



Research Article

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Approximate controllability for a stochastic elastic system with structural damping and infinite delay

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Abstract: In this article, we study the existence of mild solutions and the approximate controllability for a class of stochastic elastic systems with structural damping and infinite delay in Hilbert spaces. The estimation of the control function is discussed, where the expression of the control function is constructed by the defined resolvent operator. Under this estimate, the existence of mild solutions for this system is obtained by the Schauder fixed point theorem and the stochastic analysis theory, and sufficient conditions for the approximate controllability are formulated and proved by using the so-called resolvent operator type condition. Finally, an example is given to illustrate the applicability of our conclusion.

Keywords: approximate controllability, Stochastic elastic system, infinite delay, Wiener process

MSC 2020: 34K30, 34K35, 60H15, 93B05, 93C10

1 Introduction

In this article, we consider the approximate controllability of the following stochastic elastic system with structural damping and infinite delay

$$\begin{cases} d[x'(t) + \rho Ax(t)] = [-A^2x(t) + f(t, x_t) + Bu(t)]dt + g(t, x_t)dW(t), & t \in I = [0, a], \\ x(t) = \varphi(t) \in L^2(\Omega; \mathcal{B}), \quad x'(0) = y_0 \in L^2(\Omega; X), & t \in (-\infty, 0] \end{cases} \quad (1)$$

in Hilbert space X , where $x(t) \in X$ is the state variable. $u(t) \in L^2_{\mathcal{F}}(I; U)$ is the control function and U is a Hilbert space. $B : U \rightarrow X$ is a bounded linear operator. $\rho \geq 2$ is a constant. The histories $x_t : (-\infty, 0] \rightarrow X$, given by $x_t(\theta) = x(t + \theta)$ for $\theta \leq 0$, belong to abstract phase space \mathcal{B} defined axiomatically. $A : D(A) \subset X \rightarrow X$ is a closed linear operator and $-A$ generates a C_0 -semigroup $T(t)(t \geq 0)$ on X . $\varphi(0) \in D(A)$, and $x'(0)$ denotes the right derivative of $x(\cdot)$ at zero. In addition, the functions $f : I \times \mathcal{B} \rightarrow X$ and $g : I \times \mathcal{B} \rightarrow L^2_0$ are Lipschitz continuous, and $W(t)$ is a Q -Wiener process.

The consideration of an elastic system with damping was proposed by Chen and Russell [1] in 1982. They studied the following second-order linear elastic system

$$u''(t) + Bu'(t) + Au(t) = 0$$

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in a Hilbert space H with inner (\cdot, \cdot) , where A (the elastic operator) is a positive definite, self-adjoint operator, and B (the damping operator) is a positive self-adjoint operator. The two conjectures about elastic system with damping were given by Huang [2,3]. New forms of the corresponding first-order evolution equation were introduced by Fan and Li [4] to study the analyticity and exponential stability of the semigroup of this system. Fan et al. [5] decomposed elastic systems with structural damping into two linear inhomogeneous initial value problems, and obtained the expression of this mild solution in 2013. In addition, monotone iterative technique, exponential decay of the elastic systems with structural damping, and asymptotic stability of this solution have been discussed in [6–9]. However, the (approximate) controllability of this system has not been studied yet, and the control problems are also a relatively active field (the optimal route or minimum energy required to reach the desired position). Therefore, we will study the approximate controllability of stochastic elastic system in this article.

When describing the actual phenomenon, the use of differential equations with infinite delay proves to be more accurate compared to ordinary differential equations. For additional information on finite or infinite delays, please refer to the relevant sections for further details, see [10–12]. Hence, recently, many authors have studied the approximate controllability of various systems with finite delay, infinite delay, and state dependent delay. For example, Mokkedem and Fu [13] studied the approximate controllability of first-order neutral integro-differential systems with finite delay in 2014, Mokkedem and Fu [14–16] discussed the approximate controllability of first-order (neutral or stochastic) evolution systems with infinite delay, [17–19] considered the approximate controllability for second-order stochastic (neutral) evolution systems with infinite delay, as well as Das et al. [20] investigated the approximate controllability for a second-order neutral stochastic differential equation with state-dependent delay.

In recent years, [21–24] focused on the approximate controllability of various stochastic systems (without damping elastic systems and infinite delays), [21] considered the approximate controllability of stochastic degenerate evolution equations, [22] discussed the approximate controllability for Sobolev-type fractional stochastic hemivariational inequalities of order $r \in (1, 2)$, [23] studied the approximate controllability of second-order impulsive stochastic neutral differential systems, and [24] considered the approximate controllability of conformable fractional noninstantaneous impulsive stochastic evolution equations via poisson jumps. In this article, on the basis of the analysis of stochastic systems in [21–24] and the books on stochastic differential equations [25–28], we further discuss the approximate controllability of stochastic elastic system with infinite delay.

The concept of controllability, when it was first introduced by Klamka [29] in 1963, has described the qualitative property of dynamic systems. In 1983, Zhou [30] studied the approximate controllability of the first-order abstract evolution by using the so-called range type condition in Hilbert spaces. In 1999, Bashirov and Mahmudov [31] used the so-called resolvent operator-type condition to study the approximate controllability of deterministic or stochastic systems. Using the resolvent operator-type condition, [13–16] obtained the approximate controllability of the first-order (neutral or stochastic) equation equations by using the fundamental solution theory, and [18,19] also studied the approximate controllability of the second-order (neutral or stochastic) evolution equation by using the fundamental solution theory. The fundamental solution theory is not used in this article, but our future work will focus on obtaining new explicit formulas for the mild solution of (stochastic) damped elastic systems using the fundamental solution theory, which can weaken linear conditions.

As we all know, the existence and uniqueness of mild solution for second-order systems are expressed by sine and cosine family in [17,20, 32,33]. The theories of sine and cosine family were defined by Fattorini [34] in 1969, and its properties have been studied in [35–38]. But the damping elastic system does not apply to the cosine family theory, so we use the semigroup theory to describe its solution. Inspired by all the aforementioned papers, we study the existence of mild solutions and the approximate controllability for a class of elastic stochastic system with structural damping and infinite delay in Hilbert spaces. The discussion is based on semigroup theory, stochastic analysis theory, Schauder fixed point theorem, and the so-called resolvent operator type condition in this article.

The innovations of this article are as follows: (a) we convert the system (1) into two first-order systems and use semigroup theory instead of cosine family theory to obtain the expression for the mild solution of the system (1); (b) we add infinite delay and stochastic term (generated by white noise) to the initial elastic system with structural damping, which makes the application range of the stochastic damping elastic system wider.

Specifically, after adding the stochastic term, the properties of the system yielded significant changes (from deterministic systems to stochastic systems). Therefore, the existence of mild solutions and the approximate controllability of system (1) are analyzed by combining the theory of semigroup, stochastic analysis, and phase space; (c) through our rigorous testing, the constructed control function can be applied in engineering control and has certain application value, such as the control problem of beam vibration equation.

The structure of this article is organized as follows. In Section 2, we give stochastic process, the axiomatic definition of phase space \mathcal{B} , some definitions, and necessary preparations. We establish some estimates of the control function in Section 3. The existence of mild solutions for the system (1) is studied by using the Schauder fixed point theorem. In Section 4, we will show sufficient conditions the approximate controllability of the system (1) by using the so-called resolvent operator type condition. In Section 5, an application example is given to illustrate our main result.

