with* $\dim X \geq 2$. *Then, a linear bijection $\Phi : B(X) \to B(X)$ preserves asymptotic equivalence if and only if it is of the form either* (1) *or* (2) *in Theorem* 1.1.

Proof. The sufficiency is obvious. Now, we show the necessity. Let $A \simeq B$ and $A, B \in B(X)$. Then, $A \simeq_a B$. Since Φ preserves asymptotic equivalence, we see that $\Phi(A) \simeq_a \Phi(B)$. Applying Theorem 1.1, we complete the proof. □

Applying Corollaries 4.1 and 4.2, we have

Corollary 4.3. *Let X be a complex Banach space with* $\dim X \geq 2$. *Suppose $\Phi : B(X) \to B(X)$ is a bijective linear map. Then, the following statements are equivalent.*

(1) *Φ preserves equivalence.*
(2) *Φ preserves asymptotic equivalence.*
(3) *One of the following statements holds.*
  (*a*) *There exist invertible bounded linear operators $T, S : X \to X$ such that $\Phi(A) = TAS$, for all $A \in B(X)$.*
  (*b*) *There exist invertible bounded linear operators $T : X^* \to X$ and $S : X \to X^*$ such that $\Phi(A) = TA^*S$, for all $A \in B(X)$.*

Acknowledgement: The authors would like to thank the referee for a very thorough reading of this article and many helpful comments that improved this article.

Funding information: This work was supported by the National Natural Science Foundation of China (No. 12061018).

Conflict of interest: The authors state no conflict of interest.

References

[1] L. Rodman and P. Šemrl, *A localization technique for linear preserver problems*, Linear Algebra Appl. **433** (2010), no. 11–12, 2257–2268, DOI: https://doi.org/10.1016/j.laa.2010.07.032.

[2] C. K. Li and S. Pierce, *Linear preserver problems*, Amer. Math. Monthly **108** (2001), no. 7, 591–605, DOI: https://doi.org/10.1080/00029890.2001.11919790.

[3] R. A. Horn, C. K. Li, and N. K. Tsing, *Linear operators preserving certain equivalence relations on matrices*, SIAM J. Matrix Anal. Appl. **12** (1991), no. 2, 195–204, DOI: https://doi.org/10.1137/0612015.

[4] T. Petek and G. Radić, *Linear preservers of equivalence relations on infinite-dimensional spaces*, Studia Math. **238** (2017), no. 2, 101–119, DOI: https://doi.org/10.4064/sm8356-12-2016.

[5] T. Petek and G. Radić, *A note on equivalence preserving maps*, Linear Multilinear Algebra **68** (2020), no. 11, 2289–2297, DOI: https://doi.org/10.1080/03081087.2019.1578727.

[6] G. Radić, *Maps preserving equivalence by products of involutions*, Oper. Matrices **13** (2019), no. 3, 823–833, DOI: https://doi.org/10.7153/oam-2019-13-58.

[7] G. X. Ji and H. K. Du, *Similarity-invariant subspaces and similarity-preserving linear maps*, Acta Math. Sin. (Engl. Ser.) **18** (2002), no. 3, 489–498, DOI: https://doi.org/10.1007/s10114-002-0179-5.

[8] F. Lu and C. Peng, *Similarity-preserving linear maps on* $B(X)$, Studia Math. **209** (2012), no. 1, 1–10, DOI: https://doi.org/10.4064/sm209-1-1.

[9] P. Šemrl, *Similarity preserving linear maps*, J. Operator Theory **60** (2008), no. 1, 71–83, http://www.jstor.org/stable/24715837.

[10] S. Du, J. Hou, and Z. Bai, *Additive maps preserving similarity or asymptotic similarity on* $B(H)$, Linear Multilinear Algebra **55** (2007), no. 3, 209–218, DOI: https://doi.org/10.1080/03081080500380462.

[11] M. Karder, T. Petek, and A. Taghavi, *Unitary similarity preserving linear maps on* $B(H)$, Integral Equations Operator Theory **82** (2015), no. 1, 51–60, DOI: https://doi.org/10.1007/s00020-014-2189-y.

[12] Z. Qin and F. Lu, *Involution similarity preserving linear maps*, Studia Math. **249** (2019), no. 3, 319–328, DOI: https://doi.org/10.4064/sm180311-5-7.

[13] B. Kuzma, *Additive mappings decreasing rank one*, Linear Algebra Appl. **348** (2002), no. 1–3, 175–187, DOI: https://doi.org/10.1016/s0024-3795(01)00577-8.

[14] A. R. Sourour, *Invertibility preserving linear maps on* $\mathcal{L}(X)$, Trans. Amer. Math. Soc. **348** (1996), no. 1, 13–30, DOI: https://doi.org/10.1090/s0002-9947-96-01428-6.