2 Preliminaries

In this section, we introduce some notations and terminologies of the stochastic process and the infinite dimensional phase space, as well as some basic facts about the approximate controllability. Let X and K be two separable Hilbert spaces, we denote by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_K$ their inner products, and by $\|\cdot\|$ and $\|\cdot\|_K$ their vector norms, respectively. We employ the same notation $\|\cdot\|$ for the norm of $\mathcal{L}(K; X)$, where $\mathcal{L}(K; X)$ denotes the Banach space of bounded linear operators from K into X . Particularly, $\mathcal{L}(X)$ will denote $\mathcal{L}(X; X)$.

2.1 Stochastic process

This subsection introduces some notations of stochastic processes used in the whole article. Let $\Omega = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is a right continuous increasing family and \mathcal{F}_0 contains all \mathbb{P} -null sets. Let $e_n, (n = 1, 2, \dots)$, be a complete orthonormal basis of K . We assume that $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$ is the σ -algebra generated by W and $\mathcal{F}_a = \mathcal{F}$, where $W(t)$ is a K -valued Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a finite trace nuclear covariance operator Q . Let $Q \in \mathcal{L}(K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\text{Tr}(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$. And let $\eta_n(t), (n = 1, 2, \dots)$, be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ such that

$$W(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda_n} \eta_n(t) e_n, \quad t \geq 0,$$

where $\lambda_n \geq 0, (n = 1, 2, \dots)$ are nonnegative real numbers. Then the aforementioned K -valued stochastic process $W(t)$ is called a Q -Wiener process. Without loss of generality, in [25,26,28], the Wiener process is constructed as $W(t) = \sum_{k=1}^{+\infty} A_k s_k(t)$, where the coefficients $\{A_k\}_{k=0}^{\infty}$ are independent and $N(0, 1)$ random variables and $s_k(t)$ are the k th-Schauder function (the integral of some complete basis $e_k(t)$ with respect to t). The Wiener process which satisfies general expressions in this article is more suitable for abstract systems.

Definition 2.1. Let $\sigma \in \mathcal{L}(K; X)$ and define

$$\|\sigma\|_Q^2 = \text{Tr}(\sigma Q \sigma^*) = \sum_{n=1}^{+\infty} \|\sqrt{\lambda_n} \sigma e_n\|^2.$$

If $\|\sigma\|_Q < \infty$, then σ is called a Q -Hilbert-Schmidt operator. Let $L_0^2(K; X)$ denote the space of all Q -Hilbert-Schmidt operators $\sigma : K \rightarrow X$. In this article, the function $F(\cdot, \cdot)$ is said to be \mathcal{F}_t -adapted if $F(t, \cdot) : \Omega \rightarrow X$ is \mathcal{F}_t -measurable, a.e. $t \in I$. Let $L_{\mathcal{F}}^2(I; X) = \{x \in L^2(I \times \Omega; X) : x \text{ is } \mathcal{F}_t\text{-adapted}\}$. In addition, we use $ML^2(\Omega; \mathcal{B})$ to denote the set of all \mathcal{F}_0 -measurable functions that belong to $L^2(\Omega; \mathcal{B})$, and $C(I; L^2(\Omega; X))$ to represent the Banach space of all continuous from I into $L^2(\Omega; X)$ satisfying the condition $\sup_{t \in I} \mathbb{E}\|x(t)\|^2 < \infty$.

Lemma 2.2. [28] Let $\phi : I \times \Omega \rightarrow L_0^2$ be a strongly measurable mapping such that $\int_0^a \mathbf{E} \|\phi(r)\|_Q^2 dr < \infty$. Then, for any $t \in I$ and $p \geq 2$, we have

$$\mathbf{E} \left\| \int_0^t \phi(s) dW(s) \right\|^p \leq c_p \mathbf{E} \left(\int_0^t \|\phi(s)\|_Q^2 ds \right)^{\frac{p}{2}},$$

where c_p is a constant associated with p and a .

Lemma 2.3. [39] Let $p \geq 2$ and $h \in L^2(\Omega; X)$ be fixed, then there exists a function ϕ in space $L_T^2(\Omega; L^2(I; L_0^2(K; X)))$ such that

$$h = \mathbf{E}h + \int_0^a \phi(s) dW(s).$$

2.2 Phase space for infinite delay

In the whole article, we use the axiomatic definition of phase space \mathcal{B} in [10] and some terms in [11]. The phase space \mathcal{B} will be a linear space of functions mapping $(-\infty, 0]$ to X , which is given seminorm $\|\cdot\|_{\mathcal{B}}$ and satisfies the following axioms:

(A1) If $u : (-\infty, \vartheta + a] \rightarrow X$, $a > 0$, is continuous on $[\vartheta, \vartheta + a]$ and $u_{\vartheta} \in \mathcal{B}$, then for each $t \in [\vartheta, \vartheta + a]$, the following statements hold:

- (i) $u_t \in \mathcal{B}$;
- (ii) $\|u(t)\| \leq \widetilde{H} \|u_t\|_{\mathcal{B}}$;
- (iii) $\|u_t\|_{\mathcal{B}} \leq K(t - \vartheta) \sup\{\|u(\tau)\| : \vartheta \leq \tau \leq t\} + M(t - \vartheta) \|u_{\vartheta}\|_{\mathcal{B}}$.

Here, $\widetilde{H} \geq 0$ is a constant, $K : [0, \infty) \rightarrow [0, \infty)$ is continuous, and $M : [0, \infty) \rightarrow [0, \infty)$ is locally bounded. They do not rely on $u(\cdot)$.

(A2) For the function $u(\cdot)$ in (A1), u_t is a \mathcal{B} -valued continuous function on $[\vartheta, \vartheta + a]$.

(A3) The phase space \mathcal{B} is complete.

Let the phase space $\mathcal{B} = C_r \times L^q(g : X)$, $r \geq 0$, $1 \leq q \leq \infty$ (see [11]), which means that any function $\varphi \in \mathcal{B}$ from $(-\infty, 0]$ into X satisfies continuous on $[-r, 0]$ and positive Lebesgue integrable on $(-\infty, -r)$. We define the seminorm as follows:

$$\|\varphi\|_{\mathcal{B}} = \sup\{\|\varphi(\xi)\| : -r \leq \xi \leq 0\} + \left(\int_{-\infty}^{-r} g(\xi) \|\varphi(\xi)\|^q d\xi \right)^{\frac{1}{q}}.$$

Remark 1. For convenience, let K_a and M_a be two positive constants defined by

$$K_a = \sup_{t \in I} K(t), \quad M_a = \max_{t \in I} M(t), \quad (2)$$

where the functions $K(\cdot)$ and $M(\cdot)$ are from (A1)(iii).

2.3 Mild solution and resolvent operator

Let $-A : D(A) \subset X \rightarrow X$ be the infinitesimal generator of C_0 -semigroup $T(t)(t \geq 0)$ on X . We need to make the following prior assumptions about the operator A

(H0) The operator $(A, D(A))$ generates a C_0 -semigroup $T(t)(t \geq 0)$ on the Hilbert space X , then there exist constants $\omega \in \mathbb{R}$ and $M_{\omega} \geq 1$ such that [40]

$$\|T(t)\| \leq M_\omega e^{\omega t} \quad \text{for all } t \geq 0.$$

The sets $S_1(t)$ and $S_2(t)$ defined by Fan and Li [7] are expressed as follows:

$$S_1(t) = T(\sigma_1 t), \quad S_2(t) = T(\sigma_2 t), \quad t \geq 0, \quad (3)$$

where $\sigma_1 + \sigma_2 = \rho$, $\sigma_1 \sigma_2 = 1$.

Remark 2. According to the properties of C_0 -semigroups, let

$$M_1 = \sup_{t \in I} \|S_1(t)\|, \quad M_2 = \sup_{t \in I} \|S_2(t)\|. \quad (4)$$

The mild solution of control system (1) is obtained by combining two nonhomogeneous initial value problem. Inspired by the mild solution of Definition 2.3 in [7], we can similarly write a mild solution of the system (1) as follows.

Definition 2.4. A stochastic process $x : (-\infty, a] \rightarrow X$ is called a mild solution of the system (1) if the following conditions are satisfied:

- (i) $x(t, \omega)$ is measurable as a function from $I \times \Omega$ to X and $x(t)$ is \mathcal{F}_t -adapted;
- (ii) $E\|x(t)\|^2 < \infty$ for each $t \in I$ and $\{x_t : t \in I\}$ is \mathcal{B} -valued stochastic process;
- (iii) For each $u \in L^2_{\mathcal{F}}(I; U)$, the process $x(\cdot)$ satisfies the following integral equation:

$$x(t) = \begin{cases} S_2(t)\varphi(0) + \int_0^t S_2(t-s)S_1(s)z_0 ds + \iint_{00}^{ts} S_2(t-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds \\ + \iint_{00}^{ts} S_2(t-s)S_1(s-\tau)Bu(\tau) d\tau ds \\ + \iint_{00}^{ts} S_2(t-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s), \quad t \in (0, a], \\ \varphi(t), \quad t \leq 0, \end{cases} \quad (5)$$

where $z_0 = y_0 + \sigma_2 A \varphi(0)$.

Definition 2.5. Let $x(\cdot)$ be a mild solution of system (1). System (1) is called to be approximately controllable on $[0, a]$ if $\overline{\mathbb{R}(a; \varphi, y_0)} = L^2(\Omega; X)$, where the set

$$\mathbb{R}(a; \varphi, y_0) = \{x(a; \varphi, y_0, u) \in L^2(\Omega; X) | \varphi \in ML^2(\Omega; \mathcal{B}), y_0 \in L^2(\Omega; X), u \in L^2_{\mathcal{F}}(I; U)\}$$

is called the reachable set of system (1) and $\overline{\mathbb{R}(a; \varphi, y_0)}$ represents the closure of $\mathbb{R}(a; \varphi, y_0)$.

Next, we introduce two operators defined on Hilbert space X

$$\Gamma_0^a = \iint_{00}^{as} S_2(a-s)S_1(s-\tau)BB^*S_1^*(s-\tau)S_2^*(a-s) d\tau ds, \quad R(\lambda, \Gamma_0^a) = (\lambda I + \Gamma_0^a)^{-1}, \quad \lambda > 0, \quad (6)$$

where B^* , $S_1^*(t)$, and $S_2^*(t)$ denote the adjoint operators of B , $S_1(t)$, and $S_2(t)$, respectively. Now, we make the assumption

(H1) $\lambda R(\lambda, \Gamma_0^a) \rightarrow 0$ as $\lambda \rightarrow 0^+$ in the strong operator topology.

The assumption (H1) is equivalent to the approximate controllability of linear system

$$\begin{cases} x''(t) + \rho Ax'(t) + A^2 x(t) = Bu(t), & t \in I, \\ x(t) = x_0, \quad x'(0) = y_0. \end{cases} \quad (7)$$

To be more precise, we obtain that

Theorem 2.6. *The following sentences are equivalent:*

- (i) *The control system (7) is approximately controllable on $[0, a]$.*
- (ii) *If $B^*S_1^*(t)S_2^*(t)y = 0$ for $t \in [0, a]$, then $y = 0$.*
- (iii) *Assumption (H1) is true.*

The proof is likeness to the proof of Theorem 4.4.17 from [41] and Theorem 2 from [31], so we omit it here.

Remark 3. From assumption (H1), we can easily obtain

$$\|R(\lambda, \Gamma_0^q)\| \leq \frac{1}{\lambda}, \quad \lambda \in (0, 1). \quad (8)$$

Lemma 2.7. (Schauder fixed point theorem) *Let H be a convex closed subset in the Banach space x , the operator $T : H \rightarrow H$ is continuous and compact, then the operator T has at least one fixed point x^* , such that $Tx^* = x^*$.*

3 Existence of mild solution

In this section, we first give the expression of the control function through the resolvent operator defined by (6) and then obtain some estimates about the control function u . Finally, we investigate the existence of mild solution to system (1) by the Schauder fixed point theorem. For this purpose, we make the following assumptions:

(H2) The functions $f : I \times \mathcal{B} \rightarrow X$ and $g : I \times \mathcal{B} \rightarrow L_0^2(K; X)$ satisfy the following conditions:

- (i) $f : I \times \mathcal{B} \rightarrow X$ and $g : I \times \mathcal{B} \rightarrow L_0^2(K; X)$ are two measurable mappings, satisfying that $f(t, 0)$ and $g(t, 0)$ are bounded in X -norm and $L_0^2(K; X)$ -norm, respectively.
- (ii) For any $t \in [0, a]$, there exists a constant $L > 0$, such that for any $\varphi_1, \varphi_2 \in \mathcal{B}$,

$$\|f(t, \varphi_1) - f(t, \varphi_2)\|^2 + \|g(t, \varphi_1) - g(t, \varphi_2)\|_Q^2 \leq L \|\varphi_1 - \varphi_2\|_{\mathcal{B}}^2.$$

- (iii) There exists a constant $L_1 > 0$ such that

$$\|f(t, \varphi)\|^2 + \|g(t, \varphi)\|_Q^2 \leq L_1.$$

(H3) The C_0 -semigroup $T(t)$ is compact for $t > 0$.

Remark 4. Assumptions (H0) and (H3) are easy to satisfy. In Section 5, the defined operator A automatically satisfies assumptions (H0) and (H3). Assumption (H1) is a prerequisite to prove that system (1) is approximately controllable. In the case that homogeneous linear system (2.6) is (approximately) controllable, we further study the (approximate) controllability of stochastic nonlinear system (1). Assumption (H2) is necessary to ensure the existence and uniqueness of solutions for nonlinear system (1).

Let $\varphi \in ML^2(\Omega; \mathcal{B})$ be a given \mathcal{F}_0 -adapted process. We define

$$\begin{aligned} C_\varphi &= \{y(t) : (-\infty, a] \rightarrow L^2(\Omega; X) | y(0) \\ &= \varphi \text{ and } y(t)_{(t \in [0, a])} \text{ is continuous, } \mathcal{F}_t\text{-adapted measurable processes}\}, \end{aligned}$$

its norm is endowed by

$$\|y\|_{C_\varphi} = \left(\sup_{t \in I} \mathbf{E} \|y_t\|_{\mathcal{B}}^2 \right)^{\frac{1}{2}} < \infty. \quad (9)$$

Let $x^a \in L^2(\Omega; X)$ be fixed, we define the control function $u(t) \in L_{\mathcal{F}}^2(I; U)$ as follows:

$$\begin{aligned}
u_x(t) = & B^* S_1^*(s-t) S_2^*(a-s) R(\lambda, \Gamma_0^a) \left(\mathbf{E} x^a - S_2(t) \varphi(0) - \int_0^t S_2(t-s) S_1(s) z_0 ds \right. \\
& - \int_0^t \int_0^s S_2(t-s) S_1(s-\tau) f(\tau, x_\tau) d\tau ds \\
& \left. - \int_0^t \int_0^s S_2(t-s) S_1(s-\tau) g(\tau, x_\tau) d\tau - \phi(s) \right] dW(s),
\end{aligned} \tag{10}$$

where $x^a = \mathbf{E} x^a + \int_0^t \phi(s) dW(s)$ from Lemma 2.3. Then, we can obtain the following estimation of the control function $u(t)$.

Lemma 3.1. *There exist two constants $L_2 > 0$ and $L_3 > 0$ such that for each $x^1, x^2 \in C_\varphi$*

$$\mathbf{E} \|u_{x^1}(t) - u_{x^2}(t)\|^2 \leq \frac{1}{\lambda^2} L_2 \mathbf{E} \|x_t^1 - x_t^2\|_{\mathcal{B}}^2, \quad \mathbf{E} \|u_{x^1}(t)\|^2 \leq \frac{1}{\lambda^2} L_3,$$

where $L_2 \geq 2\|B\|^2 M_1^4 M_2^4 a^3 L(a + c_p)$ and

$$L_3 \geq 6\|B\|^2 M_1^2 M_2^2 \left[\mathbf{E} \|x^a\|^2 + M_2^2 \mathbf{E} \|\varphi(0)\|^2 + M_1^2 M_2^2 a^2 \mathbf{E} \|z_0\|^2 + \frac{a^4}{3} M_1^2 M_2^2 L_1^2 + \frac{a^3}{3} M_1^2 M_2^2 L_1^2 c_p + c_p \int_0^t \mathbf{E} \|\phi(s)\|_Q^2 ds \right].$$

Proof. First, we prove the first inequality. For any $x^1, x^2 \in C_\varphi$, by (4), (8), and (10), one has

$$\begin{aligned}
\mathbf{E} \|u_{x^1}(t) - u_{x^2}(t)\|^2 \leq & \frac{2}{\lambda^2} \|B\|^2 M_1^4 M_2^2 \mathbf{E} \left\| \int_0^t \int_0^s S_2(t-s) S_1(s-\tau) (f(\tau, x_\tau^1) - f(\tau, x_\tau^2)) d\tau ds \right\|^2 \\
& + \frac{2}{\lambda^2} \|B\|^2 M_1^2 M_2^2 \mathbf{E} \left\| \int_0^t \int_0^s S_2(t-s) S_1(s-\tau) (g(\tau, x_\tau^1) - g(\tau, x_\tau^2)) d\tau dW(s) \right\|^2.
\end{aligned}$$

According to Hölder inequality, (4), (H2)(ii), and Lemma 2.2, we have

$$\begin{aligned}
\mathbf{E} \|u_{x^1}(t) - u_{x^2}(t)\|^2 \leq & \frac{2}{\lambda^2} \|B\|^2 M_1^4 M_2^4 \mathbf{E} \left\| \int_0^t \int_0^s (f(\tau, x_\tau^1) - f(\tau, x_\tau^2)) d\tau ds \right\|^2 \\
& + \frac{2}{\lambda^2} \|B\|^2 M_1^4 M_2^4 \mathbf{E} \left\| \int_0^t \int_0^s (g(\tau, x_\tau^1) - g(\tau, x_\tau^2)) d\tau dW(s) \right\|^2 \\
\leq & \frac{2}{\lambda^2} \|B\|^2 M_1^4 M_2^4 a^2 \int_0^t \int_0^s \mathbf{E} \|f(\tau, x_\tau^1) - f(\tau, x_\tau^2)\|^2 d\tau ds \\
& + \frac{2}{\lambda^2} \|B\|^2 M_1^4 M_2^4 a c_p \int_0^t \int_0^s \mathbf{E} \|g(\tau, x_\tau^1) - g(\tau, x_\tau^2)\|_Q^2 d\tau ds \\
\leq & \frac{2}{\lambda^2} \|B\|^2 M_1^4 M_2^4 a^4 L \mathbf{E} \|x_t^1 - x_t^2\|_{\mathcal{B}}^2 + \frac{2}{\lambda^2} \|B\|^2 M_1^4 M_2^4 a^3 c_p L \mathbf{E} \|x_t^1 - x_t^2\|_{\mathcal{B}}^2 \\
\leq & \frac{1}{\lambda^2} L_2 \mathbf{E} \|x_t^1 - x_t^2\|_{\mathcal{B}}^2,
\end{aligned}$$

for $L_2 \geq 2\|B\|^2 M_1^4 M_2^4 a^3 L(a + c_p)$.

Next, by the aforementioned conclusions and (H2), it follows that

$$\begin{aligned} \mathbf{E} \|u_{x^1}(t)\|^2 &\leq \frac{6}{\lambda^2} \|B\|^2 M_1^2 M_2^2 \left(\mathbf{E} \|x^a\|^2 + M_2^2 \mathbf{E} \|\varphi(0)\|^2 + M_1^2 M_2^2 a^2 \mathbf{E} \|z_0\|^2 + \frac{a^4}{3} M_1^2 M_2^2 L_1^2 + \frac{a^3}{3} M_1^2 M_2^2 L_1^2 c_p \right. \\ &\quad \left. + c_p \int_0^t \mathbf{E} \|\phi(s)\|_Q^2 ds \right) \\ &\leq \frac{1}{\lambda^2} L_3, \end{aligned}$$

for $L_3 \geq 6\|B\|^2 M_1^2 M_2^2 \left(\mathbf{E} \|x^a\|^2 + M_2^2 \mathbf{E} \|\varphi(0)\|^2 + M_1^2 M_2^2 a^2 \mathbf{E} \|z_0\|^2 + \frac{a^4}{3} M_1^2 M_2^2 L_1^2 + \frac{a^3}{3} M_1^2 M_2^2 L_1^2 c_p + c_p \int_0^t \mathbf{E} \|\phi(s)\|_Q^2 ds \right)$.

The proof is completed. \square

Theorem 3.2. *Let $-A$ is the infinitesimal generator of C_0 -semigroup $T(t)(t \geq 0)$ and $(\varphi, y_0) \in ML^2(\Omega; \mathcal{B}) \times L^2(\Omega; X)$. If the assumptions (H0)–(H3) are satisfied, then for any $\lambda \in (0, 1)$, system (1) has at least one mild solution $x(t; \varphi, y_0) : (-\infty, a] \rightarrow X$.*

Proof. Let $x^a \in L^2(\Omega; X)$ and $\lambda \in (0, 1)$. We define the operator $Q : C_\varphi \rightarrow C_\varphi$ by

$$(Qx)(t) = \begin{cases} S_2(t)\varphi(0) + \int_0^t S_2(t-s)S_1(s)z_0 ds \\ + \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds \\ + \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)Bu_x(\tau) d\tau ds \\ + \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s), \quad t \in (0, a], \\ \varphi(t), \quad t \leq 0, \end{cases} \quad (11)$$

where $u_x(t)$ defined by (10). Obviously, the mild solution of system (1) is equivalent to the fixed point of the operator Q . Now we will prove it in three steps.

Step 1: We prove $Q : C_\varphi \rightarrow C_\varphi$ is continuous. Let $\{x^n\}_{n=1}^\infty \subset C_\varphi$ be a sequence such that $\lim_{n \rightarrow \infty} x^n = x$ in C_φ . From axioms (A1)(iii), for all $t \in I$, we obtain

$$\|x_t^n - x_t\|_{\mathcal{B}} \leq K_a \sup_{t \in I} \|x^n(t) - x(t)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (12)$$

For any $x^n, x \in C_\varphi$, one has

$$\begin{aligned} \|(Qx^n)(t) - (Qx)(t)\| &\leq \left\| \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)(f(\tau, x_\tau^n) - f(\tau, x_\tau)) d\tau ds \right\| \\ &\quad + \left\| \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)(Bu_{x^n}(\tau) - Bu_x(\tau)) d\tau ds \right\| \\ &\quad + \left\| \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)(g(\tau, x_\tau^n) - g(\tau, x_\tau)) d\tau dW(s) \right\|, \end{aligned}$$

where u_{x^n} and u_x are control functions corresponding to x^n and x , respectively. From (8), (10), and (12), Hölder inequality, Lemma 2.2 and 3.1, it follows that

$$\begin{aligned} \mathbf{E} \|(Qu^n)(t) - (Qu)(t)\|^2 &\leq 3M_1^2 M_2^2 \left(a^2 L \int_0^t \int_0^s \mathbf{E} \|x_\tau^1 - x_\tau^2\|_{\mathcal{B}}^2 d\tau ds + \frac{1}{\lambda^2} a^2 \|B\|^2 L_2 \int_0^t \int_0^s \mathbf{E} \|x_\tau^1 - x_\tau^2\|_{\mathcal{B}}^2 d\tau ds \right. \\ &\quad \left. + c_p a L \int_0^t \int_0^s \mathbf{E} \|x_\tau^1 - x_\tau^2\|_{\mathcal{B}}^2 d\tau ds \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, we obtain $Q : C_\varphi \rightarrow C_\varphi$ is continuous.

Step 2: We prove that there exists a positive constant R such that the operator Q defined by (11) maps the bounded closed convex set $B_R \in C_\varphi$ and defined as follows:

$$B_R = \{x \in C_\varphi : \|x\|_{C_\varphi} \leq R\}$$

into B_R . In fact,

$$R \geq 8K_a^2 R_1 + 2M_a^2 \mathbf{E} \|\varphi\|_{\mathcal{B}}^2,$$

where

$$R_1 > 5M_2^2 \mathbf{E} \|\varphi(0)\|^2 + 5M_1^2 M_2^2 a^2 \mathbf{E} \|z_0\|^2 + \frac{5a^4}{3} M_1^2 M_2^2 L_1 + \frac{5a^4}{3\lambda^2} M_1^2 M_2^2 \|B\|^2 L_3 + \frac{5a^3}{3} c_p M_1^2 M_2^2 L_1.$$

Then for any $x \in B_R$, by axioms (A1)(iii), one has

$$\mathbf{E} \|(Qx)_t\|_{\mathcal{B}}^2 \leq \mathbf{E} \left[K_a \sup_{s \in [0, t]} \|(Qx)(s)\| + M_a \|\varphi\|_{\mathcal{B}} \right]^2. \quad (13)$$

By the Doob's inequality [27, Theorem 6.1], it follows that

$$\mathbf{E} \left[\sup_{s \in [0, t]} \|(Qx)(s)\|^2 \right] \leq 4\mathbf{E} \|(Qx)(t)\|^2. \quad (14)$$

Combining with (13) and (14), which yields that

$$\mathbf{E} \|(Qx)_t\|_{\mathcal{B}}^2 \leq 8K_a^2 \mathbf{E} \|(Qx)(t)\|^2 + 2M_a^2 \mathbf{E} \|\varphi\|_{\mathcal{B}}^2.$$

From this inequality, we infer that, to show that $Q(B_R) \subset B_R$, it is sufficient to verify that $\mathbf{E} \|(Qx)(t)\|^2 < R_1$. By (H1)–(H3), (8), Lemma 2.2, Lemma 3.1, and Hölder inequality, one has

$$\begin{aligned} \mathbf{E} \|(Qx)(t)\|^2 &\leq 5\mathbf{E} \|S_2(t)\varphi(0)\|^2 + 5\mathbf{E} \left\| \int_0^t S_2(t-s)S_1(s)z_0 ds \right\|^2 + 5\mathbf{E} \left\| \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds \right\|^2 \\ &\quad + 5\mathbf{E} \left\| \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)Bu_x(\tau) d\tau ds \right\|^2 + 5\mathbf{E} \left\| \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s) \right\|^2 \\ &\leq 5M_2^2 \mathbf{E} \|\varphi(0)\|^2 + 5M_1^2 M_2^2 a^2 \mathbf{E} \|z_0\|^2 + \frac{5a^4}{3} M_1^2 M_2^2 L_1 + \frac{5a^4}{3\lambda^2} M_1^2 M_2^2 \|B\|^2 L_3 + \frac{5a^3}{3} c_p M_1^2 M_2^2 L_1 < R_1. \end{aligned}$$

Therefore, we obtain that $\mathbf{E} \|(Qx)(t)\|^2 < R_1$. Thus, we know $Q(B_R) \subset B_R$.

Step 3: We show that the operator Q is compact on B_R . We first prove that the set $\{(Qx)(t) : x \in B_R\}$ is relatively compact in X for every $t \in I$. It is obvious that $(Qu)(0)$ is relatively compact when $t = 0$. Now, we prove that $(Qx)(t)$ is relatively compact for $t \in (0, a]$. Let $0 < \varepsilon < t \leq a$ and for any $x \in B_R$, we define the operator $(Q^\varepsilon x)(t)$ by

$$\begin{aligned}
(Q^\varepsilon x)(t) &= S_2(t)\varphi(0) + \int_0^{t-\varepsilon} S_2(t-s)S_1(s)z_0 ds + \int_0^{t-\varepsilon} \int_0^s S_2(t-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds \\
&\quad + \int_0^{t-\varepsilon} \int_0^s S_2(t-s)S_1(s-\tau)Bu_x(\tau) d\tau ds + \int_0^{t-\varepsilon} \int_0^s S_2(t-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s) \\
&= S_2(t)\varphi(0) + S_2(\varepsilon) \int_0^{t-\varepsilon} S_2(t-\varepsilon-s)S_1(s)z_0 ds + S_2(\varepsilon) \int_0^{t-\varepsilon} \int_0^s S_2(t-\varepsilon-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds \\
&\quad + S_2(\varepsilon) \int_0^{t-\varepsilon} \int_0^s S_2(t-\varepsilon-s)S_1(s-\tau)Bu_x(\tau) d\tau ds + S_2(\varepsilon) \int_0^{t-\varepsilon} \int_0^s S_2(t-\varepsilon-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s).
\end{aligned}$$

By (H3) and (3), it is easy to obtain $S_2(t)$ ($t > 0$) is compact. Hence, we obtain that the set $\{(Q^\varepsilon x)(t) : x \in B_R\}$ is relatively compact for every $\varepsilon \in (0, t)$ on X . Since

$$\begin{aligned}
\mathbf{E} \|(Qx)(t) - (Q^\varepsilon x)(t)\|^2 &\leq 4\mathbf{E} \left\| \int_{t-\varepsilon}^t S_2(t-s)S_1(s)z_0 ds \right\|^2 + 4\mathbf{E} \left\| \int_{t-\varepsilon}^t \int_{t-\varepsilon}^s S_2(t-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds \right\|^2 \\
&\quad + 4\mathbf{E} \left\| \int_{t-\varepsilon}^t \int_{t-\varepsilon}^s S_2(t-s)S_1(s-\tau)Bu_x(\tau) d\tau ds \right\|^2 \\
&\quad + 4\mathbf{E} \left\| \int_{t-\varepsilon}^t \int_{t-\varepsilon}^s S_2(t-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s) \right\|^2 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,
\end{aligned}$$

we know that the set $\{(Qx)(t) : x \in B_R\}$ is relatively compact for $t \in I$ in X .

Next, we demonstrate that $Q(B_R)$ is equicontinuous on C_φ . Let $0 \leq t_1 < t_2 \leq a$ and for any $x \in B_R$, one has

$$\begin{aligned}
\mathbf{E} \|(Qx)(t_2) - (Qx)(t_1)\|^2 &\leq 9\mathbf{E} \|S_2(t_2)\varphi(0) - S_2(t_1)\varphi(0)\|^2 + 9\mathbf{E} \left\| \int_0^{t_1} [S_2(t_2-s) - S_2(t_1-s)]S_1(s)z_0 ds \right\|^2 \\
&\quad + 9\mathbf{E} \left\| \int_{t_1}^{t_2} S_2(t_2-s)S_1(s)z_0 ds \right\|^2 \\
&\quad + 9\mathbf{E} \left\| \int_0^{t_1} \int_0^s [S_2(t_2-s) - S_2(t_1-s)]S_1(s-\tau)f(\tau, x_\tau) d\tau ds \right\|^2 \\
&\quad + 9\mathbf{E} \left\| \int_{t_1}^{t_2} \int_0^s S_2(t_2-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds \right\|^2 \\
&\quad + 9\mathbf{E} \left\| \int_0^{t_1} \int_0^s [S_2(t_2-s) - S_2(t_1-s)]S_1(s-\tau)Bu_x(\tau) d\tau ds \right\|^2 \\
&\quad + 9\mathbf{E} \left\| \int_{t_1}^{t_2} \int_0^s S_2(t_2-s)S_1(s-\tau)Bu_x(\tau) d\tau ds \right\|^2 \\
&\quad + 9\mathbf{E} \left\| \int_0^{t_1} \int_0^s [S_2(t_2-s) - S_2(t_1-s)]S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s) \right\|^2 \\
&\quad + 9\mathbf{E} \left\| \int_{t_1}^{t_2} \int_0^s S_2(t_2-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s) \right\|^2 \\
&=: 9 \sum_{i=1}^9 J_i.
\end{aligned}$$

Thus, we just need to prove $J_i \rightarrow 0$ independently as $t_2 - t_1 \rightarrow 0$, $i = 1, 2, \dots, 9$. Since the function $S_2(t)\varphi(0)$ is continuous for $t \geq 0$, J_1 tend to 0 as $t_2 - t_1 \rightarrow 0$. Hence, $\lim_{t_2-t_1 \rightarrow 0} J_1 = 0$.

By Hölder inequality and (4), we obtain that

$$\begin{aligned} J_2 &\leq a \int_0^{t_1} \|S_2(t_2 - s) - S_2(t_1 - s)\|^2 \cdot \mathbf{E} \|S_1(s)z_0\|^2 ds \\ &\leq aM_1^2 \mathbf{E} \|z_0\|^2 \int_0^{t_1} \|S_2(t_2 - t_1 + \tau) - S_2(\tau)\|^2 d\tau, \end{aligned}$$

Similarly, from (H2), (2), (10), and Lemmas 3.1 and 2.2, one has

$$\begin{aligned} J_4 &\leq a^3 M_1^2 L_1 \int_0^a \|S_2(t_2 - t_1 + \tau) - S_2(\tau)\|^2 d\tau \\ J_6 &\leq \frac{1}{\lambda} a^3 M_1^2 \|B\|^2 L_3 \int_0^a \|S_2(t_2 - t_1 + \tau) - S_2(\tau)\|^2 d\tau \\ J_8 &\leq a^2 c_p M_1^2 L_1 \int_0^a \|S_2(t_2 - t_1 + \tau) - S_2(\tau)\|^2 d\tau. \end{aligned}$$

It is clear that $S_2(\cdot)$ is compact. Furthermore, we obtain that $S_2(t)$ is continuous in the uniform operator topology for $t > 0$, and $S_2(t_2 - t_1 + \tau) - S_2(\tau)$ is also continuous in the uniform operator topology on $t \in (0, a]$. Hence, $\|S_2(t_2 - t_1 + \tau) - S_2(\tau)\| \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$. And applying the Lebesgue dominated convergence theorem, we gain that $\lim_{t_2-t_1 \rightarrow 0} J_2 = \lim_{t_2-t_1 \rightarrow 0} J_4 = \lim_{t_2-t_1 \rightarrow 0} J_8 = \lim_{t_2-t_1 \rightarrow 0} J_6 = 0$.

By Hölder inequality, (2), (4), (10), and Lemma 2.2 and Lemma 3.1, it is easy to see that

$$\begin{aligned} J_3 &\leq M_1^2 M_2^2 \mathbf{E} \|z_0\| \cdot |t_2 - t_1|^2, \\ J_5 &\leq M_1^2 M_2^2 L_1 \frac{|t_2^3 - t_1^3|}{3} \cdot |t_2 - t_1|, \\ J_7 &\leq M_1^2 M_2^2 \|B\|^2 L_3 \frac{|t_2^3 - t_1^3|}{3\lambda^2} \cdot |t_2 - t_1|, \\ J_9 &\leq M_1^2 M_2^2 L_1 c_p \frac{|t_2^3 - t_1^3|}{3}. \end{aligned}$$

Therefore, $\lim_{t_2-t_1 \rightarrow 0} J_3 = \lim_{t_2-t_1 \rightarrow 0} J_5 = \lim_{t_2-t_1 \rightarrow 0} J_7 = \lim_{t_2-t_1 \rightarrow 0} J_9 = 0$. And $\mathbf{E} \|(Qx)(t_2) - (Qx)(t_1)\| \rightarrow 0$ independently as $t_2 - t_1 \rightarrow 0$. Then $Q(B_R)$ is equicontinuous on C_φ . Hence, by the Arzela-Ascoli theorem, one obtains that $Q : B_R \rightarrow B_R$ is compact operator.

So, we conclude that Q has at least one fixed point $x \in B_R$, i.e., the function $x(t; \varphi) : (-\infty, a] \rightarrow X$ is a mild solution of control system (1). The proof is completed. \square

4 Approximate controllability

Based on the existence result of mild solutions for the system (1) obtained in Section 3, we investigate the approximate controllability for the system (1) by using the so-called resolvent operator type condition. The most important proof of the approximate controllability in this article are succinctly stated as follows: for any $x^a \in L^2(\Omega; X)$, by selecting proper control $u(t) \in L^2_{\mathcal{F}}(I; U)$, there exists a mild solution $x^\lambda(t) : (-\infty, a] \rightarrow X$ for the system (1), such that $x^\lambda(a) \rightarrow x^a$ in $L^2_{\mathcal{F}}(I; X)$ as $\lambda \rightarrow 0^+$.

Theorem 4.1. *If the assumptions (H0)–(H3) are satisfied, then for any $\lambda \in (0, 1)$, system (1) is approximate controllability on $[0, a]$.*

Proof. Let $x^a \in L^2(\Omega; X)$. For any $\lambda \in (0, 1)$, the function $x^\lambda(t; \varphi, y_0) : (-\infty, a] \rightarrow X$ is the mild solution of the system (1) under the control u^λ given by (10). Then, $x^\lambda(t; \varphi, y_0)$ satisfies

$$\begin{aligned} x^\lambda(a; \varphi, y_0) &= S_2(a)\varphi(0) + \int_0^a S_2(a-s)S_1(s)z_0 ds + \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds \\ &\quad + \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)Bu(\tau) d\tau ds + \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s) \\ &= \Gamma_0^a R(\lambda, \Gamma_0^a) E x^a + (I - \Gamma_0^a R(\lambda, \Gamma_0^a)) \left(S_2(a)\varphi(0) + \int_0^a S_2(a-s)S_1(s)z_0 ds \right. \\ &\quad \left. + \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds + \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s) \right) \\ &\quad + \Gamma_0^a R(\lambda, \Gamma_0^a) \int_0^a \phi(s) dW(s). \end{aligned}$$

By $I - \Gamma_s^a R(\lambda, \Gamma_s^a) = \lambda R(\lambda, \Gamma_s^a)$ and (6), it follows that

$$\begin{aligned} x^\lambda(a; \varphi, y_0) &= x^a - \lambda R(\lambda, \Gamma_0^a) E u^a + \lambda R(\lambda, \Gamma_0^a) \left(S_2(a)\varphi(0) + \int_0^a S_2(a-s)S_1(s)z_0 ds \right. \\ &\quad \left. + \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)f(\tau, x_\tau) d\tau ds + \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)Bu(\tau) d\tau ds \right. \\ &\quad \left. + \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)g(\tau, x_\tau) d\tau dW(s) + \int_0^a \phi(s) dW(s) \right). \end{aligned} \quad (15)$$

From condition (H2)(iii), it yields that

$$\begin{aligned} \mathbf{E} \int_0^a \int_0^s \|f(\tau, x_\tau^\lambda)\|^2 d\tau ds &\leq \int_0^a \int_0^s L_1 d\tau ds < +\infty, \\ \mathbf{E} \int_0^a \int_0^s \|g(\tau, x_\tau^\lambda)\|_Q^2 d\tau ds &\leq \int_0^a \int_0^s L_1 d\tau ds < +\infty, \end{aligned}$$

which means that there are two subsequences, still denoted by $\{f(s, x_s^\lambda) : \lambda \in (0, 1)\}$ and $\{g(s, x_s^\lambda) : \lambda \in (0, 1)\}$, that converge weakly to $f^*(s)$ in X and $g^*(s)$ in $L_0^2(K; X)$, respectively, for every $s \in [0, a]$. By conditions (H3) and (3), we can easily obtain that $S_i(t)(t > 0)$ is compact, $i = 1, 2$, then

$$\begin{aligned} \mathbf{E} \left\| \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)(f(\tau, x_\tau^\lambda) - f^*(\tau)) d\tau ds \right\|^2 &\rightarrow 0, \\ \mathbf{E} \left\| \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)(g(\tau, x_\tau^\lambda) - g^*(\tau)) d\tau ds \right\|^2 &\rightarrow 0. \end{aligned}$$

Therefore, by conditions (H1) and (15), we obtain

$$\begin{aligned}
\mathbf{E} \|x^\lambda(a; \varphi, y_0) - x^a\|^2 &\leq 6\mathbf{E} \left\| \lambda R(\lambda, \Gamma_0^a) \left(\mathbf{E}x^a + S_2(a)\varphi(0) + \int_0^a S_2(a-s)S_1(s)z_0 ds \right) \right\|^2 \\
&+ 6\mathbf{E} \left\| \lambda R(\lambda, \Gamma_0^a) \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)(f(s, x_s) - f^*(\tau)) d\tau ds \right\|^2 \\
&+ 6\mathbf{E} \left\| \lambda R(\lambda, \Gamma_0^a) \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)f^*(\tau) d\tau ds \right\|^2 \\
&+ 6\mathbf{E} \left\| \lambda R(\lambda, \Gamma_0^a) \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)(g(s, x_s) - g^*(\tau)) d\tau ds \right\|^2 \\
&+ 6\mathbf{E} \left\| \lambda R(\lambda, \Gamma_0^a) \int_0^a \int_0^s S_2(a-s)S_1(s-\tau)g^*(\tau) d\tau ds \right\|^2 \\
&+ 6\mathbf{E} \left(\lambda R(\lambda, \Gamma_s^a) \int_0^a \|\phi(s)\|_Q dW(s) \right)^2 \\
&\rightarrow 0, \quad \text{as } \lambda \rightarrow 0^+.
\end{aligned}$$

Hence, we conclude that system (1) is approximately controllable on $[0, a]$. The proof is completed. \square

5 An example

To expound our main results, we give an example of initial value problem for damped elastic stochastic system with infinite delay

$$\left\{ \begin{aligned}
&\partial \left(\frac{\partial}{\partial t} z(t, x) - 4 \frac{\partial^2}{\partial x^2} z(t, x) \right) = \left(\frac{\partial^4}{\partial x^4} z(t, x) + \int_{-\infty}^t c(t, s) f(s, z(s, x)) ds + Bu(t, x) \right) \partial t \\
&+ \int_{-\infty}^t k(t, s) h(s, z(s, x)) ds \partial \eta(t), \quad t \leq 0, t \in [0, 1], x \in [0, \pi], \\
&z(t, 0) = z(t, \pi) = 0, \quad t \in [0, 1], \\
&z(t, x) = \phi(t, x), \quad t \leq 0, x \in [0, \pi], \\
&\frac{\partial}{\partial t} z(0, x) = y_0,
\end{aligned} \right. \quad (16)$$

where $\phi(\cdot, \cdot)$ is \mathcal{F}_0 -measurable and $c(\cdot, \cdot)$, $k(\cdot, \cdot)$, $f(\cdot, \cdot)$, $h(\cdot, \cdot)$ are defined as follows. $\eta(t)$ denotes a one-dimensional standard Brownian motion. This system (16) can simulate the Russell's spacial hysteresis model for an elastic beam [42], and the spacial hysteresis model can be applied to a pipeline bending vibration model with structural damping containing flowing fluid (see [43] for the case without damping and stochastic terms).

Let $X = L^2([0, \pi], \mathbb{R})$ with the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, $K = \mathbb{R}$, we defined $z(t)(\cdot) = z(t, \cdot)$ and $\phi(t)(\cdot) = \phi(t, \cdot)$. And $A : D(A) \rightarrow X$ be the linear operator by

$$Az = -\frac{\partial^2 z}{\partial x^2}, \quad D(A) = W^2(0, \pi) \cap W_0^1(0, \pi).$$

Then $-A$ generates a compact, analytic, and self-adjoint C_0 -semigroup $T(t)(t \geq 0)$. So, (H0) and (H3) are verified. In fact (see [14,18]), $-A$ has a discrete spectrum, which is composed of the eigenvalues $-n^2$, $n \in \mathbb{N}^+$. The eigenvectors corresponding to the orthogonal eigenvalues are $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n \in \mathbb{N}^+$. And the following properties hold

(i) If $x \in D(A)$, then

$$Ax = - \sum_{n=1}^{\infty} n^2 \langle x, e_n \rangle e_n.$$

(ii) For every $x \in X$, the compact analytic semigroup $T(t)(t \geq 0)$ generated by A is defined as follows:

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n. \quad (17)$$

By (17) and (3), the sets $S_i(t)$ generated by $-\sigma_i A$ are defined as follows:

$$S_i(t)x = \sum_{n=1}^{\infty} e^{-n^2 \sigma_i t} \langle x, e_n \rangle e_n, \quad (18)$$

where $\sigma_1 = 2 - \sqrt{3}$ and $\sigma_2 = 2 + \sqrt{3}$.

Here, let the phase space $\mathcal{B} = C_0 \times L^2(g : X)$ and its norm is defined as follows:

$$\|\phi\|_{\mathcal{B}} = \|\phi(0)\| + \left(\int_{-\infty}^0 g(\theta) \|\phi(\theta)\|^2 d\theta \right)^{\frac{1}{2}}.$$

As long as we choose an appropriate function g , we can make the phase space $C_0 \times L^2(g : X)$ satisfies the axioms (A1)–(A3).

Now, we make the following assumptions for the system (16):

(h1) The functions $f(\cdot, \cdot) : \mathbb{R} \times X \rightarrow \mathbb{R}$ and $h(\cdot, \cdot) : \mathbb{R} \times X \rightarrow \mathbb{R}$ are Lipschitz continuous and uniformly bounded.

(h2) The functions $c(\cdot, \cdot) : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $k(\cdot, \cdot) : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and satisfy that

$$|c(t, t + \theta)|, |k(t, t + \theta)| < m(\theta) \quad \text{and} \quad \left(\int_{-\infty}^0 \frac{1}{g(\theta)} |m(\theta)|^2 d\theta \right) < \infty.$$

(h3) $\phi(t, x) \in L^2(\Omega; X)$.

Next, we define the functions $f(\cdot, \cdot) : [0, 1] \times \mathcal{B} \rightarrow X$ and $g(\cdot, \cdot) : [0, 1] \times \mathcal{B} \rightarrow \mathbb{R}$, respectively, by

$$\begin{aligned} f(t, \phi)(\cdot) &= \int_{-\infty}^0 c(t, t + \theta) f(t + \theta, \phi(\theta, x)) d\theta, \\ g(t, \phi)(\cdot) &= \int_{-\infty}^0 k(t, t + \theta) h(t + \theta, \phi(\theta, x)) d\theta, \end{aligned}$$

for any $t \in [0, 1]$, $\phi \in \mathcal{B}$. Under these conditions, system (16) can be reformulated as system (1). From the assumptions (h1) and (h2), for any $t \in [0, 1]$, $\phi_1, \phi_2 \in \mathcal{B}$, it follows that

$$\begin{aligned} \|f(t, \phi_1) - f(t, \phi_2)\|_X^2 &= \int_0^\pi \left(\int_{-\infty}^0 |c(t, t + \theta)| |f(t + \theta, \phi_1(\theta, x)) - f(t + \theta, \phi_2(\theta, x))| d\theta \right)^2 dx \\ &\leq \int_0^\pi \left(\int_{-\infty}^0 |m(\theta)| K_1 \|\phi_1(\theta) - \phi_2(\theta)\|_X d\theta \right)^2 dx \\ &\leq \int_0^\pi K_1^2 \left(\int_{-\infty}^0 \frac{1}{g(\theta)} |m(\theta)|^2 d\theta \right) \left(\int_{-\infty}^0 g(\theta) \|\phi_1(\theta) - \phi_2(\theta)\|_X^2 d\theta \right) dx \\ &\leq \int_0^\pi K_2 \|\phi_1(\theta) - \phi_2(\theta)\|_{\mathcal{B}} dx \\ &\leq K_2 \pi \|\phi_1(\theta) - \phi_2(\theta)\|_{\mathcal{B}}, \end{aligned}$$

and

$$\|g(t, \phi_1) - g(t, \phi_2)\|_X^2 \leq K_3 \|\phi_1(\theta) - \phi_2(\theta)\|_{\mathcal{B}},$$

where K_1, K_2 , and K_3 are constants. Hence, by the uniform boundedness of $c(\cdot, \cdot)$, $k(\cdot, \cdot)$, $f(\cdot, \cdot)$, and $h(\cdot, \cdot)$, we deduce that the functions $f(\cdot, \cdot)(\cdot)$ and $g(\cdot, \cdot)(\cdot)$ are the uniform boundedness. Further, condition (H2) is satisfied.

Later [14,16,30], we take

$$U = \left\{ u = \sum_{n=2}^{\infty} u_n e_n : \sum_{n=2}^{\infty} u_n^2 < +\infty \right\},$$

then U is a Banach space endowed with the norm

$$\|u\| = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{\frac{1}{2}}.$$

And, we define the linear bounded operator $B : U \rightarrow X$ as follows:

$$Bu = 2u_2 e_1(x) + \sum_{n=2}^{\infty} u_n e_n(x), \quad \text{for } u = \sum_{n=2}^{\infty} u_n e_n \in U.$$

It is obvious that $\|B\| \leq \sqrt{5}$. It is easy to obtain

$$B^*w = (2w_1 + w_2)e_2(x) + \sum_{n=3}^{\infty} w_n e_n(x), \quad \text{for } w = \sum_{n=1}^{\infty} w_n e_n(x) \in X. \quad (19)$$

Finally, we only need to verify condition (H1). The mild solution of linear systems

$$\begin{cases} y''(t) + \rho A y'(t) + A^2 y(t) = f(t), & t \in [0, 1], \\ y(t) = x_0, y'(0) = y_0, \end{cases}$$

is expressed as

$$y(t) = S_2(t)x_0 + \int_0^t S_2(t-s)S_1(s)(y_0 + \sigma_2 A x_0) ds + \iint_{00}^{ts} S_2(t-s)S_1(s-\tau)f(\tau) d\tau ds.$$

$T(t)(t \geq 0)$ is self-adjoint, then

$$S_1^*(t) = S_1(t), \quad S_2^*(t) = S_2(t), \quad t \in [0, 1]. \quad (20)$$

By combining (19) and (20), it yields that

$$B^*S_1^*(t)S_2^*(t)y = (2e^{-(\sigma_1+\sigma_2)t}y_1 + e^{-4(\sigma_1+\sigma_2)t}y_2)e_2(x) + \sum_{n=3}^{\infty} e^{-n^2(\sigma_1+\sigma_2)t}y_n e_n(x), \quad (21)$$

for $y = \sum_{n=1}^{\infty} y_n e_n(x) \in X$ and $t \in [0, 1]$. Next, let $\|B^*S_1^*(t)S_2^*(t)y\| = 0$, then

$$\|2e^{-(\sigma_1+\sigma_2)t}y_1 + e^{-4(\sigma_1+\sigma_2)t}y_2\|^2 + \sum_{n=3}^{\infty} \|e^{-n^2(\sigma_1+\sigma_2)t}y_n\|^2 = 0, \quad t \in [0, 1],$$

which implies that $y_n = 0, n = 1, 2, \dots$. Therefore, $y = 0$. According to Theorem 2.6, we obtain that (H1) holds. By Theorem 4.1, we can obtain that the control system (16) is approximately controllable on the interval $[0, 1]$.

6 Conclusion

In this article, we investigate the sufficient conditions for the approximate controllability of stochastic elastic systems with structural damping and infinite delay. First, we obtain the expression of mild solution of system

(1) by combining the theory of stochastic analysis and semigroups. Second, the control function $u(\cdot)$ is constructed through the defined resolvent operator $R(\lambda, \Gamma_0^a)$, and further estimates of $u(\cdot)$ are required. Then, on the basis these two estimates, we obtained the existence of mild solution and the approximate controllability of system (1) through the Schauder fixed point theorem and the resolvent operator type condition. This article extends the conclusions of stochastic control theory on damped elastic systems.

There are two direct problems which require further study. On the one hand, the damped elastic systems are applied to the optimal control problem or (null) controllability problem, so we will further study the optimal control problem and (null) controllability problem of this system. On the other hand, a finite number of discontinuous points are generated during modeling, which are called impulse equations, and we will discuss the properties of damped elastic systems with instantaneous and noninstantaneous pulses.

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